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Ratio of Linear Function of Parameters and Testing Hypothesis of the Combination Two Split Plot Designs

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Abstract: The aim of this paper is going to analyze comparison of ratio of parameters and testing various hypotheses of the combination two split plot designs. The combination of two fixed effect split plot designs have constrains on its parameter and the model is non full rank model. First we transform the model into two models, one model related to whole plots and the other model related to split plot. By Model reduction Methods, we transform each model into unconstraint model with full column rank. Based on these unconstraint models estimation, ratio of linear function of parameters by using Fieller's Theorem and testing various hypotheses will be developed.

Key words: Split plot designs . model reduction methods . ratio of parameters . Fieller's theorem

INTRODUCTION

Inferences concerning comparison of some model have become an interesting subject in statistics. Many approaches have been conducted to compare some statistical models, Theil [13] has discussed the combination of several linear equation by GLS methods some inference concerning the estimation and testing hypotheses were derived. Mustofa [10] has discussed combination of some RCBDs where some testing hypothesis has been developed. Peterson [11] has discussed the combination of several experimental designs applied in the areas of agriculture. In this paper, inferences concerning ratio of linear function of parameter from two split plot models and testing various hypotheses will be discussed. Ratio of two variances are common in inferential statistics by using chi squares distributions, or some other methods [8], but for the ratio of means or linear combination of means are using difference approach. Fieller [4] provides a method to calculate the confidence interval (CI) for the ratio of means of bivariate normal distribution. Also, Nowadays, the application of this theorem become the area of research in many fields such as economic, biostatistics [5] and bioequivalence problem [2]. Zerbe [14] shows that the Fieller's Theorem provides widely used general procedure for the construction of the confidence limit for certain ratios of parameters and also shows that Fieller's theorem can be expressed in the matrix formulation of the general linear model.

In matrix notation the split plot design model with fixed effect model can be formulated as follow:

$$Y = X\theta + e \qquad (1)$$

Subject to $G\theta = 0$, where

$$X = [1_n \otimes 1_s, E \otimes 1_s, F \otimes 1_s, 1_n \otimes I_s, F \otimes I_s] \qquad (2)$$

E is nxb matrix and E= diag(1_v, 1_v, …, 1_v), F is nxv matrix and $F = [I_v, I_v, \dots, I_v]'$, 1n is nx1 vector of ones, In is nxn identity matrix,

$$\theta = (\mu,\ \beta',\ \tau',\ \delta',\ \gamma')'$$

$$\beta' = (\beta_1, \beta_2, \dots, \beta_b)$$

$$\tau' = (\tau_1, \tau_2, \dots, \tau_v)$$

$$\delta' = (\delta_1, \delta_2, \dots, \delta_s)$$

$$\gamma' = (\gamma_{11}, \gamma_{12}, \dots, \gamma_{1s}, \gamma_{21}, \gamma_{22}, \dots, \gamma_{2s}, \dots \gamma_{v1}, \gamma_{v2}, \dots, \gamma_{vs})$$

and $e \sim N(0, \sigma^2 V)$. τ_u denotes the fixed effect due to uth level of factor A; δ_l denotes the fixed effect due to lth level of factor B; γ_{ul} denotes the interaction effect between uth level of A and lth level of B and β_i the effect due to ith block and μ denotes the overall mean. Note that V = Blockdiag (V_1, V_2,…V_n) where $V_i = (1-\rho)I_s + \rho J_s$ for i=1,2,..,n, n=bv and J_s is sxs matrix of ones.

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$$G = \begin{bmatrix} 0 & 1_b' & 0_v' & 0_s' & 0_{vs}' \\ 0 & 0_b' & 1_v' & 0_s' & 0_{vs}' \\ 0 & 0_b' & 0_v' & 1_s' & 0_{vs}' \\ 0_v & 0_{v\times b} & 0_{v\times v} & 0_{v\times s} & I_v \otimes 1_s' \\ 0_{s-1} & 0_{s-1\times b} & 0_{s-1\times v} & 0_{s-1\times s} & 1_v' \otimes (I_{s-1} \quad 0_{s-1}) \end{bmatrix} \quad (3)$$

Moreover

$$\mathrm{cov}(e) = \sigma^2 I_n \otimes [(1-\rho)I_s + \rho J_s] \quad (4)$$

[3, 8, 10].

MODELING OF THE COMBINATION TWO SPLIT PLOT DESIGNS

Consider two split plot models

$$Y_1 = X_1\theta_1 + e_1 \text{ Subject to } G_1\theta_1 = 0 \quad (5)$$

$$Y_2 = X_2\theta_2 + e_2 \text{ Subject to } G_2\theta_2 = 0 \quad (6)$$

where Y_1 and Y_2 are bvs×1 vectors of observations, X_1 and X_2 are equal to matrix X given in (2), G_1 and G_2 are equal to matrix G given in (3) and

$$\mathrm{cov}(e_1) = \sigma^{*2} I_n \otimes [(1-\rho^*)I_s + \rho^* J_s]$$

and

$$\mathrm{cov}(e_2) = \sigma^{**2} I_n \otimes [(1-\rho^{**})I_s + \rho^{**} J_s]$$

