On Tweedie Family Distributions and Their Extensions

Khoirin Nisa, Netti Herawati, Eri Setiawan, Nusyirwan
Department of Mathematics ,University of Lampung
Jl. Prof. Dr. Soemantri Brodjonegoro No. 1 Gedung Meneng Bandar Lampung
email: khoirin.nisa@fmipa.unila.ac.id, netti.herawati@fmipa.unila.ac.id,
eri.setiawan@fmipa.unila.ac.id, nusyirwan@fmipa.unila.ac.id

Abstract. *We study the Tweedie family distributions as special cases of exponential dispersion models (EDMs) which are two-parameter distributions from the exponential family that have a scale parameter λ. According to such parameterization, the mean and variance for the Tweedie random variable X are given by $E(X) = \mu$ and Var(Y)= $\lambda\mu^p$, respectively, where p is an extra parameter that controls the variance of the distribution which is called "variance power" parameter. From this perspective, some properties of Tweedie distribution are discussed. The multivariate extensions of Tweedie family are also presented.*

Keywords. cumulant generating function, α-stable distributions, variance function

1 INTRODUCTION

Generalized Linear Model (GLM) is a flexible generalization of ordinary linear regression that allows for response variables that have other than a normal distribution. The GLM generalizes linear regression by allowing the linear model to be related to the response variable following distribution in the exponential family via link function. According to McCullagh and Nelder [1], the random component of a GLM is specified by an exponential family density of the following form:

$$p(x;\lambda) = a(x)\exp\{[\theta x - \kappa(\theta)]\}, \quad x \in \mathbb{R}$$

for suitable functions a and κ. Jørgensen [2] extended the one parameter exponential family by adding a dispersion parameter λ and he called it as "exponential dispersion model" (EDM). An exponential dispersion model with parameter $\theta \in \Theta$ and $\lambda \in \Lambda \subseteq \mathbb{R}_+$ is a family distribution for Y with probability density function of the form

$$p(x; \theta, \lambda) = a(x, \lambda) \exp \{\lambda [\theta x - \kappa(\theta)]\}, \quad x \in \mathbb{R},$$

θ is the canonical parameter and λ is the dispersion parameter. For this family of distributions, we have the well-known relationships

$$E(x) = \mu = \kappa^{'}(\theta)$$

and

$$Var(x) = \lambda\kappa^{''}(\theta) = \lambda V(\mu).$$

One of special interest is the class of EDM with power mean-variance relationships is $V(\mu) = \mu^p$ for some p. Following Jorgensen [2] we call these "Tweedie distribution" in honor of Tweedie's first comprehensive research on this subject [3]. Jorgensen showed that the Tweedie distribution exists for any p outside the interval $(0, 1)$, and most of the commonly encountered distributions are special cases of the Tweedie distribution, e.g., Normal ($p = 0$), Poisson ($p = 1$), Gamma ($p = 2$), and Inverse Gaussian ($p = 3$). For $p > 2$, the Tweedie distribution is generated by stable distributions and has support on the positive values, and for $p < 0$, the distribution has a positive mean and the support is on the whole real line. The distributions with $1 < p < 2$ are especially appealing for modeling quantity data when exact zeros are possible.

Apart from the well-known distributions with $p = 0, 1, 2, \text{or} 3$, none of the Tweedie distributions have density functions with explicit analytic forms. This complicates the use of these distributions in statistical modeling. In particular, it prevents their use with likelihood based estimation, testing or diagnostic procedures. Nevertheless, this does not prohibit their use in generalized linear models, where the inferential scheme only requires the knowledge of the first two moments [1]. On the other hand, Tweedie distributions do have simple, analytic moment generating functions.

In this paper, we discuss our study on the Tweedie family distributions in the framework of EDM and α-stable distribution [4]. The rest of the paper is organized as follow. In Section 2 we present some properties of the Tweedie family, in Section3 the multivariate extention of Tweedie family is discussed, we present a special case of multivariate Tweedie model i.e. "normal stable Tweedie" (NST) model in Section 4 and the final remark in Section 5

2 THE TWEEDIE FAMILY DISTRIBUTIONS

The well-known positive σ-stable distribution generating Lévy process $\left(X_t^\alpha\right)_{t>0}$ were introduced by [5] and defined by probability measures:

$$\xi_{\alpha,t}(dx) = \frac{1}{\pi x} \sum \frac{t^j \Gamma(1+\alpha j) sin(-j\pi\alpha)}{j!\alpha^j(\alpha-1)^{-j}\left[(1-\alpha)x\right]^{\alpha j}} \mathbb{1}_{x>0} dx = \xi_{\alpha,t}(x)dx \tag{2.1}$$

where α in (0,1) is the index parameter, $\Gamma(.)$ is the classical gamma function, and $\mathbb{I}_A$ denotes the indicator function of any given event A that takes the value 1 if the event accurs and 0 otherwise. Recall that a random variable X has an α-stable distribution if, for all $X_1, \ldots, X_n$ being independent copies of X and for all b_n belonging to $\mathbb{R}$,

$$X_1 + \ldots + X_n \stackrel{d}{=} n^{1/\alpha} X + b_n$$

where $\stackrel{d}{=}$ denotes the equality in distribution. The parameter α can be extended on $(0, 2]$. We then obtain the inverse Gaussian distribution for $\alpha = 1/2$ with density:

$$\xi_{1/2,t}(dx) = \frac{t}{\sqrt{2\pi x^3}} \exp\left(\frac{-t^2}{2x}\right) \mathbb{1}_{x>0} dx = \xi_{1/2,t}(x)dx. \tag{2.2}$$

For $\alpha \in [1, 2]$ one defines a family of (extreme) stable distributions concentrated on the real line $\mathbb{R}$ where special case are Cauchy ($\alpha = 1$) and normal ($\alpha = 2$) distributions with

$$\xi_{1,t}(dx) = \frac{t}{\pi\left(t^2 + x^2\right)} dx = \xi_{1,t}(x)dx$$

and

$$\xi_{2,t}(dx) = \frac{1}{\sqrt{2\pi t}} \exp\left(\frac{-x^2}{2t}\right) dx = \xi_{2,t}(x)dx.$$

The left limit case $\alpha \to 0$ leads to gamma distribution generated by the probability measure

$$\xi_{0,t}(dx) = \frac{x^{t-1}e^{-x}}{\Gamma(t)} I_{x>0} dx = \xi_{0,t}(x)dx.$$

Tweedie [3] proposed a classification of all stable distribution with $\alpha \in [-\infty, 2]$ introducing the "power variance" parameter p defined by :

$$(p-1)(1-\alpha) = 1$$

and equivalent to :

$$p = p(\alpha) = \frac{\alpha - 2}{\alpha - 1} \qquad \text{or} \qquad \alpha = \alpha(p) = \frac{p-2}{p-1}$$

In the case of $\alpha \to -\infty$ or $p = p(-\infty) = 1$, leads to Poisson distribution with probability mass function

$$f_{-\infty,t}(x) = \frac{t^x e^{-t}}{x!}, \quad \forall x \in N$$

Table 1: Summary of stable Tweedie models [2] with unit mean domain M_p and support S_p of distribution.

