THE NUMBER OF DISCONNECTED VERTEX LABELLED GRAPHS OF ORDER FIVE WITH MAXIMUM 3-PARALEL EDGES IS SIX AND CONTAINS NO LOOPS.

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Abstract. *Given n vertices and m edges, there are a lot of graphs that can be constructed, either connected or disconnected, simple or not simple. A graph is called connected if there is at least a path connecting every pair of vertices in that graph; and a graph is called simple if that graph does not contain any loops or parallel edges. In this paper will be discussed the number of disconnected vertex labeled graphs with order five without loops and the maximum allowable 3-parallel edges is six.*

Keywords: disconnected graph, vertex labelled, order, 3-parallel, loops

1 INTRODUCTION

There is no doubt that nowadays many real-life problems can be represented using graph theoretic concept. In network problems, for example network transportation problem, the vertices can represent the cities or the road junctions, and the edges represent the roads that connecting them. The edge can be assigned a number which can represent any nonstructural information such as distance, cost, time, etc. By representing the network transportation problem with graph, the problem can be visualized and easier to be solved. In [1] the graph theoretic concepts were explored and exposed related with the interconnection networks [1]; and a comprehensive applications of graph theory in science and engineering was given [2]. To solve the system of simultaneous linear equations that related with the current in electrical network, Kirchhoff developed the theory of trees; and Cayley enumerated the number of isomer of hydrocarbon and found that the process is related with counting rooted tree [3]. Harary and Palmer in 1973 gave the main idea for labeling and counting graph [4]. Some of the methods related with graph enumeration also investigated in [5-7].

Graphically, many graphs that can be constructed if given n vertices and m edges. The graph constructed can be simple which means does not contain any loops or parallel edges, or otherwise is not simple. Moreover, the graph constructed also can be connected which means that for every pair of vertices in the graph, there is at least one path connecting them, and otherwise is disconnected. In [8], the formula for counting the number of disconnected vertex labelled graphs with order maximal four already investigated; and in [9] the number of disconnected vertex labelled graphs with order five without parallel edges also already investigated. In this paper will be discussed the number of disconnected vertex labelled graph of order five without loops (parallel edges are permissible) with the maximum allowable parallel edges are 3-parallel, and the maximum 3-paralel edges is six. The paper is organized as follows: Introduction is given in Section 1, in Section 2 the observation and the pattern constructed will be discussed. Results and Discussion will be provided in Section 3, followed by Conclusion in Section 4.

2 OBSERVATION AND THE PATTERN CONSTRUCTED

Given a graph G(V,E) of order five (the number of vertices is 5), and the number of edges in the graph is m; let g, and t are as the following: g is the number of nonparallel edges, t is the number of edges that connect different pairs of vertices (parallel edges that connect the same pair of vertices is counted as one), p_i is the number of i-parallel edges, i = 2,3,… , then $t = \sum_{i=1} p_i + g$ and $m = \sum_{i=1} j.\, p_i + g$, $j \in \mathbb{N}$. Note that p_1 is equal with g. Please look at the Figure 1 below:

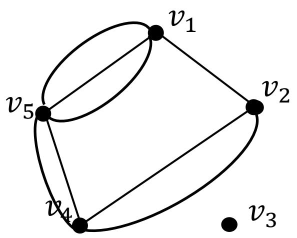


Figure 1. Example a disconnected vertex labelled graph without loops.

From Figure 1 we get g = 1, p_2 = 2, p_3 = 1, t = (2+1+ 1) = 4.
Therefore m = (2 × p_2) + p_3+ g = 2 (2) + 1(3) + 1 = 4 + 3 + 1 = 8

According to [9], the maximum number of edges of a disconnected vertex labelled graph of order five (without parallel edges or loops) is six, therefore for this problem, $1 \leq g \leq 6$. Since the maximum number of 3-paralel is six, then the maximum number of allowable edges is eighteen. Due to the space limitation we only provide some of the patterns constructed. In the construction, the isomorphic graphs are counted as one. The following tables show some of the pattern constructed.

