Reprinted from the

Far East Journal of Mathematical Sciences (FJMS)
Volume 39, Number 1, 2010, pp 129-135

IRREGULAR TOTAL LABELING ON COMPLETE BIPARTITE GRAPH AND UNION OF COMPLETE BIPARTITE GRAPHS

by

Jamal I. Daoud, Wamiliana,
Mustofa Usman and Asmiati

Pushpa Publishing House
Vijaya Niwas, 198 Mumfordganj
Allahabad 211002, INDIA
http://pphmj.com/journals/fjms.htm
fjms@pphmj.com & arun@pphmj.com
Information for Authors

Aims and Scope: The *Far East Journal of Mathematical Sciences* (FJMS) is devoted to publishing original research papers and critical survey articles in the field of Pure and Applied Mathematics, Computer Applications and Applied Statistics. The FJMS is a fortnightly journal published in twelve volumes annually and each volume comprises of two issues.

Indexing and Reviews: Scopus; Mathematical Reviews; MathSciNet and Zentralblatt für Mathematik databases.

Submission of Manuscripts: Authors may submit their papers for consideration in the *Far East Journal of Mathematical Sciences* (FJMS) by the following modes:
1. Online submission (only .pdf and .doc files): Please visit journal’s homepage at http://www.pphmj.com/journals/fjms.htm
2. Electronically (only .tex, .dvi, .pdf, .ps and .doc files): At the e-mail address: fjms@pphmj.com or kkazad@pphmj.com
3. Hard copies: Papers in duplicate with a letter of submission at the address of the publisher.

The paper must be typed only on one side in double spacing with a generous margin all round. An effort is made to publish a paper duly recommended by a referee within a period of three months. One set of galley proofs of a paper will be sent to the author submitting the paper, unless requested otherwise, without the original manuscript, for corrections.

Abstract and References: Authors are requested to provide an abstract of not more than 250 words and latest Mathematics Subject Classification. Statements of Lemmas, Propositions and Theorems should be set in italics and references should be arranged in alphabetical order by the surname of the first author.

Page Charges and Reprints: Authors are requested to arrange page charges of their papers @ USD 40.00 per page for USA and Canada, and Euro 30.00 per page for rest of the world from their institutions/research grants, if any. However, for authors in India this charge is Rs. 500.00 per page.

No extra charges for colour figures. Twenty-five reprints are provided to the corresponding author along with a copy of the issue in which the author’s paper is appeared. Additional sets of reprints may be ordered at the time of proof correction.

Copyright: It is assumed that the submitted manuscript has not been published and will not be simultaneously submitted or published elsewhere. By submitting a manuscript, the authors agree that the copyright for their articles is transferred to the Pushpa Publishing House, Allahabad, India, if and when, the paper is accepted for publication. The publisher cannot take the responsibility of any loss of manuscript. Therefore, authors are requested to maintain a copy at their end.

Subscription Information for 2010

<table>
<thead>
<tr>
<th>Service</th>
<th>Electronic Subscription</th>
<th>Print Subscription includes Online Access</th>
</tr>
</thead>
<tbody>
<tr>
<td>Institutional Price for all countries except India</td>
<td>€ 690.00</td>
<td>US$ 990.00</td>
</tr>
<tr>
<td>Print Subscription Only</td>
<td>€ 845.00</td>
<td>US$ 1245.00</td>
</tr>
</tbody>
</table>

For Institutions: On seeking a license for volume(s) of the *Far East Journal of Mathematical Sciences* (FJMS), the facility to download and print the articles will be available through the institutional 9 digits IP address to be provided by the appropriate authority. The facility to continue till the end of the next calendar year from the last issue of the volume subscribed. For having the continued facility to keep the download of the same subscribed volume for another two calendar years may be had on a considerable discounted rate.

Price in Indian Rs. (For Indian Institutions in India only)  
Print Subscription Only Rs. 12000.00

The subscription year runs from January 1, 2010 through December 31, 2010.

Information: The journals published by the "Pushpa Publishing House" are solely distributed by the "Vijaya Books and Journals Distributors".

Mode of Payment: Payment may be made by a cheque or a bank draft in Euro or USD payable to "Vijaya Books and Journals Distributors, A/c No. 035002000061882 with the Indian Overseas Bank, Allahabad".