Model (5) and (6) may be combined as

$$\mathbb{Y} = \mathbb{X}\Theta + \mathbb{E} \quad (7)$$

Subject to $\mathbb{G}\Theta = 0$
where

$$\mathbb{Y} = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}, \mathbb{X} = \begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix}, \mathbb{G} = \begin{bmatrix} G_1 & 0 \\ 0 & G_2 \end{bmatrix},$$

$$\Theta = \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} \text{ and } \mathbb{E} = \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}$$

and

$$\mathrm{cov}(\mathbb{E}) = \mathrm{diag}[\sigma^{*2} I_n \otimes [(1-\rho^*)I_s + \rho^* J_s]$$

$$\sigma^{**2} I_n \otimes [(1-\rho^{**})I_s + \rho^{**} J_s]$$

Let

$$P = \left(\left(\tfrac{1}{\sqrt{s}}\right)1_s, \; P_1'\right)'$$

be s×s orthogonal matrix. It can be shown that

$$P_1'P_1 = I_s - \left(\tfrac{1}{s}\right)J_s, \; P_1 1_s = 0_{s\times(s-1)}$$

and $P_1P_1' = I_{s-1}$. Now multiply model (7) with the orthogonal matrix

$$\mathbb{P} = I_{2n} \otimes P \quad (8)$$

and decompose the transform model into model (9) and (10) as defined below:

$$\mathbb{W} = I_2 \otimes [1_n \quad E \quad F]\Gamma + \mathbb{E}^* \quad (9)$$

Subject to

$$\sum_{i=1}^{b}\beta_{1i} = 0, \sum_{i=1}^{b}\beta_{2i} = 0$$

$$\sum_{u=1}^{v}\tau_{1u} = 0, \sum_{u=1}^{v}\tau_{2u} = 0$$

and

$$\mathbb{Z} = I_2 \otimes [1_n \otimes P_1 \quad F \otimes P_1]\Psi + \mathbb{E}^{**} \quad (10)$$

Subject to

$$\sum_{l=1}^{s}\delta_{1l} = 0, \sum_{l=1}^{s}\delta_{2l} = 0, \sum_{u=1}^{v}\gamma_{1ul} = 0$$

$$\sum_{u=1}^{v}\gamma_{2ul} = 0, \sum_{l=1}^{s}\gamma_{1ul} = 0, \sum_{l=1}^{s}\gamma_{2ul} = 0$$

Where

$$\Gamma' = (\mu_1 \quad \beta_1' \quad \tau_1' \quad \mu_2 \quad \beta_2' \quad \tau_2')'$$

$$\beta_1' = (\beta_{11}, \beta_{12}, \dots, \beta_{1b})$$

$$\beta_2' = (\beta_{21}, \beta_{22}, \dots, \beta_{2b})$$

$$\tau_1' = (\tau_{11}, \tau_{12}, \dots, \tau_{1v})$$

$$\tau_2' = (\tau_{21}, \tau_{22}, \dots, \tau_{2v})$$

$$\Psi' = (\delta_1' \quad \gamma_1' \quad \delta_2' \quad \gamma_2')'$$

$$\delta_1' = (\delta_{11}, \delta_{12}, \dots, \delta_{1s})$$

$$\delta_2' = (\delta_{21}, \delta_{22}, \dots, \delta_{2s})$$

$$\gamma_1' = (\gamma_{111}, \gamma_{112}, , \gamma_{11s}, \gamma_{121}, \gamma_{122}, \dots, \gamma_{12s}, \gamma_{1v1}, \gamma_{1v2}, \dots, \gamma_{1vs})$$

and

$$\gamma_2' = (\gamma_{211}, \gamma_{212}, , \gamma_{21s}, \gamma_{221}, \gamma_{222}, , \gamma_{22s}, \dots \gamma_{2v1}, \gamma_{2v2}, \dots, \gamma_{2vs})$$

and

$$\mathrm{cov}(\mathbb{E}^*) = \begin{bmatrix} \frac{\sigma^{*2}(1+(s-1)\rho^*)}{s} I_n & 0 \\ 0 & \frac{\sigma^{**2}(1+(s-1)\rho^{**})}{s} I_n \end{bmatrix} \quad (11)$$

and

$$cov(\mathbb{E}^{**}) = \begin{bmatrix} \sigma^{*2}(1-\rho^{*})[I_n \otimes I_{s-1}] & 0 \\ 0 & \sigma^{**2}(1-\rho^{**})[I_n \otimes I_{s-1}] \end{bmatrix} \quad (12)$$

It turn out that the procedure for testing the significance and ratios of the block effect and the main effects due to A from the combination of two split plot models are based only on the model (9) and those for testing the significance of the main effect due to B and the interaction between A and B from the combination of two split plot models are based on the model (10). It can be shown that under the assumption of connectedness, the rank of the design matrix in model (9) is 2(b+v-1) [10].