Table 1. The results for graph constructed for graph with n = 5, t = 1 and maximum 3-paralel is six.

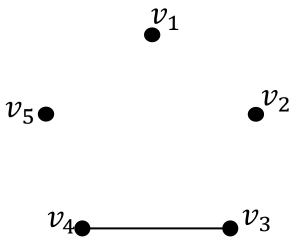
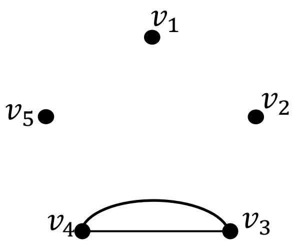
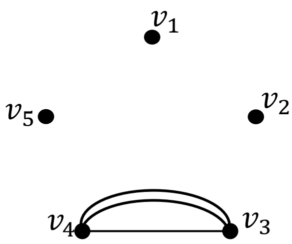
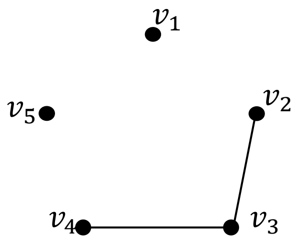
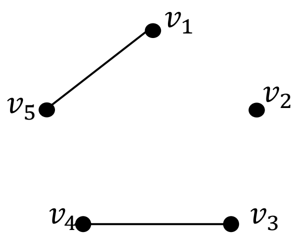
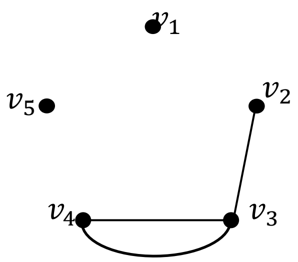
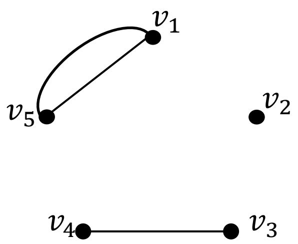
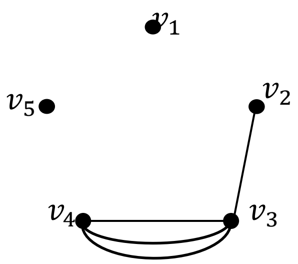
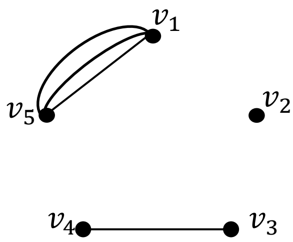
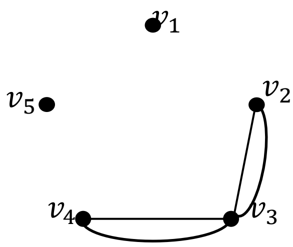
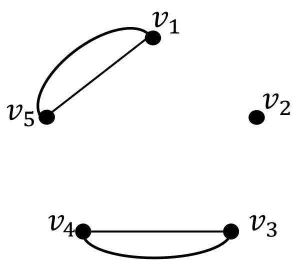
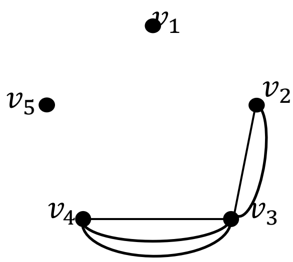
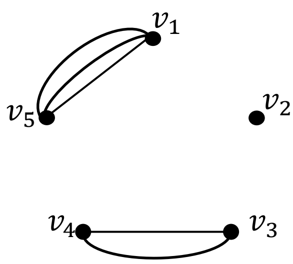
Pattern		Total
n = 5 m =1 g = 1 t = 1		10
TOTAL		10
n = 5 m = 2 g = 0 t = 1 $p_2 = 1$		10
TOTAL		10
n = 5 m = 3 g = 0 t = 1 $p_3 = 1$		10
TOTAL		10

Table 1. The results for graph constructed for graph with n = 5, t = 2 and maximum 3-paralel is six.

