MOMENT PROPERTIES OF THE GENERALIZED GAMMA DISTRIBUTION

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ABSTRACT

The generalized gamma (GG) distribution as a generalization of the gamma distribution is considered in this paper. A moment of a generating function (mgf) of the GG distribution is mathematically developed. Based on the mgf, limiting properties of the generalized beta of the second kind (GB2) distribution are discussed. Its properties related to the other well-known distributions, such as, gamma and exponential distributions are obtained.

Keywords: Moment properties, Generalized Beta of the Second Kind, Generalized Gamma distribution; Moment generating function; Gamma distribution; Exponential distribution.

INTRODUCTION

In the statistical literature, modeling of data by generalized probability models has been noted to be advantageous by numerous authors, because selecting the best probability model in a particular case is not an easy task. In survival or lifetime data, each generalization usually includes exponential, Weibull, gamma, and lognormal distributions as its either limiting or special cases. A detailed discussion of these as well as many other related distributions is provided by McDonald (1984) and McDonald and Richards (1987). A general guideline of model selection of some generalized models, such as the generalized beta of the second kind (GB2), has been outlined by them. The GB2 distribution involves four parameters and is a particularly useful family of distributions. It includes the generalized gamma (GG) distribution.

This paper proposes to discuss about the generalized gamma (GG) distribution. The main aim of this paper is to provide a gentle discussion of moment properties of the GG distribution. The properties include relationship with the GB2's moment and limiting properties to the moment of gamma and exponential distributions. In order to achieve this purpose, in Section 2 and 3 we develop moment generating functions of the GB2 and the GG distributions, respectively. Section 4 contains a description of relationship between moments of the GB2 distribution and the GG distribution. In Section 5, this paper examines limiting behavior of the GG's moment. Finally we conclude the paper in Section 6.

Moment Generating Function Of The Gb2 Distribution. The probability density function (pdf) of the GB2 distribution in this paper is given by

$$f(x;\, a,b,m_1,m_2) = \begin{cases} \left(\dfrac{a}{xB(m_1,m_2)}\right) \dfrac{\left[\left(\dfrac{x}{b}\right)^{a}\right]^{m_1}}{\left[1+\left(\dfrac{x}{b}\right)^{a}\right]^{m_1+m_2}} & \text{for} \quad x>0 \\ 0 & \text{else} \end{cases}$$

where $a>0$, $b>0$, $m_1>0$, and $m_2>0$.

The moment generating function (mgf) of the GB2 distribution is stated in the following theorem.

Theorem 1. Let X be a random variable of the GB2 (a,b,m1,m2) distribution, then the mgf of X is given by

$$M_x(t) = \sum_{n=0}^{\infty} \frac{(tb)^n}{n!} \cdot \frac{\Gamma\left(m_1 + \frac{n}{a}\right) \cdot \Gamma\left(m_2 - \frac{n}{a}\right)}{\Gamma(m_1) \cdot \Gamma(m_2)}$$

Proof:

$$M_x(t) = \int_0^{\infty} e^{tx} \left(\frac{a}{xB(m_1, m_2)} \right) \frac{\left(\left(\frac{x}{b} \right)^a \right)^{m_1}}{\left[1 + \left(\frac{x}{b} \right)^a \right]^{m_1+m_2}} dx$$

$$= \frac{a}{B(m_1, m_2)} \int_0^{\infty} \left(\frac{e^{tx}}{x} \right) \frac{\left[\left(\frac{x}{b} \right)^a \right]^{m_1}}{\left[1 + \left(\frac{x}{b} \right)^a \right]^{m_1+m_2}} dx \qquad (1)$$

By letting $y = \left(\frac{x}{b} \right)^a$ we may rerwite the equation (1) in the following form

$$M_X(t) = \frac{a}{B(m_1, m_2)} \int_0^{\infty} \left(\frac{e^{tx}}{ay} \right) \frac{y^{m1}}{(1+y)^{m1+m2}} dy \qquad (2)$$

$$= \frac{1}{B(m_1, m_2)} \int_0^{\infty} e^{tx} \frac{y^{m1-1}}{(1+y)^{m1+m2}} dy$$

