Locating-Chromatic Number of Amalgamation of Stars

Asmiati*, H. Assiyatun & E.T. Baskoro

Combinatorial Mathematics Research Group,
Faculty of Mathematics and Natural Sciences,
Institut Teknologi Bandung, Jl. Ganesa 10 Bandung.
Email: asmiati308@students.itb.ac.id

Abstract. Let $G$ be a connected graph and $c$ a proper coloring of $G$. For $i=1,2,...,k$ define the color class $C_i$ as the set of vertices receiving color $i$. The color code $c_i(v)$ of a vertex $v$ in $G$ is the ordered $k$-tuple $(d(v,C_1),...,d(v,C_k))$ where $d(v,C_i)$ is the distance of $v$ to $C_i$. If all distinct vertices of $G$ have distinct color codes, then $c$ is called a locating-coloring of $G$. The locating-chromatic number of graph $G$, denoted by $\chi_L(G)$ is the smallest $k$ such that $G$ has a locating coloring with $k$ colors. In this paper we discuss the locating-chromatic number of amalgamation of stars $S_{1,m}$. $S_{1,m}$ is obtained from $k$ copies of star $K_{1,m}$ by identifying a leaf from each star. We also determine a sufficient condition for a connected subgraph $H \subseteq S_{1,m}$ satisfying $\chi_L(H) \leq \chi_L(S_{1,m})$.

Keywords: amalgamation of stars; color code; locating-chromatic number.

1 Introduction

Let $G$ be a finite, simple, and connected graph. Let $c$ be a proper coloring of a connected graph $G$ using the colors $1,2,...,k$ for some positive integer $k$, where $c(u) \neq c(v)$ for adjacent vertices $u$ and $v$ in $G$. Thus, the coloring $c$ can be considered as a partition $\Pi$ of $V(G)$ into color classes (independent sets) $C_1,C_2,...,C_k$, where the vertices of $C_i$ are colored by $i$ for $1 \leq i \leq k$. The color code $c_i(v)$ of a vertex $v$ in $G$ is the ordered $k$-tuple $(d(v,C_1),...,d(v,C_k))$ where $d(v,C_i) = \min\{d(v,x) \mid x \in C_i\}$ for $1 \leq i \leq k$. If all distinct vertices of $G$ have distinct color codes, then $c$ is called a locating-coloring of $G$. A minimum locating-coloring uses a minimum number of colors and this number is called the locating-chromatic number of graph $G$, denoted by $\chi_L(G)$.

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*Permanent address: Mathematics Department, Faculty of Mathematics and Natural Sciences, Universitas Lampung, Jl. Brojonegoro no.1 Bandar Lampung, Lampung.
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The following definition of an amalgamation of graphs is taken from [3]. For \( i=1,2,\ldots,k \), let \( G_i \) be a graph with a fixed vertex \( v_{oi} \). The *amalgamation* \( \text{Amal}(G_i, v_{oi}) \) is a graph formed by taking all the \( G_i \)'s and identifying their fixed vertices. In this paper, we consider the amalgamation of stars. More precisely, for \( i=1,2,\ldots,k \), let \( G_i = K_{1,n_i} \) where \( n_i \geq 1 \) where \( v_{oi} \) be any leaf (a vertex of degree 1) of \( K_{1,n_i} \). We denote the amalgamation of \( k \) stars \( K_{1,n_i} \) by \( S_{k,(n_1,n_2,\ldots,n_k)}, k \geq 2 \). We call the identified vertex as the *center* (denoted by \( x \)), the vertices of distance 1 from the center as the *intermediate vertices* (denoted by \( l_i; i=1,2,\ldots,k \)), and the \( j \)-th leaf of the intermediate vertex \( l_i \) by \( l_{i,j} (j=1,2,\ldots,m-1) \). In particular, when \( n_i = m, m \geq 1 \) for all \( i \), we denote the amalgamation of \( k \) isomorphic stars \( K_{1,m} \) by \( S_{k,m} \).

The locating-chromatic number was firstly studied by Chartrand et al. [1]. They determined the locating-chromatic numbers of some well-known classes such as paths, cycles, complete multipartit graphs and double stars. Apart of paths and double stars, the locating-chromatic number of other trees are also considered by Chartrand et al. [2]. They constructed a tree of order \( n \geq 5 \) with the locating-chromatic number \( k \), where \( k \in \{3,4,\ldots,n-2,n\} \). They also showed that no tree on \( n \) vertices with locating-chromatic number \( n-1 \).