Distribution	$p = p(\alpha)$	$\alpha = \alpha(p)$	M_p	S_p
Extreme stable	$p < 0$	$1 < \alpha < 2$	$(0, \infty)$	$\mathbb{R}$
Gaussian	$p = 0$	$\alpha = 2$	$\mathbb{R}$	$\mathbb{R}$
Do not exist	$0 < p < 1$	$2 < \alpha < \infty$		
Poisson	$p = 1$	$\alpha = -\infty$	$(0, \infty)$	$\mathbb{N}$
Compound Poisson	$1 < p < 2$	$\alpha < 0$	$(0, \infty)$	$[0, \infty)$
Non-central gamma	$p = 3/2$	$\alpha = -1$	$(0, \infty)$	$[0, \infty)$
Gamma	$p = 2$	$\alpha = 0$	$(0, \infty)$	$(0, \infty)$
Positive stable	$p > 2$	$0 < \alpha < 1$	$(0, \infty)$	$(0, \infty)$
Inverse Gaussian	$p = 3$	$\alpha = 1/2$	$(0, \infty)$	$(0, \infty)$
Extreme stable	$p = \infty$	$\alpha = 1$	$\mathbb{R}$	$\mathbb{R}$

The cumulant generating function of a univariate σ-finite positive measure $\mu_{\alpha,t}$ generating the NEF $F_{p,t} = F(\xi_{\alpha,t})$ is given by

$$K_{\xi_{\alpha,t}}(\theta) = \log L_{\xi_{\alpha,t}}(\theta) = t \log L_{\xi_{\alpha,1}}(\theta) = t K_{\xi_{\alpha,1}}(\theta)$$

with $L_{\xi_{\alpha,t}}$ the Laplace transform of the NEF $F_{p,t}$. Parameters θ and μ are one-to-one connected by the following expression

$$\mu = \frac{d}{d\theta}\kappa(\theta) = \mu(\theta) \begin{cases} \exp(\theta) & \text{for} \quad p = 1 \\ [(1-p)\theta]^{1/(1-p)} & \text{for} \quad p \neq 1. \end{cases}$$

Also, the corresponding variance function is given by

$$V_{F_{p,t}}(\mu) = K''_{\xi_{\alpha,t}}(\theta) = t K''_{\xi_{\alpha,1}}(\theta) = t V_p(\mu/t),$$

where V_p is the unit variance function.

We define the univariate stable Tweedie NEFs $F_{p,t} = F(\xi_{p,t})$ generated by the σ-finite positive measures $\xi_{p,t}$ such that their cumulant functions are $K_{\nu_{p,t}} = tK_{\nu_{p,1}}$ with

$$K_{\xi_{p,1}}(\theta_0) = \begin{cases} \exp(\theta_0) & \text{for} \quad p = 1 \\ -\log(-\theta_0) & \text{for} \quad p = 2 \\ [1/(2-p)]\,[(1-p)\theta_0]^{(p-2)/(p-1)} & \text{for} \quad 1 \neq 1, 2 \end{cases} \tag{2.3}$$

for all θ_0 in their respective canonical domains

$$\Theta(\xi_{p,1}) = \begin{cases} \mathbb{R} & \text{for} \quad p = 0, 1 \\ [0, \infty) & \text{for} \quad p < 0 \quad \text{or} \quad 0 < p < 1 \\ (-\infty, 0) & \text{for} \quad 1 < p \leqslant 2 \quad \text{or} \quad p = \infty \\ (-\infty, 0] & \text{for} \quad 2 < p < \infty. \end{cases}$$

Using α parameterization, i.e. $\alpha = \alpha(p) = 1 + \dfrac{1}{(1-p)}$, then the Tweedie cumulant functions for $p \neq 1, 2$ in (2.3) can be written as

$$K_{\xi_{\alpha,1}}(\theta) = \frac{\alpha - 1}{\alpha}\left(\frac{\theta}{\alpha - 1}\right)^{\alpha}$$

for values of θ such that $\theta/(\alpha - 1) > 0$.

Dunn and Smyth [6] gave a survey of published applications showing that Tweedie distributions have been used in a diverse range of fields. A significant number of scientific papers have been dedicated to exploring various members of the Tweedie class in diverse fields of actuarial science [7, 8, 9, 10] , survival analysis [11], fishery predictions [12] and rainfall modeling [13, 14].

3 MULTIVARIATE TWEEDIE MODEL

In this section we discuss the multivariate extension of Tweedie family distribution. In order to define the multivariate Tweedie (probability) models, we consider first the intermediate weight parameter $\gamma = \lambda\kappa_{\alpha}(\theta)$, and write the cumulant generating function (CGF) as follows:

$$s \mapsto \gamma\left[\left(1 + \frac{s}{\theta}\right)^{\alpha} - 1\right] \tag{3.1}$$

Since γ and $\kappa_\alpha(\theta)$ have the same sign, it follows that the domain for γ is either $\mathbb{R}_+$ or $\mathbb{R}_-$, depending on the sign of $(\alpha-1)/\alpha$. Our starting point is the bivariate singular distribution with joint CGF

$$(s_1, s_2)^\top \mapsto \gamma\left[\left(1+\frac{s_1}{\theta_1}+\frac{s_2}{\theta_2}\right)^\alpha - 1\right],$$

whose marginals are Tweedie distributions of the form (3.1).

Now we define the multivariate Tweedie models as follow. A multivariate (additive) Tweedie model denoted $Tw_k^{*p}(\boldsymbol{\mu}, \boldsymbol{\Lambda})$ is defined by the joint CGF:

$$K(\mathbf{s};\theta,\gamma) = \sum_{i<j}\gamma_{ij}\left[\left(1+\frac{s_i}{\theta_i}+\frac{s_j}{\theta_j}\right)^\alpha - 1\right] + \sum_{i=1}^{k}\gamma_i\left[\left(1+\frac{s_i}{\theta_i}\right)^\alpha - 1\right], \qquad (3.2)$$

where $\gamma = \lambda\kappa_{\nu_\alpha}(\theta)$ and the weight parameters γ_{ij} and γ_i all have the same sign as $(\alpha-1)/\alpha$.

By taking $s_j = 0$ for $j \neq i$ in the expression (3.2) we find that the ith marginal follows a univariate Tweedie distribution with CGF (3.1) with $\theta = \theta_i$ and $\gamma = \gamma_{ii}$ defined by

$$\gamma_{ii} = \sum_{j:j\neq i}\gamma_{ij} + \gamma_i.$$

For multivariate Tweedie distributions, the exponential dispersion model weight parameters λ_{ii} and λ_{ij} are defined by

$$\lambda_{ii} = \frac{\gamma_{ii}}{\kappa_\alpha(\theta_i)} \quad \text{for} \quad i = 1,\ldots,k \qquad (3.3)$$

and

$$\lambda_{ij} = \frac{\gamma_{ij}}{\kappa_\alpha^{1/2}(\theta_i,\theta_j)} \quad \text{for} \quad i<j, \qquad (3.4)$$

where $\kappa_\alpha^{1/2}$ is a function defined by

$$\kappa_\alpha^{1/2}(\theta_i,\theta_j) = \frac{\alpha-1}{\alpha}\left(\frac{\theta_i}{\alpha-1}\right)^{\alpha/2}\left(\frac{\theta_j}{\alpha-1}\right)^{\alpha/2}.$$

Using the parameters λ_{ii} the marginal mean are of the form

$$\lambda_{ii}\mu_i = \lambda_{ii}\kappa_\alpha(\theta_i)\frac{\alpha}{\theta_i} = \lambda_{ii}\left(\frac{\theta_i}{\alpha-1}\right)^{\alpha-1} \quad \text{for} \quad i = 1,\ldots,k \qquad (3.5)$$

Table 2: Summary of Multivariate Tweedie Dispersion Models on $\mathbb{R}^k$ with Support $\mathbf{S}_{p,k}$ and Mean Domain $\mathbf{M}_{p,k}$

Distribution(s)	p	$\alpha = \alpha(p)$	$\mathbf{S}_{p,k}$	$\mathbf{M}_{p,k}$
Multivariate extreme stable	$p < 0$	$1 < \alpha < 2$	$\mathbb{R}^k$	$(0,\infty)^k$
Multivariate Gaussian	$p = 0$	$\alpha = 2$	$\mathbb{R}^k$	$\mathbb{R}^k$
[Do not exist]	$0 < p < 1$	$2 < \alpha < \infty$		
Multivariate Poisson	$p = 1$	$\alpha = -\infty$	$\mathbb{N}_0^k$	$(0,\infty)^k$
Multivarite compound Poisson	$1 < p < 2$	$\alpha < 0$	$[0,\infty)^k$	$(0,\infty)^k$
Multivariate non-central gamma	$p = 3/2$	$\alpha = -1$	$[0,\infty)^k$	$(0,\infty)^k$
Multivariate gamma	$p = 2$	$\alpha = 0$	$(0,\infty)^k$	$(0,\infty)^k$
Multivariate positive stable	$p > 2$	$0 < \alpha < 1$	$(0,\infty)^k$	$(0,\infty)^k$
Multivariate inverse Gaussian	$p = 3$	$\alpha = 1/2$	$(0,\infty)^k$	$(0,\infty)^k$
Multivariate extreme stable	$p = \infty$	$\alpha = 1$	$\mathbb{R}^k$	$\mathbb{R}^k$