ESTIMATION, RATIO OF LINEAR FUNCTION OF PARAMETERS AND TESTING HYPOTHESIS

To test the hypothesis Ho: $\tau_1 = \tau_2$ against Ha: $\tau_1 \neq \tau_2$ one need only to consider model (9). The model (9) is nonfull rank model and has a constraint on it parameters. To transform the constrained model into unconstrained model, we used Model Reduction Methods given in Hocking (1985). Define a transform matrix T, T is an orthogonal matrix and T=diag(T_1 T_2) where T_i is an orthogonal matrix i=1,2 and $T_i'T_i = I_{b+v-1}$, and $T'T = I_{2(b+v-1)}$.
Model (9) can be written as

$$\mathbb{W} = \mathbb{X}_1\Gamma^* + \mathbb{E}^* \quad (13)$$

Subject to $\mathbb{G}_1^*\Gamma^* = 0$
where

$$\mathbb{X}_1 = I_2 \otimes [E \;\; F \;\; 1_n],$$

$$\Gamma^{*\prime} = (\beta_1' \;\; \tau_1' \;\; \mu_1 \;\; \beta_2' \;\; \tau_2' \;\; \mu_2)'$$

$$\mathbb{G}_1^* = diag[G_1^* \;\; G_2^*]$$

and

$$G_i^* = \begin{bmatrix} 1_b' & 0_v' & 0 \\ 0_b' & 1_v' & 0 \end{bmatrix}, \text{ i=1,2}$$

Now we transform the model

$$\mathbb{W} = \mathbb{X}_1 T'T\Gamma^* + \mathbb{E}^* \quad (14)$$

or

$$\mathbb{W} = \mathbb{X}_1^*\Gamma_1^* + \mathbb{E}^* \quad (15)$$

Subject to $\mathbb{G}_{11}^*\Gamma_1^* = 0$
where

$$\Gamma_1^{*\prime} = [\beta_{11}\; \tau_{11}\; \beta_{(11)}'\; \tau_{(11)}'\; \mu_1\; \beta_{21}\; \tau_{21}\; \beta_{(21)}'\; \tau_{(21)}'\; \mu_2]'$$

and

$$\beta_{(11)}' = (\beta_{12}, \dots, \beta_{1b})$$

$$\beta_{(21)}' = (\beta_{22}, \dots, \beta_{2b})$$

$$\tau_{(11)}' = (\tau_{12}, \dots, \tau_{1v})$$

and

$$\tau_{(21)}' = (\tau_{22}, \dots, \tau_{2v})$$

Next, we find second transformation matrix, T* such that $T^*\Gamma_1^* = \Gamma_{11}^*$, where the vector parameter

$$\Gamma_{11}^* = [\beta_{11}\; \tau_{11}\; \beta_{21}\; \tau_{21}\; \beta_{(11)}'\; \tau_{(11)}'\; \beta_{(21)}'\; \tau_{(21)}'\; \mu_1\; \mu_1]'$$

So model (15) can be written as

$$\mathbb{W} = \mathbb{X}_{11}^*\Gamma_{11}^* + \mathbb{E}^* \quad (16)$$

Subject to $\mathbb{G}_{111}^*\Gamma_{11}^* = 0$

where

$$\mathbb{X}_{11}^* = \mathbb{X}_1^*T^*, \; \mathbb{G}_{111}^* = [I_4 \;\; \Lambda \;\; 0_2' \otimes 1_4]$$

and

$$\Lambda = diag\left[\begin{bmatrix} 1_b' & 0_v' \\ 0_b' & 1_v' \end{bmatrix}, \begin{bmatrix} 1_b' & 0_v' \\ 0_b' & 1_v' \end{bmatrix}\right]$$

Follow the procedure of model reduction method given in Hocking (1985, p.38), first partition $\mathbb{G}_{111}^*$ and $\mathbb{X}_{11}^*$ as

$$\mathbb{G}_{111}^* = [\mathbb{G}_{1111}^* \;\; \mathbb{G}_{1112}^*] \text{ and } \mathbb{X}_{11}^* = [\mathbb{X}_{111}^* \;\; \mathbb{X}_{112}^*]$$

$\mathbb{G}_{1111}^*$ is the first four column of $\mathbb{G}_{1111}^*$ and $\mathbb{G}_{1112}^*$ is the rest of the column of $\mathbb{G}_{111}^*$ and $\mathbb{G}_{1111}^* = I_4$. By applying model reduction method, then we have the unconstraint model

$$\mathbb{W}_r = \mathbb{X}_r\Gamma_r + \mathbb{E}^* \quad (17)$$

where

$$\mathbb{W}_r = \mathbb{W} - \mathbb{X}_{111}^*\mathbb{G}_{1111}^*g$$

where g is the constraint given in (16) and g=0, then

$$\mathbb{W}_r = \mathbb{W}$$

$$\mathbb{X}_r = \mathbb{X}_{112}^* - \mathbb{X}_{111}^*\mathbb{G}_{1111}^{*-1}\mathbb{G}_{1112}^*$$

since $\mathbb{G}_{1111}^* = I_4$, then we have

$$\mathbb{X}_r = \mathbb{X}_{112}^* - \mathbb{X}_{111}^*\mathbb{G}_{1112}^*$$

and

$$\Gamma_r = [\beta_{(11)}'\; \tau_{(11)}'\; \beta_{(21)}'\; \tau_{(21)}'\; \mu_1\; \mu_1]'$$

Matrix $\mathbb{X}_r$ has rank of size 2(b+v-1). Therefore model (17) is full column rank and unconstraint model. Therefore to analyze the model, namely to find the ratio of function of parameters, estimation and testing hypothesis the parameter of the model (17) we can used the standard Gauss Markov model.