After expanding e^{tx} by MacLaurin's series (Spiegel, 1968), one finds that the equation (2) is

$$M_x(t) = \frac{1}{B(m_1, m_2)} \int_0^{\infty} \left(\sum_{n=0}^{\infty} \frac{(tx)^n}{n!} \right) \frac{y^{m_1-1}}{(1+y)^{m_1+m_2}} dy$$

$$= \frac{1}{B(m_1, m_2)} \left[\int_0^{\infty} \frac{y^{m1-1}}{(1+y)^{m1+m2}} dy + \int_0^{\infty} tx \frac{y^{m1-1}}{(1+y)^{m1+m2}} dy + \right.$$

$$\left. \int_0^{\infty} \frac{(tx)^2}{2!} \frac{y^{m_1-1}}{(1+y)^{m_1+m_2}} dy + \int_0^{\infty} \frac{(tx)^3}{3!} \frac{y^{m_1-1}}{(1+y)^{m_1+m_2}} dy + \ldots\ldots \right]$$

$$= \frac{1}{B(m_1, m_2)} \left[\int_0^{\infty} \frac{y^{m1-1}}{(1+y)^{m1+m2}} dy + tb \int_0^{\infty} \frac{y^{m1+\frac{1}{a}-1}}{(1+y)^{m1+\frac{1}{a}+m2-\frac{1}{a}}} dy + \right.$$

$$\left. \frac{(tb)^2}{2!} \int_0^{\infty} \frac{y^{m_1+\frac{2}{a}-1}}{(1+y)^{m_1+\frac{2}{a}+m_2-\frac{2}{a}}} dy + \frac{(tb)^3}{3!} \int_0^{\infty} \frac{y^{m_1+\frac{3}{a}-1}}{(1+y)^{m_1+\frac{3}{a}+m_2-\frac{3}{a}}} dy + \ldots \right]$$

$$= \frac{1}{B(m_1, m_2)} \left[B(m_1, m_2) + tb \cdot B\left(m_1 + \frac{1}{a}, m_2 - \frac{1}{a} \right) + \right.$$

$$\left. \frac{(tb)^2}{2!} \cdot B\left(m_1 + \frac{2}{a}, m_2 - \frac{2}{a} \right) + \frac{(tb)^3}{3!} \cdot B\left(m_1 + \frac{3}{a}, m_2 - \frac{3}{a} \right) + \ldots \right]$$

$$= \sum_{n=0}^{\infty} \frac{(tb)^n}{n!} \cdot \frac{B\left(m_1 + \frac{n}{a}, m_2 - \frac{n}{a}\right)}{B(m_1, m_2)}$$

$$= \sum_{n=0}^{\infty} \frac{(tb)^n}{n!} \cdot \frac{\Gamma\left(m_1 + \frac{n}{\alpha}\right) \cdot \Gamma\left(m_2 - \frac{n}{\alpha}\right)}{\Gamma(m_1 + m_2)} \cdot \frac{\Gamma(m_1 + m_2)}{\Gamma(m_1) \cdot \Gamma(m_2)}$$

$$= \sum_{n=0}^{\infty} \frac{(tb)^n}{n!} \cdot \frac{\Gamma\left(m_1 + \frac{n}{\alpha}\right) \cdot \Gamma\left(m_2 - \frac{n}{\alpha}\right)}{\Gamma(m_1) \cdot \Gamma(m_2)}$$

Therefore the mgf of the GB2 distribution is:

$$M_x(t) = \sum_{n=0}^{\infty} \frac{(tb)^n}{n!} \cdot \frac{\Gamma\left(m_1 + \frac{n}{a}\right) \cdot \Gamma\left(m_2 - \frac{n}{a}\right)}{\Gamma(m_1) \cdot \Gamma(m_2)}$$

Moment Generating Function Of The Gg Distribution. The pdf of the GG distribution in this paper is given by the following form

where $a > 0$, $\gamma > 0$, and $m_1 > 0$.

$$g(x; a, \gamma, m_1) = \begin{cases} \frac{a}{x\Gamma(m_1)} \left(\left(\frac{x}{\gamma} \right)^a \right)^{m_1} e^{-\left(\frac{x}{\gamma}\right)^a} & \text{for} \quad x > 0 \\ 0 & \text{else} \end{cases}$$

The moment generating function (mgf) of the GB2 distribution is stated in the theorem 2 of this paper.