Based on the previous results, locating-chromatic number of amalgamation of stars have not been studied. Motivated by this, in this paper we determine the locating-chromatic number of amalgamation of stars.

Beside that, we also discuss the monotonicity property of the locating-chromatic number for the class of amalgamation of stars. Clearly, the locating-chromatic number of a star \( K_{1,n} \) is \( n+1 \), for any \( n \) (since all vertices must have different color codes). Since any connected subgraph \( H \) of \( K_{1,n} \) is also a star with small size, then we clearly have \( \chi_L(H) \leq \chi_L(K_{1,n}) \). However in general for any connected subgraph \( H \subseteq G \), the locating-chromatic number of \( H \) may not be necessarily smaller or equal to the locating-chromatic number of \( G \).

In this paper, we also investigate the monotonicity property of the locating-chromatic number for amalgamation of stars, \( S_{k,m} \). We derive a sufficient condition for a connected subgraph \( H \subseteq S_{k,m} \) satisfying \( \chi_L(H) \leq \chi_L(S_{k,m}) \).
The following results were proved by Chartrand et al. in [1]. The set of neighbors of a vertex \( v \) in \( G \) is denoted by \( N(v) \).

**Theorem 1.1.** Let \( c \) be a locating-coloring in a connected graph \( G \). If \( u \) and \( v \) are distinct vertices of \( G \) such that \( d(u,w)=d(v,w) \) for all \( w \in V(G)\setminus\{u,v\} \), then \( c(u) \neq c(v) \). In particular, if \( u \) and \( v \) are non adjacent vertices of \( G \) such that \( N(u)=N(v) \), then \( c(u) \neq c(v) \).

**Corollary 1.1.** If \( G \) is a connected graph containing a vertex adjacent to \( k \) leaves of \( G \), then \( \chi'_L(G) \geq k+1 \).

## 2 Main Results

We first prove some lemmas regarding the properties of locating-chromatic number of amalgamation of stars. From now on \( S_{k,m} \) denotes the amalgamation of \( k \) isomorphic stars \( K_{1,m} \).

**Lemma 2.1.** For \( k \geq 2, m \geq 2 \), let \( c \) be a proper coloring of \( S_{k,m} \), using at least \( m \) colors. The coloring \( c \) is a locating-coloring if and only if \( (c(l_i), c(l_n)) \neq (c(l_i), c(l_n)) \) for some \( i \neq n \).

**Proof.** Let \( P = \{c(l_j) \mid j = 1, 2, \ldots, m-1\} \) and \( Q = \{c(l_j) \mid j = 1, 2, \ldots, m-1\} \). Let \( c \) be a locating-coloring of \( S_{k,m} \), \( k \geq 2, m \geq 2 \) using at least \( m \) colors and \( c(l_i) = c(l_n) \), for some \( i \neq n \). Suppose that \( P = Q \). Because \( d(l_i, u) = d(l_n, u) \) for every \( u \in V \setminus \{l_j \mid j = 1, 2, \ldots, m-1\} \cup \{l_j \mid j = 1, 2, \ldots, m-1\} \) then the color codes of \( l_i \) and \( l_n \) will be the same. So \( c \) is not a locating-coloring, a contradiction. Therefore \( P \neq Q \).

Let \( \Pi \) be a partition of \( V(G) \) into color classes with \( |\Pi| \geq m \). Consider \( c(l_i) = c(l_n), i \neq n \). Since \( P \neq Q \), there are color \( x \) and color \( y \) such that \((x \in P, x \not\in Q)\) and \((y \in P, y \not\in Q)\). We will show that color codes for every \( v \in V(S_{k,m}) \) is unique.

- Clearly, \( c_{\Pi}(l_i) \neq c_{\Pi}(l_n) \) because their color codes differ in the \( x \)th-ordinate and \( y \)th-ordinate.
• If \( c(l_i) = c(l_m) \), for some \( l_i \neq l_m \), we will show that \( c_{11}(l_i) \neq c_{11}(l_m) \).

We divide into two cases.

Case 1. If \( c(l_i) = c(l_n) \) then by the premise of this theorem, \( P \neq Q \). So \( c_{11}(l_i) \neq c_{11}(l_m) \).

