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Further results on locating chromatic number for amalgamation of stars linking by one path

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Abstract

Let $G = (V, E)$ be a connected graph. Let c be a proper coloring using k colors, namely $1, 2, \cdots, k$. Let $\Pi = \{S_1, S_2, \cdots, S_k\}$ be a partition of $V(G)$ induced by c and let S_i be the color class that receives the color i. The color code, $c_\Pi(v) = (d(v, S_1), d(v, S_2), \cdots, d(v, S_k))$, where $d(v, S_i) = \min\{d(v, x)|x \in S_i\}$ for $i \in [1, k]$. If all vertices in $V(G)$ have different color codes, then c is called as the *locating-chromatic k-coloring* of G. Minimum k such that G has the locating-chromatic k-coloring is called the locating-chromatic number, denoted by $\chi_L(G)$. In this paper, we discuss the locating-chromatic number for n certain amalgamation of stars linking a path, denoted by $nS_{k,m}$, for $n \geq 1$, $m \geq 2$, $k \geq 3$, and $k > m$.

Keywords: locating chromatic number, amalgamation of stars
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1. Introduction

The locating chromatic number is a topic in graph theory, derived from the vertex-coloring and partition dimension of a graph [11]. Many paper discussed about the locating chromatic number since Chartrand et al. [9] introduced the concept in 2002.

All graphs considered are finite, undirected and simple. Let $G = (V, E)$ be a connected graph. Let c be a proper coloring using k colors, namely $1, 2, \cdots, k$. Let $\Pi = \{S_1, S_2, \cdots, S_k\}$ be a partition of $V(G)$ induced by c and let S_i be the color class that receives the color i. The

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color code, $c_\Pi(v) = (d(v,S_1), d(v,S_2), \cdots, d(v,S_k))$, where $d(v,S_i) = \min\{d(v,x)|x \in S_i\}$ for $i \in [1,k]$. If all vertices in $V(G)$ have different color codes, then c is called as the *locating-chromatic k-coloring* of G. Minimum k such that G has the locating-chromatic k-coloring is called the locating-chromatic number, denoted by $\chi_L(G)$.

Theorem 1.1. *[10] Let G be a simple connected graph and c be a locating coloring of G. If $v, w \in V(G)$ and $v \neq w$ such that $d(v,x) = d(w,x)$ for all $x \in V(G) - \{v,w\}$, then $c(v) \neq c(w)$. In particular, if v and w are non adjacent vertices of G such that neighborhood of v is equal to neighborhood of w, then $c(v) \neq c(w)$.*

Corollary 1.1. *[10] If G is a simple connected graph containing a vertex that is adjacent to k leaves of G, then $\chi_L(G) \geq k+1$.*

Chartrand et al. [9][10] obtained the locating chromatic number of some classes of graphs such that: paths, stars, double stars, caterpillars, complete graphs, bipartite graphs, and the characterization of graphs having locating chromatic number $n, (n-1)$, or $(n-2)$. Next, Asmiati et al. investigated locating chromatic number for special kind of trees, namely: amalgamation of stars [1], [4], firecracker graphs [2], banana trees [5]. Moreover, Baskoro at al. [8] determined the locating chromatic number for corona product of some graphs. Beside that, Asmiati et al. [3] characterized graphs containing cycle having locating chromatic number tree and Baskoro et al. [7] characterized all trees having locating chromatic number three.

Let S_{m+2} be a star with $(m+2)$ vertices. The amalgamation of stars, denoted by $S_{k,m}$, where $k, m \geq 2$, is obtained from $(k-1)$ stars S_{m+2}, by identifying one leaf of every stars S_{m+2}. The identified vertex is denoted as the center of $S_{k,m}$. Graph $nS_{k,m}$ is obtained from n copies $S_{k,m}$ and every center of them, denoted by x_i, for $i = 1, 2, \ldots, n$ is linked by one path, and $(n-1)$ new vertices denoted y_i, $i = 1, 2, \ldots, n-1$ are the subdivision vertices in $x_i x_{i+1}$, $i = 1, 2, \ldots, n-1$. Next, the vertices of distance 1 from the center x_i are defined as the intermediate vertices, denoted by l_j^i, $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, k-1$ and the t-th leaf of the intermediate vertices l_j^i are denoted by $l_{jt}^i (t = 1, 2, \ldots, m)$.

