Far East Journal of Mathematical Sciences (FJMS)
Volume 63, Number 1, 2012, Pages 11-23
Published Online: February 2012
Available online at http://pphmj.com/journals/fjms.htm
Published by Pushpa Publishing House, Allahabad, INDIA

THE LOCATING-CHROMATIC NUMBER OF FIRECRACKER GRAPHS

Asmiati*, E. T. Baskoro, H. Assiyatun, D. Suprijanto, R. Simanjuntak and S. Uttunggadewa

Combinatorial Mathematics Research Division
Faculty of Mathematics and Natural Sciences
Institut Teknologi Bandung
Jl. Ganesa 10 Bandung 40132, Indonesia
e-mail: asmiati308@students.itb.ac.id
ebaskoro@math.itb.ac.id; hilda@math.itb.ac.id
djoko@math.itb.ac.id; rino@math.itb.ac.id
saladin@math.itb.ac.id

**Permanent address*
Mathematics Department
Faculty of Mathematics and Natural Sciences
Lampung University
Jl.Brojonegoro No. 1, Bandar Lampung, Indonesia

Abstract

Let $G = (V, E)$ be a connected graph and c be a proper k-coloring of G. Let $\prod = \{C_1, C_2, ..., C_k\}$ be a partition of $V(G)$, where C_i is the set of vertices receiving color i. The color code $c_{\prod}(v)$ of a vertex v in

2010 Mathematics Subject Classification: 05C12, 05C15.

Keywords and phrases: color code, locating-chromatic number, firecracker graph.

This research is supported by IMHERE Research Grant 2011 and DGHE Competence Research Grant 2009-2011.

Received October 20, 2011

G is the ordered k-tuple $(d(v, C_1), ..., d(v, C_k))$, where $(d(v, C_i))$ is the distance of v to C_i. If for any two distinct vertices u, v in G, $c_\prod(u) \neq c_\prod(v)$, then c is called a locating-chromatic coloring of G. The locating-chromatic number of graph G, denoted by $\chi_L(G)$, is the smallest k such that G admits a locating coloring with k colors. A firecracker graphs $F_{n,k}$ is a graph obtained by the concatenation of n stars, each consists of k vertices, by linking one leaf from each star. In this paper, we determine the locating-chromatic number of firecracker graphs $F_{n,k}$.

1. Introduction

The locating-chromatic number of a graph was firstly studied by Chartrand et al. [2] in 2002. This concept is derived from the partition dimension and the graph coloring. The partition dimension of a graph was firstly studied by Chartrand et al. in [4]. They gave the partition dimension of some classes of trees, such as: paths, double stars, and caterpillars. Since then, many studies have been conducted to find the partition dimension of certain classes of graphs. For instances, Tomescu et al. [8] showed upper bound and lower bound on partition dimension of wheels. Next, Javaid and Shokat [6] gave the upper bounds on the number of vertices in some wheel related graphs, namely, gear graphs, helms, sunflowers, and friendships graph with partition dimension k. More recent results, Marinescu-Ghemeci and Tomescu [7] derived the *star* partition dimension of generalized gear graphs.

Let $G = (V, E)$ be a connected graph. Let c be a proper k-coloring of G with colors $1, 2, ..., k$. Let $\prod = \{C_1, C_2, ..., C_k\}$ be a partition of $V(G)$, where C_i is the set of vertices receiving color i. The color code $c_\prod(v)$ of v is the ordered k-tuple $(d(v, C_1), d(v, C_2), ..., d(v, C_k))$, where $d(v, C_i) = \min\{d(v, x) | x \in C_i\}$ for any i. If all distinct vertices of G have distinct color codes, then c is called a *locating-chromatic k-coloring* of G (*k-locating* *coloring*, in short). The locating-chromatic number, denoted by $\chi_L(G)$ is the smallest *k* such that *G* has a locating coloring with *k* colors.