Contact Person: Subscription Manager, Vijaya Books and Journals Distributors, Vijaya Niwas, 198 Mumfordganj, Allahabad 211002, India; sub@pphmj.com; arun@pphmj.com
IRREGULAR TOTAL LABELING
ON COMPLETE BIPARTITE GRAPH AND UNION OF
COMPLETE BIPARTITE GRAPHS

JAMAL I. DAoud\textsuperscript{1}, WAMILIANA\textsuperscript{2}, MUSTOF\textsuperscript{3}, USMAN\textsuperscript{2} and ASMIATI\textsuperscript{2}

\textsuperscript{1}Department of Science in Engineering
International Islamic University Malaysia
Kuala Lumpur, Malaysia
e-mail: jamalidus@yahoo.com

\textsuperscript{2}Department of Mathematics
Faculty of Mathematics and Natural Sciences
Lampung University, Indonesia

Abstract

Given a graph $G = (V, E)$ with vertex set $V$ and edge set $E$, we define a
labeling as a function, where weight of edge $xy$ is written as $w(x, y)$. Total labeling is the sum of $xy$-label and labels of the vertices that incident to $x$, thus $w(x, y) = \lambda(x) + \lambda(y)\lambda(xy)$. $\lambda$ labeling is called as edge irregular total $k$-labeling of the graph $G$ if for every two different edges $e$ and $f$ of $G$, $w(e) \neq w(f)$. The smallest $k$ in which graph $G$ can be labeled as edge irregular total $k$-labeling is called as edge total irregularity strength and is noted as $\text{tes}(G)$. In this research, we are interested in finding $\text{tes}(G)$ of the union of complete bipartite graphs $K_{1, q}$ and $K_{2, q}$.

2010 Mathematics Subject Classification: 05C78.

Keywords and phrases: irregular total labeling, $\text{tes}(G)$, bipartite graph.

*Corresponding author

Received December 17, 2009
1. Introduction

Graph is one of the mathematical branches that can be used to represent many problems in daily life. After proposed by Euler in 1736, graph is one of the important tools that used with other fields of knowledge such as in chemistry, biology, and mostly in mathematics, especially operations research. Some examples of graphs applications include: to design telecommunication networks, to design energy networks, to represent the molecules structures, to represent the DNA structures, and so on.

One of the topics in graphs is labeling. The labeling of graph was investigated first by Sallacek in 1963. Some of the labeling techniques that had been investigated such as: Graceful labeling (Ringel and Llado [5]), Harmony labeling (Graham and Sloane [3]), Magic labeling (Kotzig and Rosa [4], Stewart [7]), and anti magic labeling (Baca et al. [2]).

In 2001, Baca et al. [2] (Asmiati [1]), introduced a new type of labeling, named as edge irregular total labeling which consists of edge labeling and vertex labeling.

For a graph $G = (V, E)$ with vertex set $V$ and edge set $E$, we define a labeling as a function. The weight of edge $xy$ is written as $W(x, y)$. Total labeling is the sum of $xy$ label and labeled of the vertices that incident to $xy$. Thus $w(x, y) = \lambda(x) + \lambda(y)\lambda(xy)$.

$\lambda$ labeling is called as edge irregular total $k$-labeling of the graph $G$ if for every two different edges $e$ and $f$ of $G$, $w(e) \neq w(f)$. The smallest $k$ in which Graph $G$ can be labeled as edge irregular total $k$-labeling is called as edge total irregularity strength and is noted as $\text{tes}(G)$.

One interesting fact about the labeling is that every graph $G$ can be labeled by edge irregular total $k$-labeling. But, to find the $\text{tes}(G)$ is not that easy as finding the labeling. Therefore in this research we try to find the $\text{tes}(G)$ of some special graphs which are: the union of complete bipartite graphs $K_{1,q}$ and $K_{2,q}$.

2. Some Previous Results in Literature

**Theorem** (Baca et al. [2], Asmiati [1]). If $G = (V, E)$ is a graph, then

$$\left\lfloor \frac{|E(G) + 2|}{3} \right\rfloor \leq \text{tes}(G) \leq |E(G)|.$$
Proof. Since $w(e) = \lambda(a) + \lambda(b) + \lambda(e)$, where $a = x, b = y, e = xy$, and $w(e) \geq 3$, $w(e) \geq |E(G)| + 2$.

Notice that $\text{tes}(G) \geq \max \{\lambda(a), \lambda(b), \lambda(e)\} \geq \frac{w(e)}{3} \geq \frac{|E(G)| + 2}{3}$. Therefore $\left\lfloor \frac{|E(G)| + 2}{3} \right\rfloor \leq \text{tes}(G)$.