In [6], Asmiati et al. determined the locating chromatic number of $nS_{k,m}$ for $k \leq m$, where $k \geq 3$ and $m \geq 2$, as follows.

$$\chi_L(nS_{k,m}) = \begin{cases} m+1, & 1 \leq n \leq \lfloor \frac{m}{k-1} \rfloor; \\ m+2, & \text{otherwise.} \end{cases}$$

In this paper we will discuss the locating chromatic number of $nS_{k,m}$ for $k > m$, where $k \geq 3$ and $m \geq 2$.

2. Main Results

In this section, we will discuss about the locating chromatic number of $nS_{k,m}$ for $n \geq 1$ and $k > m, k \geq 3, m \geq 2$.

Lemma 2.1. *Let c be a coloring on $nS_{k,m}$ using $(k-a)$ colors, where $k > m, k \geq 3, m \geq 2$, $a \geq 0$, $a = k-m-1$. Coloring c is a locating coloring if and only if $c\left(l_j^i\right) = c(l_n^s)$, $j \neq n$ and $i \neq s$ such that $\left\{c(l_{jt}^i) \mid t = 1, 2, 3, \ldots, m\right\}$ and $\{c(l_{nt}^s) \mid t = 1, 2, 3, \ldots, m\}$ are two different sets.*

Proof. Consider $P=\left\{c(l_{jt}^i)\mid t=1,2,3,\ \dots,m\right\}$ and $Q=\{c(l_{nt}^s)\mid t=1,2,3,\ \dots,m\}$. Let c be a locating coloring of $nS_{k,m},\ k>m,\ k\geq 3, m\geq 2,\ a\geq 0$, dan $c\left(l_j^i\right)=c(l_n^s)$, for some $j\neq n$, and $i\neq s$. Suppose that $P=Q$. Since $d(l_j^i,u)=d(l_n^s,u)$ for each $u\in V\backslash\{\{l_{jt}^i\}\cup\{l_{nk}^s\}\}$, then the color codes of l_j^i and l_n^s are the same. So, c is not a locating coloring, a contrary. As the result, $P\neq Q$.

Let Π be a partition of $V(G)$ with $|\Pi|\geq m$. Consider $c(l_j^i)=c(l_n^s)$, $j\neq n$, dan $i\neq s$. Since $P\neq Q$, then there are two colors, namely x and y such that ($x\in P,\ x\notin Q$) or ($y\in P,\ y\notin Q$). Next, we will show that every $v\in V\left(nS_{k,m}\right)$ have different color codes.

- It is clear that $c_\Pi\left(l_j^i\right)\neq c_\Pi(l_n^s)$, since their color codes are different in the x-ordinat or y-ordinat.
- If $c(l_{jt}^i)=c(l_{nt}^s)$, for each $l_j^i\neq l_n^s$, then we divide two cases to show that $c_\Pi(l_{jt}^i)\neq c_\Pi(l_{nt}^s)$

  Case 1: If $c(l_{jt}^i)=c(l_{nt}^s)$, then based on the previous proof $P\neq Q$. So, $c_\Pi\left(l_{jt}^i\right)\neq c_\Pi(l_{nt}^s)$.

  Case 2: Consider $c\left(l_j^i\right)=p_1$ and $c\left(l_n^s\right)=p_2$, where $p_1\neq p_2$. Then $c_\Pi\left(l_{jt}^i\right)\neq c_\Pi(l_{nt}^s)$ because their color codes are different at least in the p_1-ordinat and p_2-ordinat.
- If $c\left(x_i\right)=c(l_{jt}^i)$, then the color code of $c_\Pi\left(x_i\right)$ contains at least two components with value 1, whereas in $c_\Pi(l_{jt}^i)$ contains exactly one component with value 1. So, $c_\Pi\left(x_i\right)\neq c_\Pi(l_{jt}^i)$.
- If $c\left(y_i\right)=c(l_{jt}^i)$, then the color code of $c_\Pi\left(y_i\right)$ contains at least two components with values 1. whereas in $c_\Pi(l_{jt}^i)$ contains exactly one component with value 1. So, $c_\Pi\left(y_i\right)\neq c_\Pi(l_{jt}^i)$.