The multivariate gamma distribution above is different from the one discussed by [**?**], the multivariate gamma here has the joint CGF of the form corresponds to definition : $K(\mathbf{s},\theta,\mathbf{\Lambda}) = -\sum_{i<j}\lambda_{ij}\log\left(1-\frac{s_i}{\theta_i}-\frac{s_j}{\theta_j}\right)-\sum_{i=1}^{k}\lambda_i\log\left(1-\frac{s_i}{\theta_i}\right)$.

and the variances are

$$\lambda_{ii}\kappa_\alpha(\theta_i)\frac{\alpha(\alpha-1)}{\theta_i^2} = \lambda_{ii}\mu_i^p \quad \text{for} \quad i = 1,\dots,k.$$

This defines the multivariate additive Tweedie random vector $\mathbf{X} \sim Tw_k^{*p}(\boldsymbol{\mu},\mathbf{\Lambda})$, with mean vector $\mathrm{Diag}(\mathbf{\Lambda})\boldsymbol{\mu}$ where its elements are defined by (3.5), i.e.

$$\mathrm{Diag}(\mathbf{\Lambda})\boldsymbol{\mu} = \left(\lambda_{11}\left(\frac{\theta_1}{\alpha-1}\right)^{\alpha-1},\dots,\lambda_{kk}\left(\frac{\theta_k}{\alpha-1}\right)^{\alpha-1}\right)^\top$$

and the covariance matrix for $\mathbf{X}$ has the form $\mathbf{\Lambda}\odot V(\boldsymbol{\mu})$, where the elements of $\mathbf{\Lambda} = (\lambda_{ij})_{i,j=1,\dots,k}$ are defined in (3.3) and (3.4) and $\mathbf{V}(\boldsymbol{\mu})$ has entries

$$V_{ij} = (\mu_i\mu_j)^{p/2},$$

then the covariance matrix of $\mathbf{X}$ is

$$\mathbf{\Sigma} = \mathbf{\Lambda}\odot\mathbf{V}(\boldsymbol{\mu}) = \begin{bmatrix} \lambda_{11}\mu_1^p & \lambda_{12}(\mu_1\mu_2)^{p/2} & \dots & \lambda_{1k}(\mu_1\mu_k)^{p/2} \\ \lambda_{21}(\mu_2\mu_1)^{p/2} & \lambda_{22}\mu_2^p & \dots & \lambda_{2k}(\mu_1\mu_k)^{p/2} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{k1}(\mu_k\mu_1)^{p/2} & \lambda_{k2}(\mu_k\mu_1)^{p/2} & \dots & \lambda_{kk}\mu_k^p \end{bmatrix}.$$

The multivariate additive Tweedie model $Tw_k^{*p}(\boldsymbol{\mu}, \boldsymbol{\Lambda})$ satisfies the following additive property:

$$Tw_k^{*p}(\boldsymbol{\mu}, \boldsymbol{\Lambda}_1) + Tw_k^{*p}(\boldsymbol{\mu}, \boldsymbol{\Lambda}_2) = Tw_k^{*p}(\boldsymbol{\mu}, \boldsymbol{\Lambda}_1 + \boldsymbol{\Lambda}_2)$$

To obtain the reproductive form $Tw_k(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, we need to use the following duality transformation:

$$Tw_k^{p}(\boldsymbol{\mu}, \boldsymbol{\Sigma}) = \boldsymbol{\lambda}^{-1} Tw_k^{*p}(\boldsymbol{\mu}, \boldsymbol{\Lambda}).$$

The distribution $Tw_k(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ has mean vector $\boldsymbol{\mu}$ as follow

$$\boldsymbol{\mu} = \left(\left(\frac{\theta_1}{\alpha - 1} \right)^{\alpha - 1}, \ldots, \left(\frac{\theta_k}{\alpha - 1} \right)^{\alpha - 1} \right)^{\top}$$

and covariance matrix

$$\boldsymbol{\Sigma} \odot \mathbf{V}(\boldsymbol{\mu}) = \mathrm{Diag}(\boldsymbol{\mu})^{p/2} \, \boldsymbol{\Sigma} \, \mathrm{Diag}(\boldsymbol{\mu})^{p/2}$$

where $\boldsymbol{\Sigma} = \mathrm{Diag}(\boldsymbol{\Lambda})^{-1} \, \boldsymbol{\Lambda} \, \mathrm{Diag}(\boldsymbol{\Lambda})^{-1}$.

One can see [15] and [16] for more details on multivariate Tweedie models. An interesting behaviour of negatively correlated multivariate Tweedie distribution (bivariate case) was revealed from the simulation study done by [16]. They showed that for a large negative correlation the scatter of bivariate Tweedie distribution depicted a curve which reminds us to the inverse exponential function, also the distribution lies only on the positive side of the Euclidean space. These behaviour appears to be new because of positive support of the multivariate Tweedie. While for large positive correlation the scatter of bivariate Tweedie distribution depicted a straight line with positive slope as commonly seen in the same case on multivariate Gaussian distribution.

4 NORMAL STABLE TWEEDIE MODEL

Motivated by normal gamma [17] and normal inverse Gaussian [18] models, Boubacar Maïnassara, Y. and Kokonendji [19] introduced a new form of generalized variance functions which are generated by the so-called *normal stable Tweedie* (NST) models of k-variate distributions ($k > 1$). The generating σ-finite positive measure $\mu_{\alpha,t}$ on $\mathbb{R}^k$ of NST models is composed by the probability measure $\xi_{\alpha,t}$ of univariate positive σ-stable distribution generating Lévy process in 2.1.

For a k-dimensional NST random vector $\mathbf{X} = (X_1, \ldots, X_k)^\top$, the generating σ-finite positive measure $\nu_{\alpha,t}$ is given by

$$\nu_{\alpha,t}(d\mathbf{x}) = \xi_{\alpha,t}(dx_1) \prod_{j=2}^{k} \xi_{2,x_1}(dx_j), \tag{4.1}$$

where X_1 is a univariate (non-negative) stable Tweedie variable and all other variables $(X_2, \ldots, , X_k)^\top =: \mathbf{X}_1^c$ given X_1 are $k-1$ real independent Gaussian variables with variance X_1.

By introducing "power variance" parameter p the generating σ-finite positive measure $\nu_{\alpha,t}$ on $\mathbb{R}^k$ of NST models is

$$\nu_{\alpha,t;j}(d\mathbf{x}) = \xi_{\alpha,t}(dx_j) \prod_{\ell \neq j} \xi_{2,x_j}(dx_\ell) \tag{4.2}$$

with $\alpha = \alpha(p) \in [-\infty, 0)$.

Since $(p-1)(1-\alpha) = 1$ then Equation (2.1) can be expressed in term of p namely $\xi_{p,t}$ with $\xi_{p,t} = \xi_{p(\alpha),t}$, then equation (4.2) can be written as follows

$$\nu_{p,t:j}(d\mathbf{x}) = \xi_{p,t}(dx_j) \prod_{\ell \neq j} \xi_{0,x_j}(dx_\ell). \tag{4.3}$$

For suitable univariate NEF $F_{p,t} = F(\xi_{\alpha(p),t})$ of stable Tweedie types, we can interpret the multivariate NEFs $\mathbf{G}_{p,t} = \mathbf{G}(\nu_{\alpha(p),t})$ as composed by the distribution (4.3) of the random vector $\mathbf{X} = (X_1, \ldots, X_k)^\top$ where X_1 is a univariate stable Tweedie variable generated by xi_{0,x_1} (with mean 0 and variance $x-1$). So from Table 1 in Appendix with $S_p \subseteq [0, \infty)$, we must retain α in $[-\infty, 1)$ and the associated univariate model may be called the non-negative stable Tweedie, which include normal Poisson models appearing as new multivariate distribution having one discrete component.