Chartrand et al. [2] determined the locating-chromatic numbers of some well-known classes such as paths, cycles, complete multipartite graphs and double stars. Moreover, the locating-chromatic number of some particular trees is also considered by Chartrand et al. [3]. They determined the locating-chromatic number of tree T_n order $n \geq 5$ and showed that locating-chromatic number of tree T_n is *t*, where $t \in [3, n]$ and $t \neq n - 1.$ For each value in the interval, some trees are indicated. However, there are many classes of trees which locating-chromatic numbers are still not known.

In 2011, Asmiati et al. [1] determined the locating-chromatic number of amalgamation $S_{k,m}$ of stars, namely, the graph obtained from *k* copies of star $K_{1,m}$ by identifying a leaf from each star. Motivated by this, in this paper, we determine the locating-chromatic number of firecracker graph $F_{n,k}$, namely, the graph obtained by the concatenation of *n* stars S_k by linking one leaf from each star [5].

The following results were proved by Chartrand et al. in [2]. We denote the set of neighbors of a vertex *v* in *G* by $N(v)$.

Theorem 1. *Let c be a locating coloring in a connected graph* $G = (V, E)$. *If u and v are distinct vertices of G such that* $d(u, w) = d(v, w)$ *for all* $w \in V(G) - \{u, v\}$, *then* $c(u) \neq c(v)$. *In particular, if u and v are non-adjacent vertices of G such that* $N(u) = N(v)$, *then* $c(u) \neq c(v)$.

Corollary 1. *If G is a connected graph containing a vertex adjacent to m end-vertices of G, then* $\chi_L(G) \geq m + 1$.

Corollary 1 gives a lower bound of the locating-chromatic number of a general graph *G*.

2. Dominant Vertices

Let c be a locating-chromatic coloring on graph $G = (V, E)$. Let $\prod = \{C_1, C_2, ..., C_k\}$ be the partition of $V(G)$ under such a coloring c. Then a vertex $v \in C_i$, for some i, is *dominant* if $d(v, C_j) = 1$ for $j \neq i$. Note that C_i has at most one dominant vertex. A path connecting two dominant vertices is said to be *clear* if all of its internal vertices are not dominant. Now, we show, in the following lemma, that every clear path in a graph whose locating-chromatic number three must have odd length.

Lemma 1. *Let G be a graph of locating-chromatic number three. Then, any clear path has odd length.*

Proof. Let P be a clear path connecting two dominant vertices x and y in G. Assume $c(x) = 1$ and $c(y) = 2$. Since all internal vertices of P are not dominant, the colors of these vertices must be either 2 or 1 alternatingly, and together with x and y, they form an alternating sequence. Therefore, the number of these internal vertices must be even, and it implies that the length of P is odd. □

Lemma 2. *Let G be a connected graph of locating-chromatic number three. Let G contain three dominant vertices. Then, these three dominant vertices must lie in one path.*

Proof. Let x, y, z be the three dominant vertices of G. Let P be a path connecting x and z. For a contradiction, assume y is not in a path P (as well as the extension of P). Since G is connected, there exists an internal vertex u such that u has a smallest distance (compared to other internal vertices) to y. Now consider the path L_1 connecting x to u and then to y. Such a path is a clear path. Therefore, it has odd length. Next, consider the path L_2 connecting y to u and then to z. Then, L_2 is also a clear path. Therefore, its length is odd. These two facts imply that the length of the path connecting x to u plus the length of the path connecting u to z is even, a contradiction. □

3. The Locating-chromatic Number of $F_{n,k}$

Let $F_{n,k}$ be a firecracker, for $n, k \geq 2$. In this section, we shall determine the locating-chromatic number of $F_{n,k}$. Since $|V(F_{n,k})| \geq 3$, it is clear that the locating-chromatic number $\chi_L(F_{n,k}) \geq 3$.