Theorem 2. Let X be a random variable of the GG (a, γ, and m1) distribution, then the mgf of X is given by

$$M_x(t) = \sum_{n=0}^{\infty} \frac{(t\gamma)^n}{n!} \cdot \frac{\Gamma\left(m_1 + \frac{n}{a}\right)}{\Gamma(m_1)}$$

Proof:

$$M_x(t) = \int_0^{\infty} e^{tx} \cdot \frac{a}{x\Gamma(m_1)} \left(\left(\frac{x}{\gamma} \right)^a \right)^{m_1} \cdot e^{-\left(\frac{x}{\gamma}\right)^a} dx$$

$$= \frac{a}{\Gamma(m_1)} \int_0^{\infty} \frac{e^{tx}}{x} \left(\left(\frac{x}{\gamma} \right)^a \right)^{m_1} \cdot e^{-\left(\frac{x}{\gamma}\right)^a} dx \qquad (3)$$

Similar to section 2, by letting $y = \left(\frac{x}{\gamma} \right)^a$ we may rerwite the equation (3) in the form of

$$M_X(t)=\frac{a}{\Gamma(m_1)}\int_0^\infty\left(\frac{e^{tx}}{ay}\right)\cdot y^{m1}\cdot e^{-y}\ dy=\frac{1}{\Gamma(m_1)}\int_0^\infty e^{tx}\cdot y^{m1-1}\cdot e^{-y}\ dy \qquad (4)$$

By expanding e^{tx} with MacLaurin's series, the equation (4) can be rerwitten as

$$M_X(t)=\frac{1}{\Gamma(m_1)}\int_0^\infty\left(\sum_{n=0}^\infty\frac{(tx)^n}{n!}\right)\cdot y^{m_1-1}\cdot e^{-y}\ dy$$

$$=\frac{1}{\Gamma(m_1)}\int_0^\infty\left(1+tx+\frac{(tx)^2}{2!}+\frac{(tx)^3}{3!}+...\right)\cdot y^{m_1-1}\cdot e^{-y}\ dy$$

$$=\frac{1}{\Gamma(m_1)}\left[\int_0^\infty y^{m_1-1}\cdot e^{-y}\ dy+\int_0^\infty(tx)y^{m_1-1}\cdot e^{-y}\ dy+\int_0^\infty\frac{(tx)^2}{2!}y^{m_1-1}\cdot e^{-y}\ dy+\int_0^\infty\frac{(tx)^3}{3!}y^{m_1-1}e^{-y}+\ldots\right]$$

$$=\frac{1}{\Gamma(m_1)}\left[\int_0^\infty y^{m_1-1}\cdot e^{-y}\ dy+t\gamma\int_0^\infty y^{m_1+\frac{1}{a}-1}\cdot e^{-y}\ dy+\frac{(t\gamma)^2}{2!}\int_0^\infty y^{m_1+\frac{2}{a}-1}\cdot e^{-y}\ dy+\frac{(t\gamma)^3}{3!}\int y^{m_1+\frac{3}{a}-1}\cdot e^{-y}\,dy+\ldots\right]$$

$$=\frac{1}{\Gamma(m_1)}\left[\Gamma(m_1)+t\gamma\cdot\Gamma\left(m_1+\frac{1}{a}\right)+\frac{(t\gamma)^2}{2!}\cdot\Gamma\left(m_1+\frac{2}{a}\right)+\frac{(t\gamma)^3}{3!}\cdot\Gamma\left(m_1+\frac{3}{\alpha}\right)+\ldots\right]$$

$$=\sum_{n=0}^\infty\frac{(t\gamma)^n}{n!}\cdot\frac{\Gamma\left(m_1+\frac{n}{a}\right)}{\Gamma(m_1)}$$

So, the mgf of the GG(a, γ, m_1) distribution can be written in the following form

$$M_x(t)=\sum_{n=0}^\infty\frac{(t\gamma)^n}{n!}\cdot\frac{\Gamma\left(m_1+\frac{n}{a}\right)}{\Gamma(m_1)}$$

Relationship Between Moment Of The Gg Distribution And Moment Ofthe Gb2 Distribution. The relationship between the mgf of the GG distribution and the mgf of the GB2 distribution can be seen in the following proposition.