	Pattern	Total
n = 5 m = 2 g = 2 t = 2		30
		15
	TOTAL	45
n = 5 m = 3 g = 1 t = 2 $p_2 = 1$		60
		30
	TOTAL	90
n = 5 m = 4 g = 1 t = 2 $p_3 = 1$		60
		30

		
n = 5 m = 4 g = 0 t = 2 $p_2 = 2$		30
		15
	TOTAL	135
n = 5 m = 5 g = 0 t = 2 $p_2 = 1$		60
$p_3 = 1$		30
		90

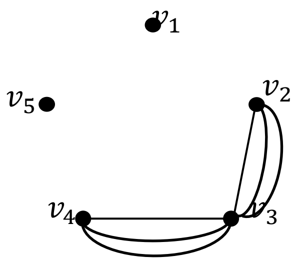
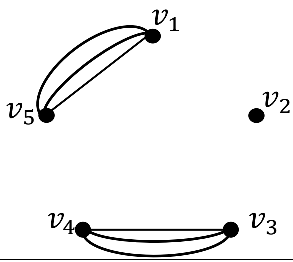
n = 5 m = 6 g = 0 t = 2 $p_3 = 2$		30
		15
	TOTAL	45

Table 3 below shows the number of disconnected graphs constructed according to t and m. Please to be noted here that there are a lot more patterns observed, but due to the space limitation then those patterns are not given here. After doing the observation, the we grouped them according to t, t = 1,2,3,4,5,6 as shown in Table 3 below.

Table 3. The number of disconnected vertex labelled graphs of order five with maximum 3- parallel is six and contain no loops.

m	The number of disconnected vertex labelled graphs of order five with maximum 3- parallel is six and contain no loops.					
	t					
	1	2	3	4	5	6
1	10					
2	10	45				
3	10	90	120			
4		135	360	85		
5		90	720	340	30	
6		45	840	850	150	5
7			720	1360	450	30
8			360	1615	900	105
9			120	1360	1350	250
10				850	1530	450
11				340	1350	630
12				85	900	705
13					450	630
14					150	450
15					30	250
16						105

17						30
18						5

3. **RESULTS AND DISCUSSION**

According to Table 3, there are some sequences of numbers occurs such as:

For t_1 : 10, 10, 10
For t_2 : 45, 90, 135, 90, 45
For t_3 : 120, 360, 720, 840, 720, 360, 120
For t_4 : 85, 340, 850, 1360, 1615, 1360, 850, 340, 85
For t_5 : 30, 150, 450, 900, 1350, 1530, 1350, 900, 450, 150, 30
For t_6 : 5, 30, 105, 250, 450, 630, 705, 630, 450, 250, 105, 30, 5

Notate:

$G^d_{n,m,t}$ as disconnected labelled graph of order n with m edges and t as the number of edges that connect different pairs of edges on that graph (parallel edges which connect the same pair of edges are counted as one). $N(G^d_{n,m,t}) = |(G^d_{n,m,t})|$ = the number of $G^d_{n,m,t}$.

In this paper, the graph considered in the notation $G^d_{n,m,t}$ is disconnected vertex labelled graph of order five (n=5) with the maximum 3-parallel edges is six and contains no loops. From Table 3 we notice that that table also can be put as in Table 4 and 5 as the following:

Table 4. Another form of Table 3.

m	The number of disconnected vertex labelled graphs of order five with maximum 3- parallel is six and contain no loops.					
	t					
	1	2	3	4	5	6
1	10x**1**					
2	10x**1**	45x**1**				
3	10x**1**	45x**2**	120x**1**			
4		45x**3**	120x**3**	85x**1**		
5		45x**2**	120x**6**	85x**4**	30x**1**	
6		45x**1**	120x**7**	85x**10**	30x**5**	5x**1**
7			120x**6**	85x**16**	30x**15**	5x**6**
8			120x**3**	85x**19**	30x**30**	5x**21**
9			120x**1**	85x**16**	30x**45**	5x**50**
10				85x**10**	30x**51**	5x**90**
11				85x**4**	30x**45**	5x**126**
12				85x**1**	30x**30**	5x**141**
13					30x**15**	5x**126**
14					30x**5**	5x**90**
15					30x**1**	5x**50**
16						5x**21**
17						5x**6**
18						5x**1**