Theorem 2. *For* $k = 2$ *or* 3, *we have that*

$$\chi_L(F_{n,k}) = \begin{cases} 3, & \text{if } 2 \leq n \leq 7, \\ 4, & \text{if } n \geq 7. \end{cases}$$

Proof. First, consider the firecracker $F_{n,k}$ for $k = 2$ or 3, and $2 \leq n < 7$. To show that locating-chromatic number is 3, we only need to give the locating coloring of such a graph. Consider the coloring *c* of $F_{6,3}$ in Figure 1. It is easy to see that *c* is a locating coloring. By using the corresponding sub-coloring (from left side) of *c*, we can obtain the locating coloring of $F_{n,k}$, for $2 \leq n < 7$ and $k = 2$ or 3, except for $F_{3,2}$ and $F_{3,3}$, replace color 2 for the rightmost vertices by 1, whereas for $F_{5,2}$ and $F_{5,3}$, replace color 3 for the rightmost vertex by 2.

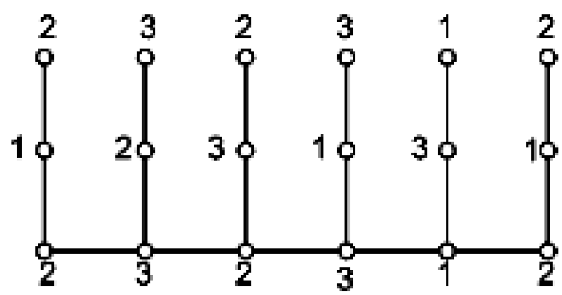

Figure 1. A minimum locating coloring of $F_{6,3}$.

Next, consider a graph $G \cong F_{n,k}$ for $n \geq 7$ and $k = 2$ or 3. We shall show that $\chi_L(G) \geq 4$. Suppose there exists a 3-locating coloring *c* on $F_{n,k}$ for $n \geq 7$, $k = 2, 3$. First, we are going to show that there are exactly three dominant vertices in $F_{n,k}$, for $n \geq 7$ and $k = 2, 3$. Since $F_{n,k}$ for $k = 2, 3$ is not a path, $F_{n,k}$ for $k = 2, 3$ has at least two dominant vertices.

Now, by a contradiction, suppose there are only two dominant vertices in $F_{n,k}$, $k = 2, 3$, say x and y. By Lemma 1, there is an odd clear path from x to y, say $(x = p_1, p_2, ..., p_r = y)$, with r is even. If all degrees of x and y are 2, then for $F_{n,2}$, the two neighbors of p_2 (other than x) will have the same color code, a contradiction. For $F_{n,3}$, if $r \leq 6$ and let v be a vertex (other than p_{r-2}) of degree 3 adjacent to p_{r-1}. Then the two neighbors of v (other than p_{r-1}) will have the same color code, a contradiction. If $r > 6$, then the two neighbors of p_3 (other than p_2) will have the same color code, a contradiction.

Now, consider the case of the degree of x is 2 and the degree of y is 3. For $F_{n,k}$ and $k = 2, 3$, if $r \leq 4$, then let z be a vertex (other than p_{r-1}) of degree 3 adjacent to y. Consider the two neighbors of z (other than y). Then, the color codes of these two vertices will be the same, a contradiction. For $F_{n,2}$, if $r > 4$, then the color codes of the two neighbors of p_2 (other than x) will be the same, whereas for $F_{n,3}$, the color codes of the two neighbors of p_{r-1} (other than y) will be the same, a contradiction.

Now, consider the degrees of x and y are 3. We may assume that y is adjacent to a vertex z with degree 3. For $F_{n,k}$ and $k = 2, 3$, if $r = 2$, then the two neighbors of z (other than y) will have the same color code, a contradiction. If $r > 2$, then two neighbors of x will have the same color code, a contradiction. Therefore, if $\chi_L(F_{n,k}) = 3$, $n \geq 7$ and $k = 2, 3$, then it has exactly 3 dominant vertices.