To find the upper bound, choose $\lambda(x) = 1, \forall x \in V(G)$ and give every edge label start from 1 until $|E(G)|$. This labeling procedure makes every edge has a different weight and therefore $\text{tes}(G) \leq |E(G)|$. Theorem 1 of Baca et al. gives the upper bound and lower bound of $\text{tes}(G)$ for every graph, and also give procedure on labeling construction.

**Theorem** (Baca et al. [2], Asmiati [1]). If $G$ is a complete bipartite graph $K_{1,q}$, then $\text{tes}(K_{1,q}) = \left\lfloor \frac{q + 1}{2} \right\rfloor$.

**Proof.** Complete bipartite graph $K_{1,q}$ has one centre vertex which is vertex with degree $q$, where $|V(k_{1,q})| = q + 1$ and $|E(k_{1,q})| = q$.

Suppose that $V(k, q) = \{x_1^1, y_1^1, y_1^2, ..., y_1^q\}$ and $E(k, q) = \{x_1^j y_1^j, x_1^j y_1^i, ..., x_1^1 y_1^q\}$. Give label vertices and edges with the following procedure:

$$
\lambda(x_1^1) = 1, \quad \lambda(y_1^j) = \left\lfloor \frac{j + 1}{2} \right\rfloor, \quad \lambda(x_1^j y_1^j) = \left\lfloor \frac{j + 1}{2} \right\rfloor, \quad j = 1, 2, 3, ..., q.
$$

Then get $w(x_1^1 y_1^j) = 2 + j$ and it can be easily seen that the weight of the edges starts from 3 to $q + 2$. Therefore $\text{tes}(K_{1,q}) = \left\lfloor \frac{q + 1}{2} \right\rfloor$.

**Theorem** (Asmiati [1]). If $G$ is a complete bipartite graph $K_{2,q}$, then $\text{tes}(K_{2,q}) = \left\lfloor \frac{2q + 2}{3} \right\rfloor$. 
3. $\text{Tes}(G)$ for Complete Bipartite Graph, and Union of Complete Bipartite Graphs

**Result 1.** If $G$ is the union of complete bipartite graph $m(K_{1,q})$, $m \in Z^+$, then

$$\text{tes}(G) = \left\lceil \frac{mq + 2}{3} \right\rceil; \quad \forall m \geq 2, m \in Z^+.$$ 

**Proof.** The union of complete bipartite graph $m(K_{1,q})$ is a disconnected graph which consists of $m$ components, where every component is a complete bipartite graph $K_{1,q}$. The $m(K_{1,q})$ graph has the sets of points:

$$V = X_1 \cup X_2 \cup \ldots \cup X_m \cup Y_1 \cup Y_2 \cup \ldots \cup Y_m,$$

$$X_i = x_i^1, \quad Y_i = \{y_i^1, y_i^2, \ldots, y_i^q\}$$

and the set of edges $E = E_1 \cup E_2 \cup \ldots \cup E_m$,

$$E_i = \{x_i^1y_i^1, x_i^1y_i^2, \ldots, x_i^1y_i^q\}, \quad i = 1, 2, 3, \ldots, m.$$ 

Given label vertices and edges of the union of complete bipartite graph $m(K_{1,q})$ using the following procedure:

![Figure 1. Labeling construction for $m(K_{1,q})$.](image)

$$\lambda(x_i^1) = \left\lceil \frac{mq + 2}{3} \right\rceil; \quad m \geq 1; \quad i = 1, 2, 3, \ldots, m,$$

$$\lambda(y_i^j) = \left\lceil \frac{mq + 2}{2} \right\rceil + \left\lceil \frac{j}{2} \right\rceil - \left\lceil \frac{q}{2} \right\rceil; \quad j = 1, 2, 3, \ldots, q; \quad m \geq 1; \quad i = 1, 2, 3, \ldots, m,$$
IRREGULAR TOTAL LABELING ON COMPLETE BIPARTITE \ldots

\[
\lambda(x_i y_j) = \left\lfloor \frac{mq + 2}{3} \right\rfloor + \left\lfloor \frac{j + 1}{2} \right\rfloor - \left\lfloor \frac{q + 1}{2} \right\rfloor, \quad j = 1, 2, \ldots, q; \quad m > 1; \quad i = 1, 2, 3, \ldots, m,
\]