Case 2. Let \( c(l_i) = r_1 \) and \( c(l_n) = r_2 \), with \( r_1 \neq r_2 \). Then \( c_{11}(l_i) \neq c_{11}(l_m) \) because their color codes are different at least in the \( r_i \)th-ordinate and \( r_j \)th-ordinate.

• If \( c(l_i) = c(l_m) \), \( l_i \neq l_m \), then \( c_{11}(l_i) \) contains at least two components of value 1, whereas \( c_{11}(l_m) \) contains exactly one component of value 1. Thus \( c_{11}(l_i) \neq c_{11}(l_m) \).

• If \( c(x) = c(l_y) \), then color code of \( c_{11}(x) \) contains at least two components of value 1, whereas \( c_{11}(l_y) \) contains exactly one component of value 1. Thus \( c_{11}(x) \neq c_{11}(l_y) \).

From all above cases, we see that the color code for each vertex in \( S_{k,m} \) is unique, thus \( c \) is a locating coloring.

Lemma 2.2. Let \( c \) be a locating coloring of \( S_{k,m} \) using \( m+a \) colors and
\[
H(a) = (m+a-1) \binom{m+a-1}{m-1}, \quad a \geq 0.
\]
Then \( k \leq H(a) \).

Proof. Let \( c \) be a locating-coloring of \( S_{k,m} \) using \( m+a \) colors. For fixed \( i \), let \( c(l_i) \) be a color of intermediate vertex \( l_i \), then the number color combinations can be used by \( \{l_j\} j=1,2,...,m-1 \) is \( \binom{m+a-1}{m-1} \). Because one color is used for coloring the center \( x \), there are \((m+a-1)\) colors for \( l_i \), for every \( i=1,2,...,k \). By Lemma 2.1, the maximum number of \( k \) is \( (m+a-1) \binom{m+a-1}{m-1} = H(a) \). So \( k \leq H(a) \).
The main result of this paper concerns about locating-chromatic number of $S_{k,m}$.

**Theorem 2.1.** For $a \geq 0, k \geq 2, m \geq 2$, let $H(a) = (m+a-1)\binom{m+a-1}{m-1}$. Then,

$$
\chi_L(S_{k,m}) = \begin{cases} 
  m & \text{for } 2 \leq k \leq H(0), m \geq 3, \\
  m+a & \text{for } H(a-1) < k \leq H(a), a \geq 1.
\end{cases}
$$

**Proof.** First, we determine the trivial lower bound. By Corollary 1.1, each vertex $l_i$ is adjacent to $(m-1)$ leaves, for $i = 1, 2, ..., k$. Thus, $\chi_L(S_{k,m}) \geq m$.

Next, we determine the upper bound of $\chi_L(S_{k,m})$ for $2 \leq k \leq H(0) = m - 1$. Let $c$ be a coloring of $V(S_{k,m})$ using $m$ colors. Without loss of generality, we can assign $c(x) = 1$ and $c(l_i) = i+1$ for $i = 1, 2, ..., k$. To make sure that the leaves will have distinct color code, we assign $\{l_j | j = 1, 2, ..., m-1\}$ by $\{1, 2, ..., m\} \setminus \{i+1\}$ for any $i$. Then, by Lemma 2.1, we have that $c$ is a locating-coloring. Thus $\chi_L(S_{k,m}) \leq m$.

Next, we shall improve the lower bound for the case of $k$ such that $H(a-1) < k \leq H(a), a \geq 1$. Since $k > H(a-1)$ then by Lemma 2.2, $\chi_L(S_{k,m}) \geq m+a$. On the other hand if $k > H(a)$ then by Lemma 2.2, $\chi_L(S_{k,m}) \geq m+a+1$. Thus $\chi_L(S_{k,m}) \geq m+a$ if $H(a-1) < k \leq H(a)$.

Next, we determine the upper bound of $\chi_L(S_{k,m})$ for $H(a-1) < k \leq H(a), a \geq 1$. Without loss of generality, let $c(x) = 1$ and color the intermediate vertices $l_i$ by $2, 3, ..., m+a$ in such a way that the number of the intermediate vertices receiving the same color $t$ does not exceed $\binom{m+a-1}{m-1}$, for any $t$. We are able to do so because $H(a-1) < k \leq H(a)$. Therefore, if $c(l_i) = c(l_n), i \neq n$ then we can manage $\{c(l_j) | j = 1, 2, ..., m-1\} \neq \{c(l_j) | j = 1, 2, ..., m-1\}$.