Since $G \cong F_{n,k}$ has three dominant vertices, for $k = 2, 3$ and $n \geq 7$, by Lemma 2, these three dominant vertices lie in one path P, say $P := (x = p_1, p_2, ..., p_t = y, ..., p_r = z)$, where x, y, z are the dominant vertices. Next, we consider the following three cases (by symmetry):

Case 1. Distances: $d(x, y) = d(y, z) = 1$.

This means that $x = p_1$, $y = p_2$ and $z = p_3$. Since $n \geq 7$, at least one of x, y or z has a neighbor w of degree 3 (which is not a dominant vertex) in G. Then, two neighbors of w will have the same color code.

Case 2. Distances: $d(x, y) = 1$ and $d(y, z) \geq 3$.

If $d(y, z) > 3$, then the two non-dominant vertices adjacent to w will have the same color code, where w is a neighbor of y which lies in the clear path from y to z, a contradiction. If $d(y, z) = 3$ and $d(z) = 3$, then two neighbors of z will have the same color code. If $d(y, z) = 3$ and $d(z) = 2$, then since $n \geq 7$, there exists a vertex w not in P of degree 3 adjacent to either x, y or p_{r-1}. Then, the color codes of the two non-dominant neighbors of w are the same, a contradiction.

Case 3. Distances: $d(x, y) \geq 3$ and $d(y, z) \geq 3$.

In this case, the degree of y must be 3 and two neighbors of y will have the same color code, a contradiction.

Therefore, from these three cases, we conclude that $\chi_L(F_{n,k}) \geq 4$, for $n \geq 7$ and $k = 2, 3$.

Next, we show that $\chi_L(F_{n,k}) \leq 4$, for $n \geq 7$, $k = 2, 3$. Label all leaves of $F_{n,2}$ by $l_1, l_2, \ldots, l_n$ and a vertex adjacent to leave l_i by x_i. Now, define a 4-coloring c on $F_{n,2}$ as follows:

- $c(x_i) = 1$ if i is odd and $c(x_i) = 2$ if i is even; and
- $c(l_1) = 4$ and $c(l_i) = 3$ for $i \geq 2$.

It is clear that the color codes of all vertices are different (by the distance to the vertex of color 4), therefore c is a locating coloring on $F_{n,2}$, $n \geq 7$.

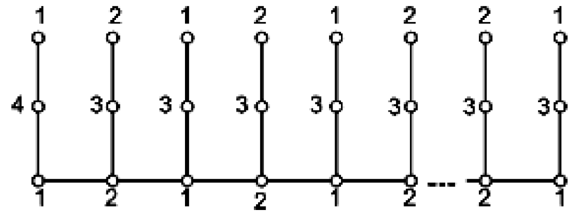

Figure 2. A minimum locating coloring of $F_{n,3}$, for $n \geq 7$.

Next, consider a graph $F_{n,3}$ for $n \geq 7$. Let $V(F_{n,3}) = \{x_i, m_i, l_i \mid i = 1, 2, \cdot, n\}$ and

$$E(F_{n,3}) = \{x_i x_{i+1} \mid i = 1, 2, ..., n-1\} \cup \{x_i m_i, m_i l_i \mid i = 1, 2, ..., n\}.$$

Define a 4-coloring c on $F_{n,3}$ as follows:

- $c(x_i) = c(l_i) = 1$ if i is odd and $c(x_i) = c(l_i) = 2$ if i is even; and
- $c(m_1) = 4$ and $c(m_i) = 3$ for $i \geq 2$.

The coloring c will create a partition $\prod$ on $V(F_{n,3})$. We show that the color codes for all vertices in $F_{n,3}$ are different. It is clear that $c_{\prod}(m_1) = (1, 2, 3, 0)$, $c_{\prod}(m_i) = (2, 1, 0, i+1)$ for even i and $c_{\prod}(m_i) = (1, 2, 0, i+1)$ for odd $i \geq 3$, whereas for x_i, $c_{\prod}(x_1) = (0, 1, 2, 1)$, $c_{\prod}(x_i) = (1, 0, 1, i)$ for even i and $c_{\prod}(x_i) = (0, 1, 1, i)$ for odd $i \geq 3$. For leaves, $c_{\prod}(l_1) = (0\ 3, 4, 1)$, $c_{\prod}(l_i) = (3, 0, 1, i+2)$ for even i and $c_{\prod}(l_i) = (0, 3, 1, i+2)$ for odd $i \geq 3$. All these color codes are different, thus c is a locating-coloring.