Proposition 1. The GB2 (a,b,m_1,m_2) distribution converges to the GG distribution as m_2 tends to ∞ and b = γ

Proof:

$$\lim_{m_2\to\infty}M_X(t)\,GB2(a,b=\gamma,m_1,m_2)=\lim_{m_2\to\infty}\sum_{n=0}^\infty\frac{(t\gamma)^n}{n!}\cdot\frac{\Gamma\left(m_1+\frac{n}{a}\right)\cdot\Gamma\left(m_2-\frac{n}{a}\right)}{\Gamma(m_1)\cdot\Gamma(m_2)}$$

$$=\lim_{m_2\to\infty}1+\lim_{m_2\to\infty}(t\gamma)\cdot\frac{\Gamma\left(m_1+\frac{1}{a}\right)\cdot\Gamma\left(m_2-\frac{1}{a}\right)}{\Gamma(m_1)\cdot\Gamma(m_2)}+\ldots$$

$$=\lim_{m_2\to\infty}1+\lim_{m_2\to\infty}\left(t\gamma(m_2)^{\frac{1}{a}}\right)\cdot\frac{\Gamma\left(m_1+\frac{1}{a}\right)\cdot\Gamma\left(m_2-\frac{1}{a}\right)}{\Gamma(m_1)\cdot\Gamma(m_2)}+\lim_{m_2\to\infty}\frac{(t\gamma)^2}{2!}\cdot\frac{\Gamma\left(m_1+\frac{2}{a}\right)\cdot\Gamma\left(m_2-\frac{2}{a}\right)}{\Gamma(m_1)\cdot\Gamma(m_2)}+$$

$$\ldots$$

$$=\lim_{m_2\to\infty} 1+(t\gamma)\cdot\frac{\Gamma\left(m_1+\frac{1}{a}\right)}{\Gamma(m_1)}\lim_{m_2\to\infty}(m_2)^{\frac{1}{a}}\cdot\frac{\Gamma\left(m_2-\frac{1}{a}\right)}{\Gamma(m_2)}+\frac{(t\gamma)^2}{2!}\cdot\frac{\Gamma\left(m_1+\frac{2}{a}\right)}{\Gamma(m_1)}\lim_{m_2\to\infty}(m_2)^{\frac{2}{a}}\cdot\frac{\Gamma\left(m_2-\frac{2}{a}\right)}{\Gamma(m_2)}+$$

The Stirling's approximation formula (Spiegel, 1968) of the gamma function is

$$\Gamma(az+b)\sim\sqrt{2\pi}\cdot e^{-az}(az)^{az+b-\frac{1}{2}}$$

and others

$$\frac{\Gamma\left(m_2-\frac{1}{a}\right)}{\Gamma(m_2)}\sim\frac{\sqrt{2\pi}\cdot e^{-m_2}(m_2)^{m_2-\frac{1}{a}-\frac{1}{2}}}{\sqrt{2\pi}\cdot e^{-m_2}(m_2)^{m_2-\frac{1}{2}}}=(m_2)^{-\frac{1}{a}}=\frac{1}{(m_2)^{\frac{1}{a}}}$$

Then the limiting moment property of the GLL(α,β,m_1,m_2) distribution can be written as:

$$\lim_{m_2\to\infty} M_X(t)\, GB2(a,b=\gamma,m_1,m_2)$$

$$=\lim_{m_2\to\infty} 1+(t\gamma)\cdot\frac{\Gamma\left(m_1+\frac{1}{a}\right)}{\Gamma(m_1)}\lim_{m_2\to\infty}(m_2)^{\frac{1}{a}}\cdot\frac{1}{(m_2)^{\frac{1}{a}}}+\frac{(t\gamma)^2}{2!}\cdot\frac{\Gamma\left(m_1+\frac{2}{a}\right)}{\Gamma(m_1)}\lim_{m_2\to\infty}(m_2)^{\frac{2}{a}}\cdot\frac{1}{(m_2)^{\frac{2}{a}}}+$$