From all cases, we can see that all vertices in $nS_{k,m}$ have different color codes, so c is a locating coloring. □

Lemma 2.2. *Let $n\geq 1,\ k>m,\ k\geq 3, m\geq 2,\ a\geq 0$, and $a=k-m-1$. If c is a locating coloring of $nS_{k,m}$ using $k-a$ colors and $H(a)=\left\lfloor\dfrac{(k-a-1)\binom{k-a-1}{m}}{k-1}\right\rfloor$, then $n\leq H\left(a\right)$.*

Proof. Let c be a $(k-a)$-locating coloring of $nS_{k,m}$. For some j, consider $c(l_j^i)$ as the color of l_j^i, then the color combination of $\left\{l_{jt}^i\mid t=1,2,3,\dots,\ m\right\}$ is $\binom{k-a-1}{m}$. Since one color has been used for the central vertex x, then there are $(k-a-1)$ colors left to be assigned to l_j^i, for each $i=1,2,\dots,n$ and $j=1,2,3,\dots,\ k-1$. By Lemma 2.1, the maximum number for n is $\left\lfloor\dfrac{(k-a-1)\binom{k-a-1}{m}}{k-1}\right\rfloor=H(a),\ a\geq 0$. □

Theorem 2.1. *Let $nS_{k,m}$ be some certain amalgamation of stars for $a\geq 0,\ k>m,\ k\geq 3,\ m\geq 2$, $a=k-m-1$. Then*

$$\chi_L(nS_{k,m})=\begin{cases} k-a, & 1\leq n\leq H(a),\\ k-a+1, & otherwise.\end{cases}$$

Proof. First, we determine the lower bound of $\chi_L(nS_{k,m})$ for $1 \leq n \leq H(a) = \left\lfloor \frac{(k-a-1)\binom{k-a-1}{m}}{k-1} \right\rfloor$.
Since every vertex l_j^i for $i = 1, 2, 3, \ldots, n$ and $j = 1, 2, 3, \ldots,\ k-1$ are adjacent to $m = k-a-1$ leaves, then by Corollary 1.1, we have $\chi_L(nS_{k,m}) \geq\ k-a$.
To determine the upper bound of $\chi_L(nS_{k,m})$ for $1 \leq n \leq H(a) = \left\lfloor \frac{(k-a-1)\binom{k-a-1}{m}}{k-1} \right\rfloor$, let c be a coloring of $V(nS_{k,m})$ using $(k-a)$ colors. We assign the coloring as follows.

- $c(x_i) = 1$, for $i = 1, 2, 3, \ldots, n$.
- $c(y_i) = 2$, for odd i and 3 for even $i = 1, 2, 3, \ldots, n$.
- Color of l_i^j for each $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, (k-1)$ given color $2, 3, \ldots, (k-a)$, respectively.
- $\{c(l_{jt}^i)\} = \{1,\ 2, 3, \ldots, k-a\} \setminus \{c(l_i^j)\}$ for $t = 1, 2, 3, \ldots, m$.

Next, we will show that all vertices in $V(nS_{k,m})$ have different color codes. Consider $u, v \in V(nS_{k,m})$ and $c(u) = c(v)$. Then we have the following cases.