So $\chi_L(F_{n,3}) \leq 4$, $n \geq 7$. □

Next, we determine the locating-chromatic number of firecracker graphs $F_{n,k}$ for $n \geq 2$ and $k \geq 4$.

Theorem 3. *Let* $F_{n,k}$ *be a firecracker graph. Then,*

(i) $\chi_L(F_{n,4}) = 4$, *for* $n \geq 2$.

(ii) *For* $k \geq 5$,

$$\chi_L(F_{n,k}) = \begin{cases} k-1, & \textit{if } 2 \leq n \leq k-1, \\ k, & \textit{otherwise.} \end{cases}$$

Proof. Let $V(F_{n,k}) = \{x_i, m_i, l_{ij} \mid i = 1, 2, ..., n;\ j = 1, 2, ..., k-2\}$, and

$$E(F_{n,k}) = \{x_i x_{i+1} \mid i = 1, 2, ..., n-1\}$$
$$\cup \{x_i m_i, m_i l_{ij} \mid i = 1, 2, ..., n;\ j = 1, 2, ..., k-2\}.$$

First, we determine the lower bound of $F_{n,4}$, for $n \geq 2$. By Corollary 1, we have that $\chi_L(F_{n,4}) \geq 3$. However, we will show that $\chi_L(F_{n,4}) \geq 4$. For a contradiction, assume that there exists a 3-locating coloring c on $F_{n,4}$, $n \geq 2$. If the colors are 1, 2 and 3, then $\{c(m_1), c(l_{11}), c(l_{12})\} = \{c(m_2), c(l_{21}), c(l_{22})\} = \{1, 2, 3\}$. Obviously, $c(m_1) \neq c(m_2)$, since otherwise, the color codes of l_{1i} and l_{2j} are the same, for some $i, j \in \{1, 2\}$, a contradiction. Now consider $c(x_i)$, for $i = 1, 2$. Since we have only 3 colors, $c(x_1) = c(l_{1j})$ for some $j = \{1, 2\}$. Regardless the color of x_2, we have that the color code of x_1 is the same as the color code of either l_{1j} or m_2, a contradiction. Therefore, $\chi_L(F_{n,4}) \geq 4$.

Next, we determine the upper bound of $F_{n,4}$ for $n \geq 2$. To show that $\chi_L(F_{n,4}) \leq 4$ for $n \geq 2$, consider the 4-coloring c on $F_{n,4}$ as follows:

- $c(x_i) = 1$ if i is odd and $c(x_i) = 3$ if i is even;
- $c(m_i) = 2$ for every i;
- for all vertices l_{ij}, define

$$c(l_{ij}) = \begin{cases} 4, & \text{if } i = 1,\ j = 1, \\ 1, & \text{if } i \geq 2,\ j = 1, \\ 3, & \text{if } j = 2. \end{cases}$$

The coloring c will create a partition $\prod$ on $V(F_{n,4})$. We shall show that the color codes of all vertices in $F_{n,4}$ are different. For odd i, we have $c_{\prod}(x_i) = (0, 1, 1, i+1)$ and for even i, $c_{\prod}(x_i) = (1, 1, 0, i+1)$. For m_i, $c_{\prod}(m_1) = (1, 0, 1, 1)$ and $c_{\prod}(m_i) = (1, 0, 1, i+2)$ for $i \geq 2$. For vertices $l_{i,j}$, we have $c_{\prod}(l_{11}) = (2, 1, 2, 0)$ and $c_{\prod}(l_{12}) = (2, 1, 0, 2)$. For $i \geq 2$, $c_{\prod}(l_{i1}) = (0, 1, 2, i+3)$ and $c_{\prod}(l_{i2}) = (2, 1, 0, i+3)$. Since the color codes of all vertices in $F_{n,4}$ are different, thus c is a locating-chromatic coloring. So $\chi_L(F_{n,4}) \leq 4$. It completes the proof for the first part of the theorem.