\(m \text{ or } q \text{ are multiplication of } 3,\)

\[
\left\lfloor \frac{mq + 2}{3} \right\rfloor + \left\lfloor \frac{j + 1}{2} \right\rfloor - \left\lfloor \frac{q}{2} \right\rfloor, \quad j = 1, 2, \ldots, q; \quad m > 1; \quad i = 1, 2, 3, \ldots, m,
\]

\(m \text{ or } q \text{ are not multiplication of } 3,\)

Thus, we get \(w(x_i y_j) = q(m - 1) + 2 + j,\) and it can be easily seen that the weight of the edges starts from 3 to \(mq + 2.\) That is clear that \(\lambda\) is the irregular total labeling of \(m(K_{1, q})\) with \(\text{tes}(K_{1, q}) = \left\lfloor \frac{mq + 2}{3} \right\rfloor.\)

Below we give an example of labeling for the union of complete bipartite graph \(2(K_{1, q}).\)

![Figure 2. Example for labeling construction for \(2(K_{1, q}).\)](image)

**Result 2.** If \(G\) is the union of complete bipartite graph \(m(K_{2, q}), m \in \mathbb{Z}^+,\) then

\[\text{tes}(G) = \left\lfloor \frac{2mq + 2}{3} \right\rfloor, \quad \forall m \geq 2, m \in \mathbb{Z}^+.\]

**Proof.** The union of complete bipartite graph \(m(K_{2, q})\) is a disconnected graph which consists of \(m\) components, where every component is a complete bipartite graph \(K_{2, q}.\) The \(m(K_{2, q})\) graph has the sets of points:

\[
V = X_1 \cup X_2 \cup \cdots \cup X_m \cup Y_1 \cup Y_2 \cup \cdots \cup Y_m,
\]

\[
X_i = \{x_i^1, x_i^2\}, \quad Y_i = \{y_i^1, y_i^2, \ldots, y_i^q\}
\]
and the set of edges $E = E_1 \cup E_2 \cup \cdots \cup E_m$,

$$E_i = \{x_i^1y_i^1, x_i^1y_i^2, \ldots, x_i^1y_i^q, x_i^2y_i^1, x_i^2y_i^2, \ldots, x_i^2y_i^q\}, \quad i = 1, 2, 3, \ldots, m.$$  

Give label every edge of $K_{2,q}$ as done by Asmiati [1]. Next, the vertices and the edges of the union of complete bipartite graph $m(K_{2,q})$ can be labeled by the following procedure:

Figure 3. Labeling construction for $m(K_{2,q})$.

$$\lambda(x_i^1) = \left\lfloor \frac{2mq + 2}{3} \right\rfloor - 1, \quad m > 1; \quad i = 2, 3, \ldots, m,$$

$$\lambda(x_i^2) = \left\lfloor \frac{2mq + 2}{3} \right\rfloor,$$

$$\lambda(y_i^1) = \begin{cases} 
\left\lfloor \frac{2mq + 2}{3} \right\rfloor + j - q, & 1 \leq j \leq q; \quad m > 1; \quad i = 2, 3, \ldots, m, \\
m or q are multiplication of 3,
\end{cases}$$

$$\lambda(y_i^q) = \begin{cases} 
\left\lfloor \frac{2mq + 2}{3} \right\rfloor + j - q, & 1 \leq j \leq q; \quad m > 1; \quad i = 2, 3, \ldots, m, \\
m or q are not multiplication of 3,
\end{cases}$$
IRREGULAR TOTAL LABELING ON COMPLETE BIPARTITE ...

\[ \lambda(x_i^j y_i^j) = \left[ \frac{2mq + 2}{3} \right] - q + 1; \quad m > 1; \quad i = 2, 3, ..., m. \]

\[ \lambda(x_i^2 y_i^j) = \left[ \frac{2mq + 2}{3} \right]; \quad m > 1; \quad i = 1, 2, 3, ..., m. \]

Thus we get \( \omega(x_i^j y_i^j) = 2q(m - 1) + 2 + j \) and \( \omega(x_i^2 y_i^j) = q(2m - 1) + 2 + j \).

It can be easily seen that the weight of the edges starts from 3 to \( 2mq + 2 \), thus that is clear that \( \lambda \) is the irregular total labeling of \( m(K_{2,q}) \) with \( \text{tes} m(K_{2,q}) \) is \( \left[ \frac{2mq + 2}{3} \right] \).