First assume that the variance in (11):

$$\sigma_1^{*2} = \frac{\sigma^{*2}(1+(s-1)\rho^*)}{s}, \sigma_2^{*2} = \frac{\sigma^{**2}(1+(s-1)\rho^{**})}{s}$$

are known and equal, say $\sigma_1^{*2} = \sigma_2^{*2} = \sigma_0^{*2}$ then the variance and covariance matrix (11) is equal to

$$\text{cov}(\mathbb{E}^*) = \sigma_0^{*2} I_{2n} \quad (18)$$

and assume that the distribution of $\mathbb{E}^*$ is multivariate normal distribution with mean zero vector and variance covariance matrix satisfied (18). Then the estimation of

$$\hat{\Gamma}_r = (\mathbb{X}_r'\mathbb{X}_r)^{-1}\mathbb{X}_r' W_r \quad (19)$$

and

$$\hat{\sigma}_1^{*2} = \frac{1}{2n-2(b+v-1)} W_r'(I_{2n} - \mathbb{X}_r\mathbb{X}_r^-)W_r \quad (20)$$

where the A^- stand for generalized invers of a matrix A [6, 12]. The estimation given in (19) and (20) has optimal property, Uniformly Minimum Variance Unbiased Estimation (UMVUE) [6].

To construct the confidence limit for the ratio

$$\varphi = \frac{K'\Gamma_r}{L'\Gamma_r}$$

where K and L are 2(b+v-1)×1 vectors of known constant.

Note that

$$T = \frac{K'\hat{\Gamma}_r - \varphi L'\hat{\Gamma}_r}{[\hat{\sigma}_1^{*2}\{K'(\mathbb{X}_r'\mathbb{X}_r)^{-1}K - 2\varphi K'(\mathbb{X}_r'\mathbb{X}_r)^{-1}L + \varphi^2 L'(\mathbb{X}_r'\mathbb{X}_r)L\}^{1/2}]}$$

Has student's t-distribution with 2n-2(b+v-1) degrees of freedom. 100%(1-α) confidence limit for φ can be determined by Fieller's argument [14]:

$$1-\alpha = P[-t \le T \le t] = P[A\varphi^2 + B\varphi + C \le 0]$$

where

$$A = (L'\hat{\Gamma}_r)^2 - t^2 L'(\mathbb{X}_r'\mathbb{X}_r)^{-1}L\hat{\sigma}_1^{*2} \quad (21)$$

$$B = 2[t^2 K'(\mathbb{X}_r'\mathbb{X}_r)^{-1}L\hat{\sigma}_1^{*2} - (K'\hat{\Gamma}_r)(L'\hat{\Gamma}_r)] \quad (22)$$

and

$$C = (K'\hat{\Gamma}_r)^2 - t^2 K'(\mathbb{X}_r'\mathbb{X}_r)^{-1}K'\hat{\sigma}_1^{*2}. \quad (23)$$

Letting a, b and c denote the observed values of the above random variables, (1-α) 100% is our confidence that φ contained by the interval

$$\left[\frac{-b-(b^2-4ac)^{1/2}}{2a}, \frac{-b+(b^2-4ac)^{1/2}}{2a}\right] \quad (24)$$

provided that a>0 and b^2-4ac > 0.

To test the hypothesis $\tau_1 = \tau_2$ against Ha: $\tau_1 \neq \tau_2$, which is equal to Ho: $\tau_{(11)} = \tau_{(21)}$ against Ha: $\tau_{(11)} \neq \tau_{(21)}$ which can be written in matrix form as follow:

$$\text{Ho}: \Delta_1\Gamma_r = 0_{(2b+2v-4)x1} \quad (25)$$

where

$$\Delta_1 = [0_{(v-1)\times(b-1)}\ I_{v-1}\ 0_{(v-1)\times(b-1)}\ I_{v-1}\ 0_{(v-1)x2}]$$

and Δ_1 is full row rank and rank(Δ_1) = v-1.

The likelihood ratio test for testing the hypotheses (25) [6] is given by

$$\lambda_1 = \frac{(\Delta_1\hat{\Gamma}_r)'[\Delta_1(\mathbb{X}_r'\mathbb{X}_r)^{-1}\Delta_1']^{-1}\Delta_1\hat{\Gamma}_r}{W_r'(I-\mathbb{X}_r\mathbb{X}_r^-)W_r} \times \frac{2n-2(b+v-1)}{v-1} \quad (26)$$

Under null hypotheses, λ_1 has F-distribution with degrees of freedom v-1 and 2n-2(b+v-1). Reject Ho if $\lambda_1 > F_{\alpha,2n-2(b+v-1)x(v-1)}$.