Next, we will show that for $k \geq 5$, $\chi_L(F_{n,k}) = k$ if $n \geq k$, and $\chi_L(F_{n,k}) = k-1$ if $2 \leq n \leq k-1$. To show this, let us consider the following two cases:

Case 1. For $k \geq 5$ and $2 \leq n \leq k-1$.

First, we determine the lower bound of $F_{n,k}$, for $k \geq 5$ and $2 \leq n \leq k-1$. Since each vertex l_i is adjacent to $(k-2)$ leaves, by Corollary 1, $\chi_L(F_{n,k}) \geq k-1$.

Next, we will show that $\chi_L(F_{n,k}) \leq k-1$ for $k \geq 5$ and $n \leq k-1$. Define a $(k-1)$-coloring c of $F_{n,k}$, as follows. Assign $c(m_i) = i$, for $i = 1, 2, ..., n$ and all the leaves: $\{l_{ij} \mid j = 1, 2, ..., k-2\}$ by $\{1, 2, ..., k-1\} \backslash \{i\}$, for any i. Next, define $c(x_i)$, for $i = 1, 2, ..., n$, equal to $3, 4, 5, ..., n, 2, 3$, respectively. Note that if $n = 2$, then $c(x_1) = 2$ and $c(x_2) = 3$. As a result, coloring c will create a partition $\prod = \{U_1, U_2, ..., U_{k-1}\}$ on $V(F_{n,k})$, where U_i is the set of all vertices with color i.

We show that the color codes for all vertices in $F_{n,k}$ for $k \geq 5$, $n \leq k-1$, are different. Let $u, v \in V(F_{n,k})$ and $c(u) = c(v)$. Then, consider the following cases:

• If $u = l_{ih}$, $v = l_{jl}$, for some i, j, h, l and $i \neq j$, then $c_{\prod}(u) \neq c_{\prod}(v)$ since $d(u, U_i) \neq d(v, U_i)$.

• If $u = l_{ih}$, $v = m_j$, for some i, j, h and $i \neq j$, then v must be a dominant vertex but u is not. Thus, $c_{\prod}(u) \neq c_{\prod}(v)$.

• If $u = l_{ih}$, $v = x_j$, for some i, j, h, then there exactly one set in $\prod$ which has the distance 1 from u and there is at least two sets in $\prod$ which have the distance 1 from v. Thus, $c_{\prod}(u) \neq c_{\prod}(v)$.

• If $u = m_i$, $v = x_j$, for some i, j, then u must be a dominant vertex but v is not. Thus, $c_{\prod}(u) \neq c_{\prod}(v)$.

• If $u = x_i$ and $x = x_j$, then $i = 1$ and $j = n$. So, $c_{\prod}(u) \neq c_{\prod}(v)$.

From all the above cases, we see that the color codes of all vertices in $F_{n,k}$ for $k \geq 5$, $n \leq k - 1$ are different, thus $\chi_L(F_{n,k}) \leq k - 1$.

For an illustration, we give the locating-chromatic coloring of $F_{4,5}$ in Figure 3:

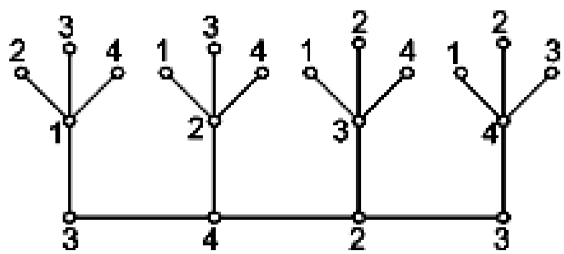

Figure 3. A locating-chromatic coloring of $F_{4,5}$.