By equation (4.3), one can obtain the cumulant function $\mathbf{K}_{\nu_{p,t}}(\boldsymbol{\theta}) = \log \int_{\mathbb{R}^k} \exp$

$\left(\boldsymbol{\theta}^T \boldsymbol{x}\right) \nu_{p,t}(d\boldsymbol{x})$:

$$\begin{aligned}
\mathbf{K}_{\nu_{p,t;j}}(\boldsymbol{\theta}) &= \log\left[\int_{\mathbb{R}} \exp(\theta_j x_j)\left(\prod_{\ell\neq j}\int_{\mathbb{R}} \exp(\theta_\ell x_\ell)\xi_{0,x_j}(dx_\ell)\right)\xi_{p,t}(dx_j)\right] \\
&= \log\left[\int_{\mathbb{R}} \exp(\theta_j x_j)\left(\prod_{\ell\neq j} \exp\frac{x_j\theta_\ell^2}{2}\right)\xi_{p,t}(dx_j)\right] \\
&= \log\left[\int_{\mathbb{R}} \exp\left(\theta_j x_j + \frac{1}{2}\sum_{\ell\neq j} x_j\theta_\ell^2\right)\xi_{p,t}(dx_j)\right]
\end{aligned}$$

$$\begin{aligned}
&= \log\left[\int_{\mathbb{R}} \exp\left\{x_j\left(\theta_j + \frac{1}{2}\sum_{\ell\neq j}\theta_\ell^2\right)\right\}\xi_{p,t}(dx_j)\right] \\
&= tK_{\xi_{p,t}}\left(\theta_j + \frac{1}{2}\sum_{\ell\neq j}\theta_\ell^2\right)
\end{aligned}$$

Here $K_{\xi_{p,t}}$ is the cumulant function of univariate stable Tweedie NEF $F_{\xi_{p,t}}$ generated by σ-finite positive measures $\xi_{p,t}$ as follow:

$$K_{\xi_{p,t}} = tK_{\xi_{p,1}}$$

with

$$K_{\xi_{p,1}}(\theta_0) = \begin{cases} \exp(\theta_0) & \text{for} \quad p = 1 \\ -\log(-\theta_0) & \text{for} \quad p = 2 \\ [1/(2-p)]\,[(1-p)\theta_0]^{(p-2)/(p-1)} & \text{for} \quad 1 \neq 1, 2 \end{cases} \tag{4.4}$$

for all θ_0 in their respective canonical domains

$$\Theta(\xi_{p,1}) = \begin{cases} \mathbb{R} & \text{for} \quad p = 0, 1 \\ [0, \infty) & \text{for} \quad p < 0 \quad \text{or} \quad 0 < p < 1 \\ (-\infty, 0) & \text{for} \quad 1 < p \leqslant 2 \quad \text{or} \quad p = \infty \\ (-\infty, 0] & \text{for} \quad 2 < p < \infty. \end{cases}$$

The function $\mathbf{K}_{\nu_{p,t;j}}(\boldsymbol{\theta})$ is finite for all $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_k)^\top$ in the canonical domain

$$\boldsymbol{\Theta}(\nu_{p,t;j}) = \left\{\boldsymbol{\theta} \in \mathbb{R}^k; \boldsymbol{\theta}^\top \tilde{\boldsymbol{\theta}}_j^c := \left(\theta_j + \frac{1}{2}\sum_{\ell\neq j}\theta_\ell^2\right) \in \Theta_p\right\} \tag{4.5}$$

where $\tilde{\boldsymbol{\theta}}_j^c = (\theta_1, \ldots, \theta_{j-1}, 1, \theta_{j+1}, \ldots, \theta_k)$ and

$$\Theta_p = \begin{cases} \mathbb{R} & \text{for} \quad p = 0, 1 \\ [0, \infty) & \text{for} \quad p < 0 \quad \text{or} \quad 0 < p < 1 \\ (-\infty, 0) & \text{for} \quad 1 < p \leqslant 2 \quad \text{or} \quad p = \infty \\ (-\infty, 0] & \text{for} \quad 2 < p < \infty. \end{cases}$$

For fixed $p \geqslant 1$ and $t > 0$, the multivariate NEF generated by $\nu_{p,t;j}$ is the set

$$\mathbf{G}_{p,t;j} = \mathbf{P}(\boldsymbol{\theta}; p, t); \boldsymbol{\theta} \in \boldsymbol{\Theta}(\nu_{p,t;j}) \tag{4.6}$$

of probability distributions

$$\mathbf{P}(\boldsymbol{\theta}; p, t)(d\mathbf{x}) = \exp\left\{\boldsymbol{\theta}^\top \mathbf{x} - \mathbf{K}_{\nu_{p,t;j}}(\boldsymbol{\theta})\right\} \nu_{p,t;j}(dx). \tag{4.7}$$

The mean vector and the covariance matrix of $\mathbf{G}_{p,t;j}$ can be calculated using the first and the second derivatives of the cumulant function, i.e.

$$\boldsymbol{\mu} = \mathbf{K}'_{\nu_{p,t;j}}(\boldsymbol{\theta})$$

and

$$\mathbf{V}_{\mathbf{G}_{p,t;j}}(\boldsymbol{\mu}) = \mathbf{K}''_{\nu_{p,t;j}}(\boldsymbol{\theta}(\boldsymbol{\mu})).$$

The followings are four examples illustrate some of the issues that may be encountered when applying equation (4.3).

Normal Poisson. For $p = 1 = p(-\infty)$ in (4.3) we represent the normal Poisson generating measure by

$$\nu_{1,t;j}(d\mathbf{x}) = \frac{t^{x_j}(x_j!)^{-1}}{(2\pi x_j)^{(k-1)/2}} \exp\left(-t - \frac{1}{2x_j} \sum_{\ell \neq j} x_\ell^2\right) \mathbb{1}_{x_j \in \mathbb{N} \setminus \{0\}} \delta_{x_j}(dx_j) \prod_{\ell \neq j} dx_\ell,$$

where $\mathbb{1}_A$ is the indicator function of the set A. Since it is also possible to have $x_j = 0$, the corresponding normal Poisson distribution is degenerated as δ_0. Normal Poisson is the only NST model which has a discrete component and correlated to the continuous components. The characterization and generalized variance estimations of this model can be been in [20, 21, 22],

Normal gamma. For $p = 2 = p(0)$ in (4.3) one has the generating measure of the normal gamma as follow:

$$\nu_{2,t;j}(d\boldsymbol{x}) = \frac{x_j^{t-1}}{(2\pi x_j)^{(k-1)/2}\Gamma(t)} \exp\left(-x_j - \frac{1}{2x_j} \sum_{\ell \neq j} x_\ell^2\right) \mathbb{1}_{x_j > 0} dx_1 dx_2 \cdots x_k$$

Normal gamma was introduced by [23] as a member of simple quadratic NEFs; she called it as "gamma-Gaussian". This model was characterized by [24]. The bivariate case is used as prior distribution of Bayesian inference for normal distribution [17].

Normal inverse Gaussian. Setting $p = 3 = p(1/2)$ in (4.3) one has the so-called normal invers Gaussian family generated by

$$\nu_{3,t;j}(d\mathbf{x}) = \frac{tx_j^{-(k+2)/2}}{(2\pi)^{k/2}} \exp\left\{\frac{-1}{2x_j}\left(t^2 + \sum_{\ell \neq j} x_\ell^2\right)\right\} \mathbb{1}_{x_j>0} dx_1 dx_2 \cdots x_k$$

It already appeared as a variance-mean mixture of a multivariate Gaussian with a univariate inverse Gaussian distribution [18]. It can be considered as a distribution of the position of multivariate Brownian motion at a certain stopping time. See [25, 26] and [27] for more details and interesting applications in stochastic volatility modelling and heavy-tailed modelling, respectively.