$$=1+(t\gamma)\cdot\frac{\Gamma\left(m_1+\frac{1}{a}\right)}{\Gamma(m_1)}+\frac{(t\gamma)^2}{2!}\cdot\frac{\Gamma\left(m_1+\frac{2}{a}\right)}{\Gamma(m_1)}+\quad=\sum_{n=0}^{\infty}\frac{(t\gamma)^n}{n!}\cdot\frac{\Gamma\left(m_1+\frac{n}{a}\right)}{\Gamma(m_1)}$$

This is the moment generating function of the GG. Therefore, the GB2 (a,b,m_1,m_2) distribution converges to the GG distribution as m_2 tends to ∞ and b = γ.

Limiting Moment Of The Gg Distribution. In this section we discuss the limiting behaviors of the GG family of distributions. The limiting behaviors of the GG distribution are assesed in the propotion 2 and 3.

Proposition 2. The GG distribution converges to the gamma distribution as a = 1.

Proof:
Letting a=1, the mgf of the GG distribution becomes

$$M_x(t)=1+t\gamma\cdot\frac{\Gamma\left(m_1+\frac{1}{a}\right)}{\Gamma(m_1)}+\frac{(t\gamma)^2}{2!}\cdot\frac{\Gamma\left(m_1+\frac{2}{a}\right)}{\Gamma(m_1)}+\frac{(t\gamma)^3}{3!}\cdot\frac{\Gamma\left(m_1+\frac{3}{a}\right)}{\Gamma(m_1)}+\ldots$$

$$=1+t\gamma\cdot\frac{\Gamma(m_1+1)}{\Gamma(m_1)}+\frac{(t\gamma)^2}{2!}\cdot\frac{\Gamma(m_1+2)}{\Gamma(m_1)}+\frac{(t\gamma)^3}{3!}\cdot\frac{\Gamma(m_1+3)}{\Gamma(m_1)}+\ldots$$

Using MacLaurin's expansion of the $(1-\gamma t)^{-m1}$ function, one finds that $M_x(t)=(1-\gamma t)^{-m_1}$
This is the moment generating function of the gamma distribution provided by Casella and Berger (1990).

Proposition 3. The GG distribution converges to the exponential distribution as a = 1 and m_1 =1

Proof:

For a = 1 and m_1 =1, the mgf of the GG distribution becomes

$$M_x(t) = 1 + t\gamma \cdot \frac{\Gamma\left(m_1 + \frac{1}{a}\right)}{\Gamma(m_1)} + \frac{(t\gamma)^2}{2!} \cdot \frac{\Gamma\left(m_1 + \frac{2}{a}\right)}{\Gamma(m_1)} + \frac{(t\gamma)^3}{3!} \cdot \frac{\Gamma\left(m_1 + \frac{3}{a}\right)}{\Gamma(m_1)} + \ldots$$

$$= 1 + t\gamma \cdot \frac{\Gamma(1+1)}{\Gamma(1)} + \frac{(t\gamma)^2}{2!} \cdot \frac{\Gamma(1+2)}{\Gamma(1)} + \frac{(t\gamma)^3}{3!} \cdot \frac{\Gamma 1(1+3)}{\Gamma(1)} + \ldots$$

$$= 1 + t\gamma \cdot \frac{\Gamma(2)}{\Gamma(1)} + \frac{(t\gamma)^2}{2!} \cdot \frac{\Gamma(3)}{\Gamma(1)} + \frac{(t\gamma)^3}{3!} \cdot \frac{\Gamma 1(4)}{\Gamma(1)} + \ldots$$

Similar to the proof of proposition 2, we may find that $M_x(t) = (1-\gamma t)^{-1}$. This is the moment generating function of the exponential distribution provided by Casella and Berger (1990)

CONCLUSION

The moment of the generalized gamma distribution is the limiting moment of the generalized beta of the second kind distribution. Moreover, the moments of the gamma and exponential distributions are special cases of the moment of the generalized gamma distribution.

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