Table 5. Another form of Table 4 by taking only the bold number in each cell.

m	The number of disconnected vertex labelled graphs of order five with maximum 3- parallel is six and contain no loops.					
	t					
	1	2	3	4	5	6
1	1					
2	1	1				
3	1	2	1			
4		3	3	1		
5		2	6	4	1	
6		1	7	10	5	1
7			6	16	15	6
8			3	19	30	21
9			1	16	45	50
10				10	51	90
11				4	45	126
12				1	30	141
13					15	126
14					5	90
15					1	50
16						21
17						6
18						1

From Table 5 we derive the following sequence of numbers:

For t_1 : 1, 1, 1
For t_2 : 1, 2, 3, 2, 1
For t_3 : 1, 3, 6, 7, 6, 3, 1
For t_4 : 1, 4, 10, 16, 19, 16, 10, 4, 1
For t_5 : 1, 5, 15, 30, 45, 51, 45, 30, 15, 5, 1
For t_6 : 1, 6, 21, 50, 90, 126, 141, 126, 90, 50, 21, 6, 1

Result 1: For t = 1 , $1 \leq m \leq 3$ $N(G^d_{5,m,1}) = 10$
Proof: Obvious

Results 2: For t = 2, $2 \leq m \leq 6$, $N(G^d_{5,m,2}) = \begin{cases} 45(m-1) \ ; \ 2 \leq m \leq 4 \\ -45(m-7) \ ; \ 4 \leq m \leq 6 \end{cases}$

Proof: For t =2, the sequence of numbers is 1, 2, 3, 2, 1. By splitting the sequence into two 1, 2, 3 and 3, 2, 1 then both sequences relate with an arithmetic sequence of order one. Therefore for, $2 \leq m \leq 4$ we get:

$45 = 2a_1 + a_0$ (1)
$90 = 3a_1 + a_0$ (2)
$135 = 4a_1 + a_0$ (3)

Solving that linear equation we get $a_1 = 45$, $a_0 = -45$, and therefore

$a_m = 45m - 45 = 45\,(m - 1)$ (4)
For $4 \leq m \leq 6$ we get
$135 = 4a_1 + a_0$ (5)
$90 = 5a_1 + a_0$ (6)
$45 = 6a_1 + a_0$ (7)
Solving that linear equation we get $a_1 = -45$, $a_0 = 315$, and therefore
$a_m = -45m + 315 = -45(m - 7)$ (8)
By (4) and (8) then $N(G^d_{5,m,2}) = \begin{cases} 45(m-1)\;;\; 2 \leq m \leq 4 \\ -45(m-7)\;;\; 4 \leq m \leq 6 \end{cases}$

Result 3 : For t = 3, $3 \leq m \leq 9$, $N(G^d_{5,m,3}) = \begin{cases} -60\,(m^3 - 13\,m^2 + 50\,m - 62)\;;\; 3 \leq m \leq 6 \\ 60\,(m^3 - 23\,m^2 + 170\,m - 394)\;;\; 6 \leq m \leq 9 \end{cases}$