Normal noncentral gamma. For $p = 3/2$ in (4.3) one has the generating measure of normal noncentral gamma which can be expressed as follows:

$$\nu_{3/2,t;j}(d\mathbf{x}) = \frac{x_j^{-1}}{(2\pi x_j)^{(k-1)/2}} \left(\sum_{\ell \neq j} \frac{(4tx_j)^j}{\ell!\Gamma(\ell)}\right) \exp\left(-\frac{1}{2x_j}\sum_{\ell \neq j} x_\ell^2\right) \mathbb{1}_{x_j>0} dx_1 dx_2 \cdots x_k$$

Since it is possible to have $x_j = 0$ (as in normal-Poisson models) the corresponding normal distributions are degenerated as δ_0.

4.1 Generalized variance function and Lévy measures

Now let $p \geqslant 1$ and $t > 0$, denote $(\mathbf{e})_{1=1,\ldots,k}$ an orthonormal basis of $\mathbb{R}^k$, and $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_k)^\top$ the mean vector of $\mathbf{X}$. [19] showed the variance functions of $\mathbf{G}_{p,t;j} = \mathbf{G}(\nu_{p,t;j})$ generated by $\nu_{p,t;j}$ is given by

$$\mathbf{V}_{\mathbf{G}_{p,t;j}}(\boldsymbol{\mu}) = t^{1-p}\mu_j^{p-2} \cdot \boldsymbol{\mu}\boldsymbol{\mu}^\top + \mathrm{Diag}_k(\mu_j, \ldots, \mu_j, 0_j, \mu_j, \ldots, \mu_j), \tag{4.8}$$

on its support

$$\mathbf{M}_{F_{t;j}} = \left\{\boldsymbol{\mu} \in \mathbb{R}^k; \mu_j > 0 \text{ and } \mu_\ell \in \mathbb{R} \text{ for } \ell \neq j\right\}. \tag{4.9}$$

for all $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_k)^\top$ in $\mathbf{M}_{\mathbf{G}_{p,t;j}} = (0, \infty) \times \mathbb{R}^{k-1}$.

For $p = 1$ (normal Poisson) and $j = 1$, the covariance matrix of $\mathbf{X}$ can be expressed as below

$$V_{F_{t;j}}(\boldsymbol{\mu}) = \begin{bmatrix} \mu_1 & \mu_2 & \cdots & \mu_i & \cdots & \mu_k \\ \mu_2 & \mu_1^{-1}\mu_2^2 + \mu_1 & \cdots & \mu_1^{-1}\mu_2\mu_i & \cdots & \mu_1^{-1}\mu_k\mu_2 \\ \vdots & \vdots & \ddots & \vdots & & \vdots \\ \mu_i & \mu_1^{-1}\mu_i\mu_2 & \cdots & \mu_1^{-1}\mu_i^2 + \mu_1 & \cdots & \mu_1^{-1}\mu_i\mu_k \\ \vdots & \vdots & & \vdots & \ddots & \vdots \\ \mu_k & \mu_1^{-1}\mu_k\mu_2 & \cdots & \mu_1^{-1}\mu_k\mu_i & \cdots & \mu_1^{-1}\mu_k^2 + \mu_1 \end{bmatrix}.$$

The generalized variance function of any multivariate NST model presents a very simpler expression than its variance function in 4.8. More precisely, it depends solely on the first component of the mean vector with the power variance parameter p and the dimension k.

Let $p = p(\alpha) \geq 1$ and $t > 0$. Then the generalized variance function of a normal stable Tweedie model in $\mathbf{G}_{p,t;j} = \mathbf{G}(\nu_{p,t;j})$ generated by $\nu_{p,t;j}$ is given by

$$\det \mathbf{V}_{\mathbf{G}_{p,t;j}}(\boldsymbol{\mu}) = t^{1-p}\left(\boldsymbol{e}_j^T\boldsymbol{\mu}\right)^{p+k-1} = t^{1-p}\mu_j^{p+k-1} \tag{4.10}$$

for $\boldsymbol{\mu} \in \mathbf{M}_{\mathbf{G}_{p,t}} = (0,\infty) \times \mathbb{R}^{k-1}$.

The proof of equation (4.10) for $j = 1$ has been described in [19] using the Schur representation of determinant:

$$\det\begin{pmatrix} \gamma & \mathbf{a}^\top \\ \mathbf{a} & \mathbf{A} \end{pmatrix}^{-1} = \gamma \quad \det(\mathbf{A} - \gamma^{-1}\mathbf{a}\mathbf{a}^\top), \tag{4.11}$$

where $\gamma \neq 0$ is a scalar, $\mathbf{A}$ is a $k \times k$ matrix, $\mathbf{a}$ and $\mathbf{b}$ are two vectors on $\mathbb{R}^k$.

Applying the Schur representation (4.11) for $j = 1$ with

$$\gamma = t^{1-p}(\boldsymbol{e_1}^\top \boldsymbol{\mu})^p,$$

$$\mathbf{a} = \mathbf{b} = t^{1-p}(\boldsymbol{e_1}^\top \boldsymbol{\mu})^{p-1}(\mu_2, \ldots, \mu_k)^\top,$$

$$\begin{aligned}\mathbf{A} &= t^{1-p}(\boldsymbol{e_1}^\top \boldsymbol{\mu})^{p-2}\,(\mu_2, \ldots, \mu_k)(\mu_2, \ldots, \mu_k)^\top + (\boldsymbol{e_1}^\top \boldsymbol{\mu}).\mathbf{I}_{k-1}\\ &= \gamma^{-1}\mathbf{a}\mathbf{a}^\top + (\boldsymbol{e_1}^\top \boldsymbol{\mu}).\mathbf{I}_{k-1},\end{aligned}$$

we obtain

$$\begin{aligned}\det \mathbf{V}_{\mathbf{G}_{p,t;1}}(\boldsymbol{\mu}) &= \gamma \det[(\boldsymbol{e_1}^\top \boldsymbol{\mu}).\mathbf{I}_{k-1}]\\ &= t^{1-p}\left(\boldsymbol{e}_1^T \boldsymbol{\mu}\right)^{p+k-1}.\end{aligned}$$

Then, it is trivial to show that for $j \in \{1, \ldots, k\}$ the generalized variance of normal stable Tweedie model is given by

$$\det \mathbf{V}_{G_{p,t;j}}(\boldsymbol{\mu}) = t^{1-p}\left(\boldsymbol{e}_j^T \boldsymbol{\mu}\right)^{p+k-1} \text{ with } \boldsymbol{\mu} \in \mathbf{M}_{F_{t,j}}. \tag{4.12}$$

It is easy to deduce the generalized unit variance function $\det \mathbf{V}_{\mathbf{G}_p}(\boldsymbol{\mu}) = \left(\boldsymbol{e}_j^T \boldsymbol{\mu}\right)^{p+k-1}$ with $t = 1$. Table 3 summarizes all k-variate NST models with support $\mathbf{S}_p$ of distributions, where $p = p(\alpha) \geq 1$ is thepower variance parameter, the parameter $\eta = 1 + k/(p-1) = \eta(p, k)$ is the corresponding modified Lévy measure $\rho(\nu_{p,t;j})$ of NST models of the normal gamma type for $p > 1$ and degenerated for $p = 1$ introduced by [19] as described in 4.1 below.

Proposition 4.1. *Let $\nu_{p,t;j}$ be a generating measure of an NST family for given $p = p(\alpha)$ and $t > 1$. Denote $\eta = 1+k/(p-1) = \eta(p, k) > 1$ the modified Lévy measure parameter. Then*

$$\rho(\nu_{p,t;j}) = \begin{cases} t^k(p-1)^{-\eta(p,k)}.\nu_{2,\eta(p,k)} & for \quad p > 1 \\ t^k.(\delta_{e_1}\prod_{\ell \neq j}\xi_{0,1})^{*k} & for \quad p = 1 \end{cases} \tag{4.13}$$

From the cumulant function (4.4) of univariate non-negative stable Tweedie we obtain the first and the second derivatives as follow

$$K'_{\xi_{p,1}}(\theta_0) = \begin{cases} \exp(\theta_0) & \text{for} \quad p = 1 \\ [(\theta)(1-p)]^{-1/(p-1)} & \text{for} \quad p \geq 1 \end{cases}$$

Table 3: Summary of k-variate NST models with power variance parameter $p = p(\alpha) \geq 1$, modified Lévy measure parameter $\eta := 1 + k/(p-1)$ and support of distributions $\mathbf{S}_p$ fixing $j = 1$.