If the variance covariance matrix are known but unequal, namely $\sigma_1^{*2} \neq \sigma_2^{*2}$, then we have

$$\text{cov}(\mathbb{E}^*) = \Sigma \otimes I_n \quad (27)$$

where

$$\Sigma = \begin{pmatrix} \sigma_1^{*2} & 0 \\ 0 & \sigma_2^{*2} \end{pmatrix}$$

To deal with this kind of problem, we can use generalized linear model [1, 6, 13]. The estimation and testing hypothesis are given below:

$$\hat{\Gamma}_r = (\mathbb{X}_r'(\Sigma^{-1}\otimes I_n)\mathbb{X}_r)^{-1}\mathbb{X}_r'(\Sigma^{-1}\otimes I_n)\mathbb{W} \quad (28)$$

with the covariance matrix

$$var(\Gamma_r) = [\mathbb{X}_r'(\Sigma^{-1}\otimes I_n)\mathbb{X}_r]^{-1}$$

Under this assumption, to test the hypothesis given in (25), the likelihood ratio test is

$$\lambda_2 = \frac{(\Delta_1\hat{\Gamma}_r)'[\Delta_1(\mathbb{X}_r'(\Sigma^{-1}\otimes I_n)\mathbb{X}_r)^{-1}\Delta_1']^{-1}\Delta_1\hat{\Gamma}_r}{(W_r-\mathbb{X}_r\hat{\Gamma}_r)'(I-\mathbb{X}_r\mathbb{X}_r^-)(W_r-\mathbb{X}_r\hat{\Gamma}_r)} \times \frac{2n-2(b+v-1)}{v-1} \quad (29)$$

Under null hypotheses, λ_2 has F-distribution with degrees of freedom v-1 and 2n-2(b+v-1). Reject Ho if $\lambda_2 > F_{\alpha,(v-1)x2n-2(b+v-1)}$. Where $F_{\alpha,(v-1)x2n-2(b+v-1)}$ is upper α point of F-distribution with (v-1) and 2n-2(b+v-1) degrees of freedom.

To find the ratio of linear function of parameter Ψ one need only to consider model (10). The model(10) is nonfull rank model and has a constraint on it parameters. To transform the constrained model into unconstrained model, we used Model Reduction Methods given in Hocking [7]. Define a transform matrix T^*, T^* is an orthogonal matrix and $T^{*\prime}T^* = I_{2s(v+1))}$. Such that

$$\Psi^* = T^*\Psi$$

where

$$\Psi^* = (\delta_{11}\ \gamma_{111}\ \gamma_{121} \cdots \gamma_{1v-1,1}\ \gamma'_{1v}\ \delta_{21}\ \gamma_{211}\ \gamma_{221} \cdots \gamma_{2v-1,1}\ \gamma'_{2v}\ \delta'_{(11)}\ \gamma'_{(111)}\ \gamma'_{(121)} \cdots \gamma'_{(1v-1,1)}\ \delta'_{(21)}\ \gamma'_{(211)}\ \gamma'_{(221)} \cdots \gamma'_{(2v-1,1)})'$$

Now model (10) can be written as

$$\mathbb{Z} = \mathbb{X}_2\Psi + \mathbb{E}^{**} \qquad (30)$$

Subject to $\mathbb{G}_2^*\Psi = 0$
where $\mathbb{X}_2 = I_2 \otimes [1_n \otimes P_1 \quad F \otimes P_1]$ and $\mathbb{G}_2^*$ is a matrix such that the constraint (30) are satisfied.
Now the model (30) can be written as

$$\mathbb{Z} = \mathbb{X}_2\, T^{*\prime} T^* \Psi + \mathbb{E}^{**}$$

or

$$\mathbb{Z} = \mathbb{X}_2^*\, \Psi^* + \mathbb{E}^{**}$$

Subject to $\mathbb{G}_{22}^*\Psi^* = 0$
where

$$\mathbb{X}_2^* = \mathbb{X}_2 T^{*\prime}$$

and

$$\mathbb{G}_{22}^* = [\mathrm{diag}(\mathbb{G}_{221}^*\ \ \mathbb{G}_{221}^*), \mathrm{diag}(\mathbb{G}_{222}^*\ \ \mathbb{G}_{222}^*)]$$

$$\mathbb{G}_{221}^* = \begin{bmatrix} I_v & 0_{vxs} \\ 0'_v & 1'_s \\ 0_{(s-1)x(v+1)} & I_{s-1} \end{bmatrix}$$

$$\mathbb{G}_{222}^* = \begin{bmatrix} I_v \otimes 1'_{s-1} \\ 0'_{v(s-1)} \\ 0_{(s-1)x(s-1)} \quad 1'_{v-1} \otimes I_{s-1} \end{bmatrix}$$