Proof: For t =3, the sequence of numbers is 1, 3, 6, 7, 6, 3, 1. By splitting the sequence into two sequences 1, 3, 6, 7 and 7, 6, 3, 1 then those sequences are arithmetic sequences of order three and the related polynomial is $a_m = a_3 m^3 + a_2 m^2 + a_1 m + a_0$.
Therefore for $3 \leq m \leq 6$ we get:
$120 = 27a_3 + 9a_2 + 3a_1 + a_0$ (9)
$360 = 64a_3 + 16a_2 + 4a_1 + a_0$ (10)
$720 = 125a_3 + 25a_2 + 5a_1 + a_0$ (11)
$840 = 216a_3 + 36a_2 + 6a_1 + a_0$ (12)
Solving that linear equation we get $a_3 = -60$, $a_2 = 780$, $a_1 = -3000$, $a_0 = 3720$ and therefore
$a_m = -60\,m^3 + 780m^2 - 3000m + 3720 = -60\,(m^3 - 13m^2 + 50m - 62)$ (13)
For $6 \leq m \leq 9$ we get:
$840 = 216a_3 + 36a_2 + 6a_1 + a_0$ (14)
$720 = 343a_3 + 49a_2 + 7a_1 + a_0$ (15)
$360 = 512a_3 + 64a_2 + 8a_1 + a_0$ (16)
$120 = 729a_3 + 81a_2 + 9a_1 + a_0$ (17)
Solving that linear equation we get $a_3 = 60$, $a_2 = -1380$, $a_1 = 10200$, $a_0 = -23640$ and therefore
$a_m = 60\,m^3 - 1380m^2 + 10200m - 23640 = 60\,(m^3 - 23m^2 + 170m - 394)$ (18)
By (13) and (18) then $(G^d_{5,m,3}) = \begin{cases} -60\,(m^3 - 13\,m^2 + 50\,m - 62)\;;\; 3 \leq m \leq 6 \\ 60\,(m^3 - 23\,m^2 + 170\,m - 394)\;;\; 6 \leq m \leq 9 \end{cases}$

Result 4: For t = 4, $4 \leq m \leq 12$, $N(G^d_{5,m,4}) = \begin{cases} \frac{-510}{12}(m^3 - 18\,m^2 + 95\,m - 10658)\;;\; 4 \leq m \leq 8 \\ \frac{510}{12}\,m^3 - \frac{153000}{12}m^2 + \frac{176370}{12}m - \frac{548785500}{12};\; 8 \leq m \leq 12 \end{cases}$

Proof: For t =4, the sequence of numbers is 1, 4, 10, 16, 19, 16, 10, 4, 1. By splitting the sequence into two sequences 1, 4, 10, 16, 19 and 19, 16, 10, 4, 1 then those sequences are arithmetic sequences of order three and the related polynomial is $a_m = a_3 m^3 + a_2 m^2 + a_1 m + a_0$.
Therefore for $4 \leq m \leq 8$ we get:
$85 = 64a_3 + 16a_2 + 4a_1 + a_0$ (18)
$340 = 125a_3 + 25a_2 + 5a_1 + a_0$ (19)
$850 = 216a_3 + 36a_2 + 6a_1 + a_0$ (20)
$1360 = 343a_3 + 49a_2 + 7a_1 + a_0$ (21)
Solving that linear equation we get $a_3 = \frac{-510}{12}$, $a_2 = \frac{9180}{12}$, $a_1 = \frac{-48450}{12}$, $a_0 = \frac{5435580}{12}$ and therefore
$a_m = \frac{-510}{12}m^3 + \frac{9180}{12}m^2\ \frac{-48450}{12}m + \frac{5435580}{12} = \frac{-510}{12}(m^3 - 18\,m^2 + 95m - 10658)$ (22)
For $8 \leq m \leq 12$ we get
$1615 = 512a_3 + 64a_2 + 8a_1 + a_0$ (23)

1360 $= 729a_3 + 81a_2 + 9a_1 + a_0$ (24)
850 $= 1000a_3 + 100a_2 + 10a_1 + a_0$ (25)
340 $= 1331a_3 + 121a_2 + 11a_1 + a_0$ (26)
Solving that linear equation we get $a_3 = \frac{510}{12}$, $a_2 = \frac{-153000}{12}$, $a_1 = \frac{176370}{12}$, $a_0 = \frac{-548785500}{12}$ and therefore $a_m = = \frac{510}{12}m^3 - \frac{153000}{12}m^2 + \frac{176370}{12}m - \frac{548785500}{12}$ (27)

By (22) and (27) we get $N(G^d_{5,m,4}) = \begin{cases} \frac{-510}{12}(m^3 - 18m^2 + 95m - 10658)\,; & 4 \le m \le 8 \\ \frac{510}{12}m^3 - \frac{153000}{12}m^2 + \frac{176370}{12}m - \frac{548785500}{12}\,; & 8 \le m \le 12 \end{cases}$ (28)