- If $u = x_i$, $v = x_k$ for some i, k and $i \neq k$, then $c_\Pi(u) \neq c_\Pi(v)$ because $c(l_j^i) \neq c(l_j^k)$ for each $i = 1, 2, \ldots, (k-1)$.
- If $u = x_i$, $v = l_{jt}^h$ for some $i, h,\ j, t$, then in $c_\Pi(u)$ does not have component value four, whereas in $c_\Pi(v)$, exactly one component has value 4. So, $c_\Pi(u) \neq c_\Pi(v)$.
- If $u = y_i$, $v = l_j^i$, for some $i,\ j$, then in $c_\Pi(u)$ exactly two components have value 1, whereas in $c(v)$, at least three components have value 1. So, $c_\Pi(u) \neq c_\Pi(v)$.
- If $u = y_i$, $v = l_j^k$, for some $i, k,\ j$ and $i \neq k$, then in $c_\Pi(u)$ exactly two components have value 1, whereas in $c(v)$, at least three components have value 1. So, $c_\Pi(u) \neq c_\Pi(v)$.
- If $u = y_i$, $v = l_{jt}^i$ for some $i, j,\ t$, then in $c_\Pi(u)$, exactly two components have value 1, whereas in $c(v)$, exactly one component has value 1. As a result, $c_\Pi(u) \neq c_\Pi(v)$.
- If $u = y_i$, $v = l_{jt}^k$ for some $i, k, j,\ t$ and $i \neq k$, then in $c_\Pi(u)$ at least two components have value 1, whereas in $c(v)$, exactly one component has value 1. So, $c_\Pi(u) \neq c_\Pi(v)$.
- If $u = l_j^i$, $v = l_{jt}^i$ for some i, j, t, then in $c_\Pi(u)$ at least two components have value 1, whereas in $c(v)$, exactly one component has value 1. As a result, $c_\Pi(u) \neq c_\Pi(v)$
- If $u = l_j^i$, $v = l_{ht}^k$ for some i, j, k, h, t and $i \neq k$, $j \neq h$, then in $c_\Pi(u)$, at least two components have value 1, whereas in $c(v)$, exactly one component has value 1. So, $c_\Pi(u) \neq c_\Pi(v)$
- if $u = l_{jt}^i, v = l_{ht}^i$ for some i, j, h, t ,$j \neq h$. Since $\{c(l_{jt}^i)\} \neq \{c(l_{ht}^i)\}$, then $c_\Pi(u) \neq c_\Pi(v)$.

- If $u = l^i_{jt}$, $v = l^k_{jt}$ for some i, j, k, t ,$i \neq k$. Since $c\left(l^i_j\right) \neq c\left(l^k_j\right)$, then $c_\Pi(u) \neq c_\Pi(v)$.

Since all vertices have different color codes, then c is a locating coloring on $nS_{k,m}$. Thus, $\chi_L\left(nS_{k,m}\right) \leq k - a$ for $n \leq H\left(a\right)$.
Next, we discuss the locating chromatic number of $nS_{k,m}$ for $n > H\left(a\right)$.
By Corollary 1.1, we have the trivial lower bound, $\chi_L\left(S_{k,m}\right) \geq k - a$ for $> H\left(a\right)$. Suppose that c is a locating coloring using $(k - a)$ colors on $nS_{k,m}$ for $k > m$, $k \geq 3$, $m \geq 2$, and $n > H\left(a\right)$. Since $n > H\left(a\right)$, then there are i, j, k, t, $i \neq k$ and $\left\{c\left(l^i_j\right), c\left(l^i_{jt}\right)\right\} = \left\{c\left(l^k_j\right), c\left(l^k_{jt}\right)\right\} = \{1, 2, 3, \ldots, k - a\}$ such that $c_\Pi\left(l^i_j\right) = c_\Pi\left(l^k_j\right)$ for some $j = 1, 2, 3, \ldots, k-1$, $t = 1, 2, 3, \ldots, m$, a contrary. Thus, $\chi_L\left(S_{k,m}\right) \geq k - a + 1$ for $n > H\left(a\right)$.
Let c be a coloring on $nS_{k,m}$ using $(k - a + 1)$ colors. We assign the coloring as follows.