Type(s)	$p = p(\alpha)$	$\eta = 1 + k/(p-1)$	$\mathbf{S}_p$
Normal Poisson	$p = 1$	$\eta = \infty$	$\mathbb{N} \times \mathbb{R}^{k-1}$
Normal compound Poisson	$1 < p < 2$	$\eta > k + 1$	$[0, \infty) \times \mathbb{R}^{k-1}$
Normal noncentral gamma	$p = 3/2$	$\eta = 2k + 1$	$[0, \infty) \times \mathbb{R}^{k-1}$
Normal gamma	$p = 2$	$\eta = k + 1$	$(0, \infty) \times \mathbb{R}^{k-1}$
Normal positive stable	$p > 2$	$1 < \eta < k + 1$	$(0, \infty) \times \mathbb{R}^{k-1}$
Normal inverse Gaussian	$p = 3$	$\eta = 1 + k/2$	$(0, \infty) \times \mathbb{R}^{k-1}$

and also

$$K''_{\xi_{p,1}}(\theta_0) = \begin{cases} \exp(\theta_0) = K'_{\xi_{p,1}}(\theta_0) & \text{for} \quad p = 1 \\ [(\theta)(1-p)]^{-p/(p-1)} = \left[K'_{\xi_{p,1}}(\theta_0)\right]^p & \text{for} \quad p \geq 1 \end{cases}$$

Lemma 4.1. *Let $f : \mathbb{R} \to \mathbb{R}$ and $\mathbf{g}, \mathbf{h} : \mathbb{R}^k \to \mathbb{R}$ be three functions, each is twice differentiable and such that $\mathbf{h} = f \circ \mathbf{g}$. Then*

$$\mathbf{h}'(\mathbf{x}) = f'(\mathbf{g}(\mathbf{x})) \times \mathbf{g}'(\mathbf{x}) \quad \textit{and} \quad \mathbf{h}''(\mathbf{x}) = f''(\mathbf{g}(\mathbf{x})) \times \mathbf{g}'(\mathbf{x})\mathbf{g}'(\mathbf{x})^\top + f'(\mathbf{g}(\mathbf{x})) \times \mathbf{g}''(\mathbf{x})$$

Then fixing $j = 1$ and using Lemma 4.1 with $\mathbf{h} = \mathbf{K}_{\nu_{p,t}}$, $f = tK_{\xi_{p,1}}$ and $\mathbf{g}(\boldsymbol{\theta}) = (\theta_1 + \sum_{j=2}^k \theta_j^2/2)$ such that $\mathbf{g}'(\boldsymbol{\theta}) = (1, \theta_2, \ldots, \theta_k)^\top$ and $\mathbf{g}''(\boldsymbol{\theta}) = \mathrm{Diag}(0, 1, \ldots, 1)$, we can write

$$\mathbf{K}''_{\nu_{p,t}}(\boldsymbol{\theta}) = \left[\frac{\partial^2}{\partial\theta_i \partial\theta_j} tK_{\xi_{p,1}}(\mathbf{g}(\boldsymbol{\theta}))\right]_{i,j=1,\ldots,k} = t.\begin{pmatrix} \gamma & \mathbf{a}^\top \\ \mathbf{a} & \mathbf{A} \end{pmatrix} \tag{4.14}$$

with $\gamma = K_{\xi_{p,1}}(\mathbf{g}(\boldsymbol{\theta}))$, $\mathbf{a} = \gamma(\theta_2, \ldots, \theta_k)^\top$ and $\mathbf{A} = \gamma^{-1}\mathbf{a}\mathbf{a}^\top + K'_{\xi_{p,1}}(\mathbf{g}(\boldsymbol{\theta}))\mathbf{I}_{k-1}$. Therefore, using (4.11) it follows that

$$\det \mathbf{K}''_{\nu_{p,t}}(\boldsymbol{\theta}) = t^k = K''_{\xi_{p,1}}(\mathbf{g}(\boldsymbol{\theta})) \left\{K'_{\xi_{p,1}}(\mathbf{g}(\boldsymbol{\theta}))\right\}^{k-1}$$

$$= \begin{cases} t^k \exp\{k\mathbf{g}(\boldsymbol{\theta})\} & \text{for} \quad p = 1 \\ [\mathbf{g}(\boldsymbol{\theta})(1-p)]^{-1-k/(p-1)} & \text{for} \quad p \geq 1 \end{cases}$$

Taking $\eta(p,k) = 1 + k/(p-1)$ and $\mathbf{K}_{\rho_{(p,t)}}(\boldsymbol{\theta}) = \log\det \mathbf{K}''_{\nu_{(p,t)}}(\boldsymbol{\theta})$ which is

$$\mathbf{K}_{\rho_{(p,t)}}(\boldsymbol{\theta}) = \begin{cases} k(\theta_1 + \frac{1}{2}\sum_{j=2}^{k}) + \log t^k & \text{for} \quad p = 1 \\ \\ -\eta(p,k)\log(\theta_1 - \frac{1}{2}\sum_{j=2}^{k}\theta_j^2 + \log c_{p,k,t} & \text{for} \quad p > 1 \end{cases}$$

for $\theta \in \boldsymbol{\Theta}(\rho(\nu_{p,t})) = \boldsymbol{\Theta}(\nu_{p,t})$ with $c_{p,k,t} = t^k(p-1)^{-\eta(p,k)}$, this leads to (4.13).

Recall that the Monge-Ampère equation which is generally stated as $\det\psi''(\theta) = r(\theta)$ where ψ is an unknown smooth function and r is a given positive function. Then from the modified Lévy measure of $\nu_{p,t;j}$, the Monge-Ampère equation below is considered to be the problem of the characterization of multivariate NST models through generalized variance function

$$\det\mathbf{K}''(\boldsymbol{\theta}) = \exp\left\{\mathbf{K}_{\rho(\nu_{p,t;j})}(\boldsymbol{\theta})\right\}, \quad p \geq 1 \tag{4.15}$$

where $\mathbf{K}$ is unknown cumulant function to be determined. See [24] for normal gamma model and some references of particular cases.

4.2 Generalized Variance Estimations of some NST models

Generalized variance; i.e. the determinant of covariance matrix expressed in term of mean vector; has important roles in statistical analysis of multivariate data. It is a scalar measure of multivariate dispersion and used for overall multivariate scatter.

The estimation of the generalized variance, mainly from a decision theoretic point of view, attracted the interest of many researchers in the past four decades; see for example [28, 29] for estimation under multivariate normality. In the last two decades the generalized variance has been extended for non-normal distributions in particular for natural exponential families (NEFs); see [30, 31, 32, 33] who worked in the context of NEFs.

Here we discuss the ML and UMVU estimators of generalized variance of normal gamma, normal inverse Gaussian (NIG) and normal Poisson models. Bayesian estimator of the generalized variance for normal Poisson is also discussed. Here we only present the analytical proof, the empirical proof through simulation study can be seen in [34].

4.2.1 Maximum Likelihood Estimator

Let $\mathbf{X}_1, \ldots, \mathbf{X}_n$ be random vectors i.i.d with distribution $\mathbf{P}(\boldsymbol{\theta}; p, t) \in \mathbf{G}(\nu_{p,t;j})$ in a given NST family, i.e. for fixed for fixed $j \in \{1, 2, \ldots, k\}$, $p \geq 1$ and $t > 0$. Denoting $\overline{\mathbf{X}} = (\mathbf{X}_1 + \cdots + \mathbf{X}_n)/n = (\overline{X}_1, \ldots, \overline{X}_k)^\top$ the sample mean.