Case 2. For $k \geq 5$ and $n \geq k$.

We first determine the lower bound for $k \geq 5$ and $n \geq k$. By Corollary 1, we have that $\chi_L(F_{n,k}) \geq k - 1$. However, we will show that $k - 1$ colors are not enough. For a contradiction, assume that there exists a $(k - 1)$-

locating coloring c on $F_{n,k}$ for $k \geq 5$ and $n \geq k$. Since $n \geq k$, there are two $i, j, i \neq j$ such that $\{c(l_{ih}) | h = 1, 2, ..., k-2\} = \{c(l_{jl}) | l = 1, 2, ..., k-2\}$. Therefore, the color codes of m_i and m_j are the same, a contradiction.

Next, we determine the upper bound of $F_{n,k}$ for $k \geq 5$, $n \geq k$. To show that $F_{n,k} \leq k$ for $k \geq 5$ and $n \geq k$, consider the locating coloring c on $F_{n,k}$ as follows:

- $c(x_i) = 1$ if i is odd and $c(x_i) = 3$ if i is even;
- $c(m_i) = 2$ for every i;
- If $A = \{1, 2, ..., k\}$, define:

$$\{c(l_{ij}) | j = 1, 2, ..., k-2\} = \begin{cases} A \backslash \{1, 2\}, & \text{if } i = 1, \\ A \backslash \{2, k\}, & \text{otherwise.} \end{cases}$$

It is easy to verify that the color codes of all vertices are different. Therefore, c is a locating-chromatic coloring on $F_{n,k}$, and so $\chi_L(F_{n,k}) \leq k$, for $n \geq k$. This completes the proof. □

Figure 4 gives the locating-chromatic coloring of $F_{6,5}$:

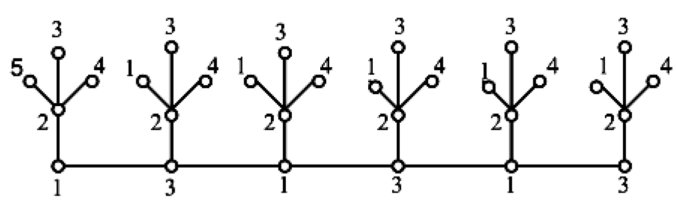

Figure 4. A minimum locating coloring of $F_{6,5}$.

References

[1] Asmiati, H. Assiyatun and E. T. Baskoro, Locating-chromatic number of amalgamation of stars, ITB J. Sci. 43A (2011), 1-8.

[2] G. Chartrand, D. Erwin, M. A. Henning, P. J. Slater and P. Zang, The locating-chromatic number of a graph, Bull. Inst. Combin. Appl. 36 (2002), 89-101.

[3] G. Chartrand, D. Erwin, M. A. Henning, P. J. Slater and P. Zang, Graph of order n with locating-chromatic number $n-1$, Discrete Math. 269 (2003), 65-79.

[4] G. Chartrand, P. Zhang and E. Salehi, On the partition dimension of graph, Congr. Numer. 130 (1998), 157-168.

[5] W. C. Chen, H. I. Lü and Y. N. Yeh, Operations of interlaced trees and graceful trees, Southeast Asian Bull. Math. 21 (1997), 337-348.

[6] I. Javaid and S. Shokat, On the partition dimension of some wheel related graphs, Journal of Prime in Mathematics 4 (2008), 154-164.

[7] R. Marinescu-Ghemeci and I. Tomescu, On star partition dimension of generalized gear graph, Bull. Math. Soc. Sci. Roumanie Tome 53(101)3 (2010), 261-268.

[8] I. Tomescu, I. Javaid and Slamin, On the partition dimension and connected partition dimension of wheels, Ars Combin. 84 (2007), 311-317.