Follow the procedure of model reduction method given in Hocking [7], first partition $\mathbb{G}_{22}^*$ and $\mathbb{X}_2^*$ as

$$\mathbb{G}_{22}^* = [\mathbb{G}_{2211}^*\ \ \mathbb{G}_{2222}^*] \text{ and } \mathbb{X}_2^* = [\mathbb{X}_{21}^*\ \ \mathbb{X}_{22}^*]$$

$\mathbb{X}_{21}^*$ is the first 2(v+s) columns of $\mathbb{X}_2^*$ and $\mathbb{X}_{22}^*$ is the rest of the columns of $\mathbb{X}_2^*$. $\mathbb{G}_{2211}^*$ is the first 2(v+s) columns of $\mathbb{G}_{22}^*$ and $\mathbb{G}_{2222}^*$ is the rest of the columns of $\mathbb{G}_{22}^*$ and $\mathbb{G}_{2211}^* = \mathrm{diag}(\mathbb{G}_{221}^*\ \ \mathbb{G}_{221}^*)$. By applying model reduction method, then we have the unconstraint model

$$\mathbb{Z}_r = \mathbb{X}_r^*\Psi_r^* + \mathbb{E}^{**} \qquad (31)$$

where

$$\mathbb{Z}_r = \mathbb{Z} - \mathbb{X}_{21}^*\mathbb{G}_{2211}^* g$$

where g is the constraint given in (30) and g=0, then

$$\mathbb{Z}_r = \mathbb{Z}$$

$$\mathbb{X}_r^* = \mathbb{X}_{22}^* - \mathbb{X}_{21}^*\mathbb{G}_{2211}^{*-1}\mathbb{G}_{2222}^*$$

and

$$\Psi_r^* = (\delta'_{(11)}\ \gamma'_{(111)}\ \gamma'_{(121)} \cdots \gamma'_{(1v-1,1)}\ \delta'_{(21)}\ \gamma'_{(211)}\ \gamma'_{(221)} \cdots \gamma'_{(2v-1,1)})'$$

where

$$\delta'_{(11)} = (\delta_{12}, \delta_{13}, \ldots, \delta_{1s}),$$

$$\delta'_{(21)} = (\delta_{22}, \delta_{23}, \ldots, \delta_{2s})$$

$$\gamma'_{(111)} = (\gamma_{112}, \gamma_{113}, \ldots, \gamma_{11s})$$

$$\gamma'_{(121)} = (\gamma_{122}, \gamma_{123}, \ldots, \gamma_{12s})$$

$$\gamma'_{(1v-1,1)} = (\gamma_{1v-1,2}, \gamma_{1v-1,3}, \ldots, \gamma_{1v-1,s}),$$

$$\gamma'_{(211)} = (\gamma_{212}, \gamma_{213}, \ldots, \gamma_{21s})$$

$$\gamma'_{(221)} = (\gamma_{222}, \gamma_{223}, \ldots, \gamma_{22s})$$

$$\gamma'_{(2v-1,1)} = (\gamma_{2v-1,2}, \gamma_{2v-1,3}, \ldots, \gamma_{2v-1,s}).$$

After some calculation, it can be shown that (the argument can be seen in Mustofa [10]

$$\mathbb{X}_r^* = I_2 \otimes (FD \otimes P_1 B)$$

where
F is given in (2), P_1 is given in (8),

$$D = \begin{bmatrix} 1_{v-1} & I_{v-1} \\ 1 & -1'_{v-1} \end{bmatrix} \text{ and } B = \begin{bmatrix} -1'_{s-1} \\ I_{s-1} \end{bmatrix}.$$

Also model (31) is full rank model since $\mathbb{X}_r^*$ is full column rank. To show that $\mathbb{X}_r^*$ is full column rank, it is sufficient to show that $\mathbb{X}_r^{*\prime}\ \mathbb{X}_r^*$ is nonsingular. Using the fact that $P_1'P_1 B = B$, one obtains

$$\mathbb{X}_r^{*\prime}\mathbb{X}_r^* = (I_2 \otimes (FD \otimes P_1 B))'(I_2 \otimes (FD \otimes P_1 B)) = I_2 \otimes (D'RD \otimes B'B)$$

Since I_2, D, R = F′F and $B'B = (I_{s-1} + J_{s-1})$ are nonsingular matrix, hence $\mathbb{X}_r^{*\prime}\mathbb{X}_r^*$ is nonsingular.

First assume that the variance and covariance matrix in (12) and let $\sigma_{11}^{*2} = \sigma^{*2}(1-\rho^*)$ and $\sigma_{22}^{*2} = \sigma^{**2}(1-\rho^{**})$ are known and equal, say $\sigma_{11}^{*2} = \sigma_{22}^{*2} = \sigma_{00}^{*2}$ then the variance and covariance matrix (12) is equal to

$$\mathrm{cov}(\mathbb{E}^{**}) = \sigma_{00}^{*2}\,(I_2 \otimes I_n \otimes I_{s-1}) \tag{32}$$

and assume that the distribution of $\mathbb{E}^{**}$ is multivariate normal distribution with mean zero vector and variance covariance matrix satisfied (31). Then the estimation of Ψ_r^* is

$$\hat{\Psi}_r^* = (\mathbb{X}_r^{*\prime}\mathbb{X}_r^*)^{-1}\mathbb{X}_r^{*\prime}\mathbb{Z}_r \tag{33}$$

and

$$\hat{\sigma}_{00}^{*2} = \frac{1}{2n(s-1)-2v(s-1)}\,\mathbb{Z}_r'(I_{2n(s-1)} - \mathbb{X}_r^*\mathbb{X}_r^{*-})\mathbb{Z}_r \tag{34}$$

The estimation given in (33) and (34) has optimal property, in the sense of Gauss Markov model, Uniformly Minimum Variance Unbiased Estimation (UMVUE) [6].