Result 5: For t=5, $5 \le m \le 15$, $N(G^d_{5,m,5}) = \begin{cases} -5(m^4 - 25m^3 + 215m^2 - 785m + 1044)\,; & 5 \le m \le 10 \\ -5(m^4 - 55m^3 + 1115m^2 - 9815m + 31344)\,; & 10 \le m \le 15 \end{cases}$

Proof: For t =5, the sequence of numbers is 1, 5, 15, 30, 45, 51, 45, 30, 15, 5, 1. By splitting the sequence into two sequences 1, 5, 15, 30, 45, 51, and 51, 45, 30, 15, 5, 1, then those sequences are arithmetic sequences of order four and the related polynomial is $a_m = a_4 m^4 + a_3 m^3 + a_2 m^2 + a_1 m + a_0$. For $5 \le m \le 10$ we get:
30 $= 625a_4 + 125a_3 + 25a_2 + 5a_1 + a_0$ (29)
150 $= 1296a_4 + 216a_3 + 36a_2 + 6a_1 + a_0$ (30)
450 $= 2401a_4 + 343a_3 + 49a_2 + 7a_1 + a_0$ (31)
900 $= 4096a_4 + 512a_3 + 64a_2 + 8a_1 + a_0$ (32)
1350 $= 6561a_4 + 729a_3 + 81a_2 + 9a_1 + a_0$ (33)
Solving that linear equation we get a_4= -5, a_3= 125, a_2= -1075, a_1= 3925, a_0 = -5220 and therefore $a_m = -5m^4 + 125m^3 - 1075m^2 + 3925m - 5220 = -5(m^4 - 25m^3 + 215m^2 - 785m + 1044)$ (34)
For $10 \le m \le 15$ we get:
1530 $= 10000a_4 + 1000a_3 + 100a_2 + 10a_1 + a_0$ (35)
1350 $= 14641a_4 + 1331a_3 + 121a_2 + 11a_1 + a_0$ (36)
900 $= 20736a_4 + 1728a_3 + 144a_2 + 12a_1 + a_0$ (37)
450 $= 28561a_4 + 2197a_3 + 169a_2 + 13a_1 + a_0$ (38)
150 $= 38416a_4 + 2744a_3 + 196a_2 + 14a_1 + a_0$ (39)
Solving that linear equation we get a_4=-5, a_3= 275, a_2= -5575, a_1= 40975, a_0=-156720 and therefore $a = -5m^4 + 275m^3 - 5575m^2 + 40975m - 156720 = -5(m^4 - 55m^3 + 1115m^2 - 9815m + 31344)$ (40)
By (34) and (40) we get $N(G^d_{5,m,5}) = \begin{cases} -5(m^4 - 25m^3 + 215m^2 - 785m + 1044)\,; & 5 \le m \le 10 \\ -5(m^4 - 55m^3 + 1115m^2 - 9815m + 31344)\,; & 10 \le m \le 15 \end{cases}$

Result 6: For t=6 , $6 \le m \le 18$,

$$N(G^d_{5,m,6}) == \begin{cases} \frac{75}{720}\left(m^6 - 53m^5 + 1141m^4 - 12803m^3 + 79402m^2 - 259160m + 348720\right); & 6 \le m \le 12 \\ \frac{75}{720}\left(m^6 - 91m^5 + 3421m^4 - 67933m^3 + 750802m^2 - 4376392m + 10516080\right); & 12 \le m \le 18 \end{cases}$$