Theorem 4.1. *The maximum likelihood estimator (MLE) of the generalized variance* $\det \boldsymbol{V}_{\mathbf{G}_{p,t;j}}(\boldsymbol{\mu})$ *is given by:*

$$T_{n;k;p,t;j} = \det \boldsymbol{V}_{\mathbf{G}_{p,t;j}}(\overline{X}) = t^{1-p} \left(\overline{X}_j\right)^{p+k-1} \tag{4.16}$$

Proof. The ML estimator (4.16) is directly obtained from (4.12) by substituting μ_j with its ML estimator $\overline{X}_j$. □

Then for each model one has:

$$T_{n;k;t;j} = \det \mathbf{V}_{\mathbf{G}_{p,t;j}}(\overline{X}) = \begin{cases} \overline{X}_j^k, & \text{for normal Poisson} \\ (1/t)\overline{X}_j^{k+1}, & \text{for normal gamma} \\ (1/t^2)\overline{X}_j^{k+2}, & \text{for normal inverse Gaussian} \end{cases}$$

For all $p \geq 1$, $T_{n;k,p,t;j}$ is a biased estimator of $\det \mathbf{V}_{\mathbf{G}_{p,t;j}}(\boldsymbol{\mu}) = t^{1-p}(\mu_j)^{p+k-1}$. For example, for $p = 1$ we have $\det \mathbf{V}_{\mathbf{G}_{p,t;j}}(\boldsymbol{\mu}) = \mu_j^k$, to obtain an unbiased estimator for this we need to use the intrinsic factorial moment formula

$$E\left(\overline{X}_j(\overline{X}_{j-1})(\overline{X}_{j-2}) \cdots (\overline{X}_{j-k+1})\right) = \mu_j^k,$$

where $\overline{X}$ follows the univariate Poisson distribution with mean μ_j.

4.2.2 Uniformly Minimum Variance Unbiased Estimator

In order to avoid the lack of good properties by estimating $\det \mathbf{V}_{\mathbf{G}_{p,t}}(\boldsymbol{\mu}) = t^{1-p}\mu_j^{k+p-1}$ with $T_{n;k,p,t}$, we are able to obtain directly the uniformly minimum variance and unbiased (UMVU) estimator $U_{n;k,p,t}$ of $\det \mathbf{V}_{\mathbf{G}_{p,t}}(\boldsymbol{\mu})$.This is done through the following techniques for all integers $n > k$ [30, 33, 35] :

$$U_{n;k,p,t} = C_{n,k,p,t}(n\overline{\mathbf{X}}) \tag{4.17}$$

where $C_{n,k,p,t} : \mathbb{R}^k \to [0, \infty)$ satisfies

$$\nu_{n,k,p,t}(d\mathbf{x}) = C_{n,k,p,t}(\boldsymbol{x})\nu_{p,nt}(d\boldsymbol{x}) \tag{4.18}$$

and $\nu_{n,k,p,t}(d\boldsymbol{x})$ is the image measure of

$$\frac{1}{(k+1)!}\left(\det\begin{bmatrix} 1 & 1 & \cdots & 1 \\ \boldsymbol{x}_1 & \boldsymbol{x}_2 & \cdots & \boldsymbol{x}_{k+1}\end{bmatrix}\right)^2 \nu_{p,t}(d\boldsymbol{x}_1)\cdots\nu_{p,t}(d\boldsymbol{x}_n)$$

by the map $(\boldsymbol{x}_1,\cdots\boldsymbol{x}_n)\mapsto \boldsymbol{x}_1+\cdots+\boldsymbol{x}_n$. Boubacar Maïnassara and Kokonendji (2013) presented the ways for getting expression of $C_{n,k,p,t}(\boldsymbol{x})$ for computing the UMVU estimator $U_{n;k,p,t}$for $p=p(\alpha)\in[1,\infty)$ as stated in the following Theorem.

Theorem 4.2. *Let $\boldsymbol{X}_1,\cdots,\boldsymbol{X}_n$ be random vectors i.i.d with distribution $\boldsymbol{P}(\boldsymbol{\mu},\mathbf{G}_{p,t;j})\in \boldsymbol{G}_{(\nu_{p,t;j})}$ in a given NST family, i.e. for fixed $p\geq 1, t>0$, and having modified Lèvy measure $\rho(\nu_{p,t})$ satisfies (4.13) with parameter $\eta(p,k)=p+k-1$. Then*

$$C_{n,k,p,t}(\boldsymbol{x})=\frac{\nu_{p,nt}*\rho(\nu_{p,t})(d\boldsymbol{x})}{\nu_{p,nt}(d\boldsymbol{x})}$$

in particular, $C_{n,k,p,t}(\boldsymbol{x})$ is

$$\begin{cases} n^{-k}x_j(x_j-1)(x_j-2)\cdots(x_j-k+1), \quad x_j\geq k & \textit{for normal Poisson}\\ t^k\Gamma(nt)[\Gamma(nt+k+1)]^{-1}x_j^{k+1} & \textit{for normal gamma}\\ t^k 2^{-1-k/2}[\Gamma(1+k/2)]^{-1}x_j^{3/2}\exp\left\{(nt)^2/(2x_j)\right\}\times & \\ \int_0^{x_j} y_j^{k/2}(x_j-y_j)^{-3/2}\exp\left\{-y_j-(nt)^2/[2(x_j-y_j)]\right\}dy_j & \textit{for normal Inverse-Gaussian}\end{cases}$$

Proof. From (4.18) we write:

$$C_{n,k,p,t}(\boldsymbol{x})=\frac{\nu_{n;k;p,t}(d\boldsymbol{x})}{\nu_{p,nt}(d\boldsymbol{x})}$$

Following [33, Theorem 1] and using (4.13) we have:

$$\begin{aligned}\mathbf{K}_{\nu_{n;k;p,t}}(\boldsymbol{\theta}) &= n\mathbf{K}_{\nu_{p,t}}(\boldsymbol{\theta})+\log\det\mathbf{K}''_{\nu_{p,t}}\\ &= \mathbf{K}_{\nu_{p,nt}}(\boldsymbol{\theta})+\mathbf{K}_{\rho(\nu_{p,t})}(\boldsymbol{\theta})\end{aligned}$$

for all $\boldsymbol{\theta}\in\boldsymbol{\Theta}(\nu_{p,1})$. Then it immediately follows that $\nu_{n;k;p,t}=\nu_{p,nt}*\rho(\nu_{p,t})$ is the convolution product of $\nu_{p,nt}$ by $\rho(\nu_{p,t})$. The proof for $C_{n,k,p,t}(\boldsymbol{x})$ is established by considering each group of the NST models with respect to the different values of $p\geq 1$ and using (4.13).

Indeed, for $p = 1$ and fixing $j = 1$ we have $\rho(\nu_{1,t}) = t^k.(\delta_{e_1} \prod_{j=2}^{k} \xi_{0,1})^{*k}$ and

$$\begin{aligned}
C_{n,k,1,t}(\boldsymbol{x}) &= t^k \frac{\nu_{1,nt} * (\delta_{e_1} \prod_{j=2}^{k} \xi_{0,1})^{*k}(d\mathbf{x})}{\nu_{1,nt}(d\mathbf{x})} \\
&= t^k \frac{\xi_{1,nt}(x_1 - k)}{\xi_{1,nt}(x_1)} \left[\prod_{j=2}^{k} \int_{\mathbb{R}} \frac{\xi_{0,x_1-k}(x_j - y_j)\xi_{0,k}(y_j)}{\xi_{0,x_1}(x_j)} dy_j \right] \\
&= t^k \frac{x_1!(nt)^{x_1-k} \exp(-nt)}{(x_1 - k)!(nt)^{x_1} \exp(-nt)} \times 1 \\
&= \frac{(x_1)(x_1 - 1) \ldots (x_1 - k + 1)}{n^k};
\end{aligned}$$

because for fixed $j = 2, \ldots, k$ the expression

$$W(j, x_1, k) = \int_{\mathbb{R}} \frac{\xi_{0,x_1-k}(x_j - y_j)\xi_{0,k}(y_j)}{\xi_{0,x_1}(x_j)} dy_j \tag{4.19}$$

is finally

$$\begin{aligned}
W(j, x_1, k) &= \int_{\mathbb{R}} \frac{\frac{1}{\sqrt{2\pi(x_1-k)}} \exp\left\{\frac{-(x_j-y_j)^2}{2(x_1-y_1)}\right\} \frac{1}{\sqrt{2\pi k}} \exp\left(\frac{-y_j^2}{2k}\right)}{\frac{1}{\sqrt{2\pi x_1}} \exp\left(\frac{-x_j^2}{2x_1}\right)} dy_j \\
&= \frac{\sqrt{x_1}}{\sqrt{2\pi}\sqrt{k(x_1 - k)}} \exp\left(\frac{x_j^2}{2x_1}\right) \int_{\mathbb{R}} \exp\left(\frac{-y_j^2 + 2\frac{kx_j}{x_1} y_j - \frac{kx_j^2}{x_1}}{2\frac{k}{x_1}(x_1 - k)}\right) dy_j \\
&= \exp(0) \int_{\mathbb{R}} \xi_{0,\frac{k(x_1-k)}{x_1}} \left(y_j - \frac{kx_j}{x_1}\right) dy_j \\
&= 1
\end{aligned}$$