By Lemma 2.1, $c$ is a locating-coloring on $S_{k,m}$. So $\chi_L(S_{k,m}) \leq m+a$ for $H(a-1) < k \leq H(a)$. □
The following figures show minimum locating-colorings on $S_{4,6}$ and $S_{9,3}$.

![Figure 1](image1.png)  
**Figure 1** A minimum locating-coloring of $S_{4,6}$.

![Figure 2](image2.png)  
**Figure 2** A minimum locating-coloring of $S_{9,3}$.

Next, we discuss the monotonicity property of locating-chromatic number for the amalgamation of stars.

**Theorem 2.2** If $2 \leq k \leq m - 1$, then $\chi_L(G) \leq \chi_L(S_{k,m})$ for every $G \subseteq S_{k,m}$ and $G \neq K_{1,m}$.

**Proof.** Let $c$ be a minimum locating-coloring of $S_{k,m}$ obtained from Theorem 2.1. Let $G$ be any connected subgraph of $S_{k,m}$. Define a coloring $c'$ on $G$ by
preserving colors used in $S_{k,m}$ for the corresponding vertices, namely $c'(v') = c(v)$ if $v$ is the corresponding vertex of $v'$ in $S_{k,m}$. We show that $c'$ is a locating-coloring of $G$.

If there exist $l_i$, $l_a$ such that $\{c'(l_j) \mid j = 1, 2, ..., r\} = \{c'(l_{a_j}) \mid j = 1, 2, ..., r\}$, with $1 \leq r \leq m-1$, then color codes of $l_j$ and $l_{a_j}$ for every $j = 1, 2, 3, ..., m$ is unique because $c'(l_j) \neq c'(l_{a_j})$ for every $l_j \neq l_{a_j}$. If $c'(l_i) = c'(l_{a_i}) \neq c'(x)$, then the first component of $c'(l_i)$ has value 1, whereas for $c'(l_{a_i})$ it has value 2. So color code of $l_i$ and $l_{a_i}$ are different. Next, if $c'(x) = c'(l_{a_i})$, $G \neq P_3$ then their color codes are different because $c'(l_i) \neq c'(l_{a_i})$ for every $l_i \neq l_{a_i}$. For the case $G = P_3$, $v_i \in V(P_3)$ for each $i$ is colored by 1, 2, and 3 respectively. Because the color codes for every $v \in V(G)$ is unique, then $c'$ is a locating-coloring of $G$. So $\chi_L(G) \leq \chi_L(S_{k,m})$ for every $G \subseteq S_{k,m}$, $G \neq K_{1,m}$. □

Let $S_{k,(n_1, n_2, ..., n_k)} \subseteq S_{k,m}$. Define $A = \{i \mid n_i = 1\}$. For $k \geq m$, we must restrict subgraphs of $S_{k,m}$ so that satisfy monotonicity property.

**Theorem 2.3** If $k \geq m$ and $|A| \leq \chi_L(S_{k,m}) - 1$ then $\chi_L(S_{k,(n_1, n_2, ..., n_k)}) \leq \chi_L(S_{k,m})$.

**Proof.** Let $k \geq m$ and from Theorem 2.1, we have that $\chi_L(S_{k,m}) = m + a$ for $H(a-1) < K \leq H(a)$, $a \geq 1$. Let $G = S_{k,(n_1, n_2, ..., n_k)}$ be any subgraph obtained from $S_{k,m}$ with $1 \leq n_i \leq m$. If $2 \leq n_i \leq m$ for each $i$, then color vertices of $G$ follow the proof of Theorem 2.1. Clearly, the coloring of $G$ is a locating-coloring. Otherwise, we have $n_i = 1$ for some $i$, and so $|A| \geq 1$. If $|A| \leq \chi_L(S_{k,m}) - 1$, then the center $x$ is given color 1, $l_i \in A$ for each $i$ is colored by 2, 3, ..., $\chi_L(S_{k,m})$, respectively and the colors for the other vertices follow the proof of Theorem 2.1. Observe that the color codes of $l_i$ for each $l_i \in A$ has value 1 in the 1st-ordinate, 0 in the $i$th-ordinate, and 2 otherwise, these color codes are unique. For the remaining of the vertices, the color codes are also unique as proven in Theorem 2.1. As the result, the coloring of $G$ is a locating-coloring. So $\chi_L(S_{k,(n_1, n_2, ..., n_k)}) \leq \chi_L(S_{k,m})$. □
References