Proof: For t =6, the sequence of numbers is 1, 6, 21, 50, 90, 126, 141, 126, 90, 50, 21, 6, 1. By splitting the sequence into two sequences 1, 6, 21, 50, 90, 126, 141, and 141, 126, 90, 50, 21, 6, 1, then those sequences are arithmetic sequences of order six and the related polynomial is $a_m = a_6 m^6 + a_5 m^5 + a_4 m^4 + a_3 m^3 + a_2 m^2 + a_1 m + a_0$. For $6 \le m \le 12$ we get:
5 $= 46656a_6 + 777a_5 + 1296a_5 + 216a3 + 36a_2 + 6a_1 + a_0$ (41)
30 $= 117649a_6 + 16807a_5 + 2401a_5 + 343a3 + 49a_2 + 7a_1 + a_0$ (42)
105 $= 262144a_6 + 32768a_5 + 4096a_5 + 512a3 + 64a_2 + 8a_1 + a_0$ (43)
250 $= 531441a_6 + 59049a_5 + 6561a_5 + 729a3 + 81a_2 + 9a_1 + a_0$ (44)
450 $= 100000a_6 + 100000a_5 + 10000a_5 + 1000a3 + 100a_2 + 10a_1 + a_0$ (45)

630 $= 1771561a_6 + 161051a_5 + 14641a_5 + 1331a3 + 121a_2 + 11a_1 + a_0$ (46)
705 $= 2985984a_6 + 248832a_5 + 20736a_5 + 1728a3 + 144a_2 + 12a_1 + a_0$ (47)
Solving that linear equation we get $a_6 = \frac{75}{720}$, $a_5 = \frac{-3975}{720}$, $a_4 = \frac{85575}{720}$, $a_3 = \frac{-960225}{720}$, $a_2 = \frac{5955150}{720}$, $a_1 = \frac{-19437000}{720}$ $a_0 = \frac{26154000}{720}$ and therefore

$$a_m = \frac{75}{720} m^6 - \frac{3975}{720} m^5 + \frac{85575}{720} m^4 - \frac{960225}{720} m^3 + \frac{5955150}{720} m^2 - \frac{19437000}{720} m + \frac{26154000}{720}$$

$$= \frac{75}{720} (m^6 - 53 m^5 + 1141 m^4 - 12803 m^3 + 79402 m^2 - 259160 m + 348720) \quad (48)$$

For $12 \leq m \leq 18$ we get:
705 $= 2985984a_6 + 248832a_5 + 20736a_4 + 1728a3 + 144a_2 + 12a_1 + a_0$ (49)
630 $= 4826809a_6 + 371293a_5 + 28561a_4 + 2197a3 + 169a_2 + 13a_1 + a_0$ (50)
450 $= 7529536a_6 + 537824a_5 + 38416a_4 + 2744a3 + 196a_2 + 14a_1 + a_0$ (51)
250 $= 11390625a_6 + 759375a_5 + 50625a_4 + 3375a3 + 225a_2 + 15a_1 + a_0$ (52)
105 $= 16777216a_6 + 1048576a_5 + 65536a_4 + 4096a3 + 256a_2 + 16a_1 + a_0$ (53)
30 $= 24137569a_6 + 1419857a_5 + 83521a_5 + 4913a3 + 289a_2 + 17a_1 + a_0$ (54)
5 $= 340112224a_6 + 1889568a_5 + 104976a_5 + 5832a3 + 324a_2 + 18a_1 + a_0$ (55)
Solving that linear equation we get $a_6 = \frac{75}{720}$, $a_5 = \frac{-6825}{720}$, $a_4 = \frac{256575}{720}$, $a_3 = \frac{-5094975}{720}$, $a_2 = \frac{56310150}{720}$, $a_1 = \frac{-328229400}{720}$ $a_0 = \frac{788706000}{720}$ and therefore

$$a_m = \frac{75}{720} m^6 - \frac{68255}{720} m^5 + \frac{256575}{720} m^4 - \frac{5094975}{720} m^3 + \frac{56310150}{720} m^2 - \frac{328229400}{720} m + \frac{788706000}{720}$$

$$= \frac{75}{720} (m^6 - 91 m^5 + 3421 m^4 - 67933 m^3 + 750802 m^2 - 4376392 m + 10516080) \quad (56)$$