To construct the confidence limit for the ratio of linear function of parameters Ψ_r^* we follow the same argument as given above, let

$$\phi = \frac{M'\Psi_r^*}{N'\Psi_r^*}$$

where M and N are 2v(s-1)×1 vectors of known constant.
Now that

$$T^* = \frac{M'\hat{\Psi}_r^* - \phi\, N'\hat{\Psi}_r^*}{[\hat{\sigma}_{00}^{*2}\{M'(\mathbb{X}_r^{*\prime}\mathbb{X}_r^*)^{-1}M - 2\phi\, M'(\mathbb{X}_r^{*\prime}\mathbb{X}_r^*)^{-1}N + \phi^2 N'(\mathbb{X}_r^{*\prime}\mathbb{X}_r^*)N\}^{1/2}]}$$

has student's t-distribution with 2n(s-1)-2v(s-1) degrees of freedom. 100%(1-α) confidence limit for ϕ can be determined by Fieller's argument (Zerbe, 1978):

$$1-\alpha = P[-t \le T^* \le t] = P[A^*\phi^2 + B^*\phi + C^* \le 0]$$

where

$$A^* = (N'\hat{\Psi}_r^*)^2 - t^2 N'(\mathbb{X}_r^{*\prime}\mathbb{X}_r^*)^{-1}N\hat{\sigma}_{00}^{*2} \tag{35}$$

$$B^* = 2[\,t^2 M'(\mathbb{X}_r^{*\prime}\mathbb{X}_r^*)^{-1}N\hat{\sigma}_{00}^{*2} - (M'\hat{\Psi}_r^*)(N'\hat{\Psi}_r^*)] \tag{36}$$

and

$$C^* = (M'\hat{\Psi}_r^*)^2 - t^2 M'(\mathbb{X}_r'\mathbb{X}_r^*)^{-1}M'\hat{\sigma}_{00}^{*2} \tag{37}$$

Letting a*, b* and c* denote the observed values of the above random variables, (1-α)100% is our confidence that ϕ contained by the interval

$$\left[\frac{-b^* - (b^{*2} - 4a^*c^*)^{1/2}}{2a^*},\ \frac{-b^* + (b^{*2} - 4a^*c^*)^{1/2}}{2a^*}\right] \tag{38}$$

provided that a*>0 and b^{*2}-4a*c* > 0.

$$\text{Ho: } \Delta_3\Psi_r^* = 0_{(s-1)x1} \tag{39}$$

where

$$\Delta_3 = [I_{s-1}\ \ 0_{(s-1)\times(s-1)(v-1)}\ \ I_{s-1}\ 0_{(s-1)\times(s-1)(v-1)}]$$

and Δ3 is full row rank and rank((3)= s-1.

The likelihood ratio test for testing the hypotheses (39) [6] is given by

$$\lambda_3 = \frac{(\Delta_3\hat{\Psi}_r^*)'\left[\Delta_3(\mathbb{X}_r^{*\prime}\mathbb{X}_r^*)^{-1}\Delta_3'\right]^{-1}\Delta_3\hat{\Psi}_r^*}{\mathbb{Z}_r'(I - \mathbb{X}_r^*\mathbb{X}_r^{*-})\mathbb{Z}_r} \times \frac{2n(s-1)-2v(s-1)}{s-1} \tag{40}$$

Under null hypotheses, λ_3 has F-distribution with degrees of freedom s-1 and 2n(s-1)-2v(s-1). Reject Ho if $\lambda_3 > F_{\alpha, s-1,\ 2n(s-1)-2v(s-1)}$.

To test the hypothesis Ho: $\gamma_1 = \gamma_2$ against Ha: $\gamma_1 \neq \gamma_2$ is equivalent to test the hypothesis

$$\text{Ho:} (\gamma_{(111)}'\ \gamma_{(121)}' \cdots \gamma_{(1v-1,1)}')' = (\gamma_{(211)}'\ \gamma_{(221)}' \cdots \gamma_{(2v-1,1)}')'$$

against

$$\text{Ha: } (\gamma_{(111)}'\ \gamma_{(121)}' \cdots \gamma_{(1v-1,1)}')' \neq (\gamma_{(211)}'\ \gamma_{(221)}' \cdots \gamma_{(2v-1,1)}')'$$

which can be written in matrix form as follow:

$$\text{Ho: } \Delta_4\Psi_r^* = 0_{(v-1)(s-1)x1} \tag{41}$$

where

$$\Delta_4 = [0_{(v-1)(s-1)\times(s-1)}\ I_{v-1}\otimes I_{s-1}\ \ 0_{(v-1)(s-1)\times(s-1)}\ \ -I_{v-1}\otimes I_{s-1}]$$

and Δ_4 is full row rank and rank (Δ_4) = (v-1)(s-1).