- $c(x_i) = 1$, for $i = 1, 2, 3, \ldots, n$.
- $c(y_i) = 2$, for odd i and 3 for even $i = 1, 2, 3, \ldots, n$.
- For $j = 1, 2, 3, \ldots, (k - 1)$, $c(l^j_i) = 2$, for odd i and 3 for even $i = 1, 2, 3, \ldots, n$.
- If $A = \{1, 2, \ldots, k - a + 1\}$, define:

$$\{c(l^i_{jt}) | t = 1, 2, \ldots, m)\} = \begin{cases} A\backslash\{1, k - a\} & \text{if } i = 1, \\ A\backslash\{k - a + 1\} & \text{otherwise.} \end{cases}$$

The maximum number of colored p is $\binom{k-a-1}{m}$ for any p. We can do that because $n > H\left(a\right)$. So, $c\left(l^i_j\right) = c(l^s_n)$, $j \neq n$, dan $i \neq s$. Thus, we can arrange such that $\left\{c(l^i_{jt}) \mid t = 1, 2, 3, \ \ldots, \ m\right\} \neq \left\{c(l^s_{nt}) \mid t = 1, 2, 3, \ \ldots, \ m\right\}$. As the result, by Lemma 2.1, c is a locating coloring. Thus, $\chi_L\left(nS_{k,m}\right) \leq k - a + 1$ for $n > H\left(a\right)$. As the conclusion, we obtain that $\chi_L\left(nS_{k,m}\right) = k - a + 1$. □
For an illustration, we give the locating-chromatic coloring of $nS_{5,3}$ for $1 \leq n \leq 4$ in Figure 1 and $nS_{5,3}$ for $n > 4$ in Figure 2.

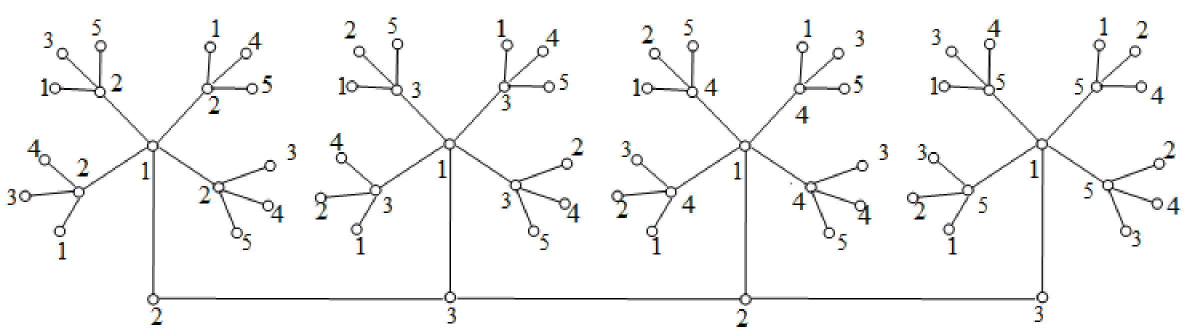

Figure 1. A minimum locating coloring of $4S_{5,3}$

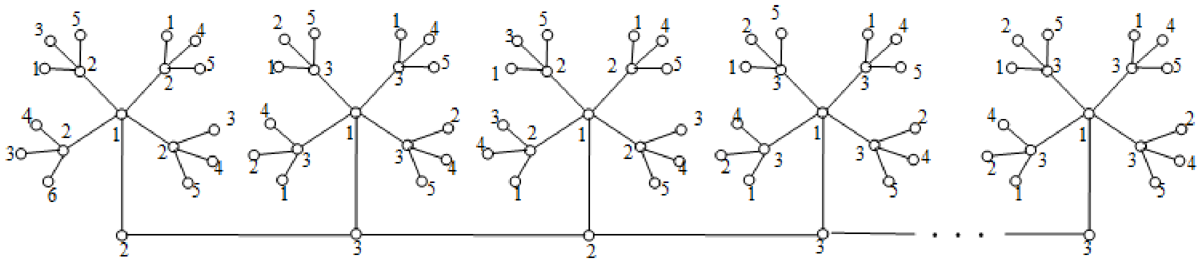

Figure 2. A minimum locating coloring of $nS_{5,3}$ for $n > 4,\ a = 0$

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