By (48) and (56) we get

$$N(G^d_{5,m,6}) = \begin{cases} \frac{75}{720}\left(m^6 - 53 m^5 + 1141 m^4 - 12803 m^3 + 79402 m^2 - 259160 m + 348720\right); & 6 \leq m \leq 12 \\ \frac{75}{720}\left(m^6 - 91 m^5 + 3421 m^4 - 67933 m^3 + 750802 m^2 - 4376392 m + 10516080\right); & 12 \leq m \leq 18 \end{cases}$$

CONCLUSION

Based on the discussion above, we can conclude that the number of disconnected vertex labelled graphs of order five with maximum number of 3-paralel edges is six and contains no loops $N(G^d_{5,m,t})$, $1 \leq t \leq 6$ are:

a. $N(G^d_{5,m,1}) = 10 \quad ; 1 \leq m \leq 3$
b. $N(G^d_{5,m,2}) = \begin{cases} 45(m - 1) ; & 2 \leq m \leq 4 \\ -45(m - 7) ; & 4 \leq m \leq 6 \end{cases}$
c. $N(G^d_{5,m,3}) = \begin{cases} -60 (m^3 - 13 m^2 + 50 m - 62) ; & 3 \leq m \leq 6 \\ 60 (m^3 - 23 m^2 + 170 m - 394) ; & 6 \leq m \leq 9 \end{cases}$
d. $N(G^d_{5,m,4}) = \begin{cases} \frac{-510}{12} (m^3 - 18 m^2 + 95 m - 10658) ; & 4 \leq m \leq 8 \\ \frac{510}{12} m^3 - \frac{153000}{12} m^2 + \frac{176370}{12} m - \frac{548785500}{12}; & 8 \leq m \leq 12 \end{cases}$
e. $N(G^d_{5,m,5}) = \begin{cases} -5 (m^4 - 25 m^3 + 215 m^2 - 785 m + 1044) ; & 5 \leq m \leq 10 \\ -5 (m^4 - 55 m^3 + 1115 m^2 - 9815 m + 31344) ; & 10 \leq m \leq 15 \end{cases}$
f. $N(G^d_{5,m,6}) =$

$$\begin{cases} \frac{75}{720}(m^6 - 53 m^5 + 1141 m^4 - 12803 m^3 + 79402 m^2 - 259160 m + 348720); & 6 \leq m \leq 12 \\ \frac{75}{720}(m^6 - 91 m^5 + 3421 m^4 - 67933 m^3 + 750802 m^2 - 4376392 m + 10516080); & 12 \leq m \leq 18 \end{cases}$$

REFERENCES

[1] Vasudev, C. *Graph Theory with Application*. New Age International Limited, 2006.
[2] Hsu, L.H., and Lin, C.K. *Graph Theory and interconnection network*. Taylor and Francis Group, LLC, New York, 2009.
[3] Foulds, L.R. *Graph Theory Applications*. Springer-Verlag, New York, USA, 1992.
[4] Harary F, and E. M. Palmer, *Graphical Enumeration*. Academic Press, New York, 1973.
[5] Stanley, R.P ,. *Enumerative Combinatorics,* **1**, no. 49 of Cambridge Studies in Advanced Mathematics. Cambridge University Press, New York. 1997
[6] Stanley, R.P , *Enumerative Combinatorics,* **2** no. 62 of Cambridge Studies in Advanced Mathematics. Cambridge University Press, New York, 1999.
[7] Agnarsson, G., and R. D. Greenlaw, *Graph Theory Modelling, Application, and Algorithms*. Pearson/Prentice Education, Inc., New Jersey.
[8] Amanto, Wamiliana, Mustofa Usman, and Reni Permata Sari. Counting the Number of Disconnected Vertex Labelled Graphs of Order Maximal Four. *Science International (Lahore),* **29**, no. 6, 1181-1186, 2017.
[9] Wamiliana, Amanto, and G. T. Nagari. Counting the Number of Disconnected labelled Graphs of Order Five Without Parallel Edges. *International Series on Interdisciplinary Science and Technology (INSIST)*, **1**, no. 1, p. 1 – 6, 2016.