The likelihood ratio test for testing the hypotheses (40) [6] is given by

$$\lambda_4 = \frac{(\Delta_4\hat{\Psi}_r^*)'\left[\Delta_4(\mathbb{X}_r^{*\prime}\mathbb{X}_r^*)^{-1}\Delta_4'\right]^{-1}\Delta_4\hat{\Psi}_r^*}{\mathbb{Z}_r'(I - \mathbb{X}_r^*\mathbb{X}_r^{*-})\mathbb{Z}_r} \times \frac{2n(s-1)-2v(s-1)}{(v-1)(s-1)} \tag{42}$$

Under null hypotheses, λ_4 has F-distribution with degrees of freedom (v-1)(s-1) and 2n(s-1)-2v(s-1). Reject Ho if $\lambda_4 > F_{\alpha,(v-1)(s-1), 2n(s-1)-2v(s-1)}$.

If the variance covariance matrix are known but unequal, namely $\sigma_{11}^{*2} \neq \sigma_{21}^{*2}$, then we have

$$\mathrm{cov}(\mathbb{E}^{**}) = \Sigma^* \otimes I_{n(s-1)} \quad (43)$$

where

$$\Sigma^* = \begin{pmatrix} \sigma_{11}^{*2} & 0 \\ 0 & \sigma_{21}^{*2} \end{pmatrix}$$

To deal with this kind of problem, we can use generalized linear model [1, 6, 13]. The estimation and testing hypothesis are given below:

$$\hat{\Psi}_r^* = \left(\mathbb{X}_r^{*\prime}\left(\Sigma^{*-1} \otimes I_{n(s-1)}\right)\mathbb{X}_r^*\right)^{-1} \mathbb{X}_r^{*\prime}\left(\Sigma^{*-1} \otimes I_{n(s-1)}\right)\mathbb{Z}_r \quad (44)$$

with the covariance matrix

$$\mathrm{var}(\Psi_r^*) = \left[\mathbb{X}_r^{*\prime}\left(\Sigma^{*-1} \otimes I_{n(s-1)}\right)\mathbb{X}_r^*\right]^{-1}$$

Under this assumption, to test the hypothesis given in (39), the likelihood ratio test is

$$\lambda_5 = \frac{(\Delta_3 \hat{\Psi}_r^*)'\left[\Delta_3\left(\mathbb{X}_r^{*\prime}\left(\Sigma^{*-1} \otimes I_{n(s-1)}\right)\mathbb{X}_r^*\right)^{-1}\Delta_3'\right]^{-1}\Delta_3 \hat{\Psi}_r^*}{(\mathbb{Z}_r - \mathbb{X}_r^* \hat{\Psi}_r^*)'(I - \mathbb{X}_r^* \mathbb{X}_r^{*-})(\mathbb{Z}_r - \mathbb{X}_r^* \hat{\Psi}_r^*)} \times \frac{2n(s-1) - 2v(s-1)}{s-1} \quad (45)$$

Under null hypotheses, λ_5 has F-distribution with degrees of freedom s-1 and 2n(s-1)-2v(s-1). Reject Ho if $\lambda_5 > F_{\alpha,(s-1),2n(s-1)-2v(s-1)}$.

Where $F_{\alpha,(s-1),2n(s-1)-2v(s-1)}$ is upper α point of F-distribution with (s-1) and 2n(s-1)-2v(s-1) degrees of freedom.

Under the assumption (43), to test the hypothesis given in (41), the likelihood ratio test is

$$\lambda_6 = \frac{(\Delta_4 \hat{\Psi}_r^*)'\left[\Delta_4\left(\mathbb{X}_r^{*\prime}\left(\Sigma^{*-1} \otimes I_{n(s-1)}\right)\mathbb{X}_r^*\right)^{-1}\Delta_4'\right]^{-1}\Delta_4 \hat{\Psi}_r^*}{(\mathbb{Z}_r - \mathbb{X}_r^* \hat{\Psi}_r^*)'(I - \mathbb{X}_r^* \mathbb{X}_r^{*-})(\mathbb{Z}_r - \mathbb{X}_r^* \hat{\Psi}_r^*)} \times \frac{2n(s-1) - 2v(s-1)}{(s-1)(v-1)} \quad (46)$$

Under null hypotheses, λ_6 has F-distribution with degrees of freedom (s-1)(v-1) and 2n(s-1)-2v(s-1). Reject Ho if $\lambda_5 > F_{\alpha,(s-1)(v-1),2n(s-1)-2v(s-1)}$. Where $F_{\alpha,(s-1)(v-1),2n(s-1)-2v(s-1)}$ is upper α point of F-distribution with (s-1)(v-1) and 2n(s-1)-2v(s-1) degrees of freedom.

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