Let $p = 2$, then $\rho(\nu_{2,t}) = t^k \nu_{2,\eta(2,k)}$, one obtains

$$\begin{aligned}
C_{n,k,2,t}(\boldsymbol{x}) &= t^k \frac{\nu_{2,nt} * \nu_{2,\eta(2,k)}(d\mathbf{x})}{\nu_{2,nt}(d\mathbf{x})} \\
&= t^k \frac{\nu_{2,nt+\eta(2,k)}(d\mathbf{x})}{\nu_{2,nt}(d\mathbf{x})} \\
&= t^k \frac{\xi_{2,nt+\eta(2,k)}(dx_1) \prod_{j=2}^k \xi_{0,x_1}(dx_j)}{\xi_{2,nt}(dx_1) \prod_{j=2}^k \xi_{0,x_1}(dx_j)} \\
&= t^k \frac{x_1^{nt+\eta(2,k)-1}}{\Gamma[nt+\eta(2,k)]} \times \frac{\Gamma(nt)}{x_1^{nt-1}} \\
&= \frac{\Gamma(nt)}{\Gamma[nt+\eta(2,k)]} x_1^{\eta(2,k)}
\end{aligned}$$

with the modified Lévy measure parameter $\eta(2,k) = k+1$.

For $p = 3$ we have $\rho(\nu_{p,t}) = t^k 2^{-\eta(3,k)} \nu_{2,\eta(3,k)}$, then

$$\begin{aligned}
C_{n,k,3,t}(\boldsymbol{x}) &= t^k 2^{-\eta(3,k)} \frac{\nu_{3,nt} * \nu_{2,\eta(3,k)}(d\mathbf{x})}{\nu_{3,nt}(d\mathbf{x})} \\
&= t^k 2^{-1-k/2} \int_{\mathbb{R}^k} \frac{\nu_{3,nt}(\mathbf{x}-\mathbf{y}) \nu_{2,\eta(3,k)}(\mathbf{y})}{\nu_{3,nt}(\mathbf{x})} d\mathbf{y} \\
&= t^k 2^{-1-k/2} \int_0^{x_1} \frac{\xi_{3,nt}(x_1-y_1)\xi_{2,\eta(3,k)}(y_1)}{\xi_{3,nt}(x_1)} \left[\prod_{j=2}^k \int_{\mathbb{R}} \frac{\xi_{0,x_1-y_1}(x_j-y_j)\xi_{0,y_1}(y_j)}{\xi_{0,x_1}(x_j)} \right] dy_1 \\
&= t^k 2^{-1-k/2} \int_0^{x_1} \frac{\xi_{3,nt}(x_1-y_1)\xi_{2,\eta(3,k)}(y_1)}{\xi_{3,nt}(x_1)} \times 1 dy_1 \\
&= t^k 2^{-1-k/2} \int_0^{x_1} \frac{\frac{nt}{\sqrt{2\pi(x_1-y_1)^3}} \exp\left\{\frac{-(nt)^2}{2(x_1-y_1)}\right\} \times \frac{y_1^{\eta(3,k)-1}}{\Gamma[\eta(3,k)]} \exp(-y_1)}{\frac{nt}{\sqrt{2\pi x_1^3}} \exp\left(\frac{-(nt)^2}{2x_1}\right)} dy_1 \\
&= t^k 2^{-1-k/2} \frac{x_1^{3/2}}{\Gamma[\eta(3,k)]} \exp\left\{\frac{(nt)^2}{2x_1}\right\} \int_0^{x_1} \frac{y_1^{\eta(3,k)-1}}{(x_1-y_1)^{3/2}} \exp\left\{-y_1 - \frac{(nt)^2}{2(x_1-y_1)}\right\} dy_1
\end{aligned}$$

$\square$

4.2.3 Bayesian Estimator

We introduce the Bayesian estimator of normal-Poisson generalized variance using the conjugate prior of Poisson distribution namely gamma distribution.

Theorem 4.3. *Let $\boldsymbol{X}_1, \cdots, \boldsymbol{X}_n$ be random vectors i.i.d with distribution $\boldsymbol{P}(\boldsymbol{\mu}, \mathbf{G}_{1,t;j}) \in \boldsymbol{G}_{(\nu_{1,t;j})}$ a normal Poisson model. For $t > 0$ and $j \in \{1, 2, \ldots, k\}$ fixed, under assumption of prior gamma distribution of μ_j with parameter $\alpha > 0$ and $\beta > 0$, the Bayesian estimator of* $\det \mathbf{V}_{F_{t;j}}(\boldsymbol{\mu}) = \mu_j^k$ *is given by*

$$B_{n,t;j,\alpha,\beta} = \left(\frac{\alpha + n\overline{X}_j}{\beta + n} \right)^k . \tag{4.20}$$

Proof. Let $X_{j1}, \ldots, X_{jn}$ given μ_j be Poisson distribution with mean μ_j, then the probability mass function is given by

$$p(x_{ji}|\mu_j) = \frac{\mu_j^{x_{ji}}}{x_{ji}!} \exp(-\mu_j) \quad \forall x_{ji} \in \mathbb{N}.$$

Assuming that μ_j follows gamma(α, β), then the prior probability distribution function of μ_j is written as

$$f(\mu_j; \alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} \mu_j^{\alpha-1} \exp(-\beta\mu_j), \quad \forall \mu_j > 0,$$

with $\Gamma(\alpha) := \int_0^\infty x^{\alpha-1} e^{-x} dx$. Using the classical Bayes theorem, the posterior distribution of μ_j given an observation x_{ji} can be expressed as

$$\begin{aligned} f(\mu_j|x_{ji}; \alpha, \beta) &= \frac{p(x_{ji}|\mu_j) f(\mu_j; \alpha, \beta)}{\int_{\mu_j>0} p(x_{ji}|\mu_j) f(\mu_j; \alpha, \beta) d\mu_j} \\ &= \frac{(\beta+1)^{\alpha+x_{ji}}}{\Gamma(\alpha + x_{ji})} \mu_j^{\alpha+x_{ji}-1} \exp\{-(\beta+1)\mu_j\} \end{aligned}$$

which is the gamma density with parameters $\alpha' = \alpha + x_{ji}$ and $\beta' = \beta + 1$. Then with random sample $X_{j1}, \ldots, X_{jn}$ the posterior will be gamma$(\alpha + n\overline{X}_j, \beta + n)$. Since Bayesian estimator of μ_j is given by the expected value of the posterior distribution i.e. $(\alpha + n\overline{X}_j)/(\beta + n)$, this concludes the proof. $\square$

5 FINAL REMARKS

Study on Tweedie family as a special case of EDM and at the same time as an extention of α-stable family distribution is an interesting topic and a challenging problem. The univariate Tweedie distributions which admit the power variance function: $Var(\mu) = V\mu^p$ with $p \in (-\infty, 0] \cup [1, \infty)$ with domain $\mu \in \mathbb{R}$ for $p = 0$ and $\mu \in \mathbb{R}_+$ for other values of p, have the cumulant function and mean that can be found by equating $\kappa''(\theta) = d\mu/d\theta = \mu^p$ and solving for μ and κ. This special characteristic is very helpful in the extensions of the family such as the multivariate Tweedie model and NST model which we have discussed in this paper.

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