Conclusion

The irregular total edge labeling for union bipartite graphs \( m(K_{1,q}) \) is \( \left[ \frac{mq + 2}{3} \right] \); and for \( m(K_{2,q}) \) is \( \left[ \frac{2mq + 2}{3} \right], \forall m \geq 2, \text{ where } m \in \mathbb{Z}^+ \).

References


Editorial Board

Brazil  Piccione, Paolo
Canada  Binding, Paul A.
        Li, Yuanlin
        Singh, S. P.
China  Chen, Mu-Fa
       Feng, Qi
       Gao, Wei Dong
       Li, Yangming
       Tang, Chun-Lei
       Zhang, Pu
France  Adly, Samir
Germany Tarkhanov, Nikolai N.
Greece  Rassias, Themistocles M.
        Rassias, John Michael
India  Agrawal, Gunjan
       Thandapani, E.
       Tripathy, B. C.
Iran  Hesaaraki, Mahmoud
Italy  Campanino, Massimo
       Carbone, Antonio
       Ferrara, Massimiliano
Japan  Akou, Tadashi
       Horiuchi, Toshio
       Kitada, Hitoshi
       Nogura, Tsugunori
       Noiri, Takashi
       Owa, Shigeyoshi
       Saito, Kimiaki
       Saitoh, Saburou
       Uchiyama, Mitsuero
       Watanabe, Shuji
Jordan Refai, Mashhoor
Korea Choi, Q-Heung
       Jun, Young Bae

New Zealand Gauld, David
Portugal Staicu, Vasile
Spain  Girela, Daniel
       Granja, Angel
       Jorba, Angel
       Sanz, Miguel Angel
South Africa Cross, Ronald William
        Knopfmacher, Arnold
        Xu, Hong-Kun
Taiwan  Cheng, Sui Sun
        Chu, Liang-Ju
UK  Duggal, B. P.
USA  Biswas, Animikh
       Campbell, James T.
       Chen, Yong-Zhuo
       Coskun, Hasan
       Dilworth, Stephen
       Ebanks, Bruce R.
       Hiremath, G. R.
       Horner, William E.
       Jahangiri, Jay M.
       Kayll, P. Mark
       Lu, Guozhen
       Oh, Yun Myung
       Pedersen, Steen
       Riahi, Daniel N.
       Rouhani, Behzad Djafari
       Silverman, Herb
       Thomas, Marc
       Tonev, Thomas V.
       Tulovsky, Vladimir
       Wong, Peter

Principal Editor
Azad, K. K. (India)
Our Publications

1. Advances and Applications in Discrete Mathematics (ISSN: 0974-1658)
2. Advances and Applications in Fluid Mechanics (ISSN: 0973-4666)
3. Advances and Applications in Statistics (ISSN: 0972-3617)
4. Advances in Computer Science and Engineering (ISSN: 0973-6999)
5. Advances in Differential Equations and Control Processes (ISSN: 0974-3243)
6. Advances in Fuzzy Sets and Systems (ISSN: 0973-421X)
7. Current Development in Theory and Applications of Wavelets (ISSN: 0973-5607)
8. Far East Journal of Applied Mathematics (ISSN: 0972-0960)
10. Far East Journal of Electronics and Communications (ISSN: 0973-7006)
11. Far East Journal of Experimental and Theoretical Artificial Intelligence (ISSN: 0974-3261)
12. Far East Journal of Mathematical Education (ISSN: 0973-5631)
13. Far East Journal of Mathematical Sciences (FJMS) (ISSN: 0972-0871)
14. Far East Journal of Ocean Research (ISSN: 0973-5593)
15. Far East Journal of Theoretical Statistics (ISSN: 0972-0863)
18. International Journal of Materials Engineering and Technology (ISSN: 0975-0444)
19. International Journal of Numerical Methods and Applications (ISSN: 0975-0452)
20. JP Journal of Algebra, Number Theory and Applications (ISSN: 0972-6655)
21. JP Journal of Biostatistics (ISSN: 0973-5143)
22. JP Journal of Fixed Point Theory and Applications (ISSN: 0973-4228)
23. JP Journal of Geometry and Topology (ISSN: 0972-415X)
24. JP Journal of Heat and Mass Transfer (ISSN: 0973-5763)
25. JP Journal of Solids and Structures (ISSN: 0973-5615)
26. Far East Journal of Mechanical Engineering and Physics