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Characterizing all graphs containing cycles with locating-chromatic number 3

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Abstract. Let G be a connected graph G. Let c be a k-coloring of $V(G)$ which induces an ordered partition $\Pi = \{S_1, S_2, \ldots, S_k\}$ of $V(G)$, where S_i is the set of vertices receiving color i. The *color code* $c_\Pi(v)$ of vertex v is the ordered k-tuple $(d(v,S_1), d(v,S_2), \ldots, d(v,S_k))$, where $d(v,S_i) = \min\{d(v,x)|x \in S_i\}$, for $1 \leq i \leq k$. If the color codes of all vertices are different, then c is called a *locating-coloring* of G. The *locating-chromatic number* of G, denoted by $\chi_L(G)$ is the smallest k such that G has a locating k-coloring. In this paper, we investigate graphs with locating-chromatic number 3. In particular, we determine all maximal graphs having cycles (in terms of the number of edges) with locating-chromatic number 3. From this result, we then characterize all graphs on n vertices containing cycles with locating-chromatic number 3.

Keywords: locating-chromatic number, graph
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INTRODUCTION

Chartrand, Salehi, and Zhang [10] introduced the concept of graph partition dimension in 1998 and since then many studies have been devoted to determine the partition dimension of graphs. In general, determining the partition dimension of a graph is an *NP*-hard problem. Later, in [11] they characterized all graphs of order n having partition dimension 2, n, or $n-1$. In 2002, Chartrand *et al.* introduced the locating-chromatic number of graphs. This notion is a combined concept between graph partition dimension and graph coloring.

Let G be a finite, simple, and connected graph. Let c be a proper k-coloring of a connected graph G, namely $c(u) \neq c(v)$ for any adjacent vertices u and v in G. Let $\Pi = \{S_1, S_2, \cdots, S_k\}$ be a partition of $V(G)$ induced by c on $V(G)$, where S_i is the set of vertices receiving color i. The *color code* $c_\Pi(v)$ of v is the ordered k-tuple $(d(v,S_1), d(v,S_2), \ldots, d(v,S_k))$ where $d(v,S_i) = \min\{d(v,x)|x \in S_i\}$ for any i. If all distinct vertices of G have distinct color codes, then c is called a *locating-chromatic k-coloring* of G (*locating k-coloring*, in short). The *locating-chromatic number*, denoted by $\chi_L(G)$, is the smallest k such that G has a locating k-coloring.

Chartrand *et al.* [8] determined the locating-chromatic numbers for some well-known classes: paths, cycles, complete multipartite graphs and double stars. The locating-chromatic number of a path P_n is 3, for $n \geq 3$. The locating-chromatic number of a cycle C_n is 3 if n is odd and 4 if n is even. Furthermore, Chartrand *et al.* [9] considered the locating-chromatic number of trees in general. As results, they showed that for any $k \in \{3, 4, ..., n-2, n\}$, there exists a tree on n vertices with locating-chromatic number k. They also showed that no tree on n vertices with locating-chromatic number $n-1$. Recently, Asmiati *et al.* [1, 2], determined the locating-chromatic number for an amalgamation of stars and firecracker graphs. Behtoei *et al.* [5] derived the locating-chromatic number for Kneser graphs.

Some authors also consider the locating chromatic number for graphs produced by a graph operation. For instances, Baskoro *et al.* [3] determined the locating-chromatic number for the corona product of two graphs. Behtoei *et al.* [6] studied for Cartesian product of graphs and Behtoei *et al.* [7] for join product of graphs. Moreover, they [7] also determined the locating chromatic number of the fans, wheels and friendship graphs.

Clearly, the only graph on n vertices with locating chromatic number n is K_n. Furthermore, Chartrand *et al.* [9] characterized all graphs on n vertices with locating-chromatic number $n-1$. In the same paper, they also gave conditions under which $n-2$ is an upper bound for $\chi_L(G)$, namely if G is a connected graph of order $n \geq 5$ containing an induced subgraph $F \in \{2K_1 \cup K_2, P_2 \cup P_3, H_1, H_2, H_3, P_2 \cup K_3, P_2, C_5, C_5 + e\}$, then $\chi_L(G) \leq n-2$. Recently, we characterized all trees with locating-chromatic number 3 [4]. Motivated by these results, in this paper, we study connected graphs on n vertices whose locating-chromatic number is 3. In particular, we will characterize all such graphs with a cycle inside.

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DOMINANT VERTICES

Let c be a locating k-coloring on graph $G(V,E)$. Let $\Pi = \{S_1, S_2, \cdots, S_k\}$ be the ordered partition of $V(G)$ under c. A vertex $v \in G$ is called a *dominant vertex* if $d(v, S_i) = 0$ if $v \in S_i$ and 1 otherwise. A path connecting two dominant vertices in G is called a *clear path* if all of its internal vertices are not dominant. Then, we have the following lemma as a direct consequence from the definition.

Lemma 1. *Let G be a graph with $\chi_L(G) = k$. Then, there are at most k dominant vertices in G and all of them must receive different colors.* □

Asmiati *et al.* [2] showed that the length of any clear path in a graph with locating-chromatic number 3 must be odd. The proof is as follows.

Lemma 2. *[2] Let G be a graph with $\chi_L(G) = 3$. Then, the length of any clear path in G is odd.*

Proof. Let c be a locating 3-coloring on G. Let P be a clear path connecting two dominant vertices x and y in G. Assume $c(x) = 1$ and $c(y) = 2$. Since all internal vertices of P are not dominant then the colors of these vertices must be either 1 or 2, and together with x and y they form an alternating sequence. Therefore, the number of these internal vertices must be even, and it implies that the length of P is odd. □

Lemma 3. *Let G be a graph containing a cycle with $\chi_L(G) = 3$. Let c be any locating 3-coloring of G. Then, the following statements hold:*

a. *If G contains an odd cycle then G has exactly three dominant vertices and three of them are in some odd cycle.*
b. *If G contains only even cycles then G has at most three dominant vertices in which two of them must be two adjacent vertices in some of the cycles. Furthermore, each of dominant vertices in such a cycle has other neighbor which is not in the cycle.*

Proof. By Lemma 1, G has at most three dominant vertices. Since c is a locating 3-coloring of G, then there exists at least three vertices in G receiving different colors. If G contains an odd cycle, say C, then C must contain 3 colors. Now, select three vertices receiving different colors in C so that each of them has two adjacent vertices with different colors. This selection can be done, since C is an odd cycle in G. Therefore, we get exactly three dominant vertices from this selection. So, part (a) is proved.

Now if G contains only even cycles then by Lemmas 1 and 2 all vertices of any cycle C in G must receive only two colors alternatingly. Therefore, there are at most two dominant vertices in C. Assume that there is only one dominant vertex in C, say x, then vertex x must have at least the third neighbor (outside cycle C) receiving the third color different than those in C. But, now the two adjacent vertices of x in C will have the same color code, a contradiction. Therefore, there are exactly two dominant vertices x and y in C. If they are not adjacent then the two neighbors of x in C will have the same color code, a contradiction. Additionally, each of $\{x, y\}$ is required to have another neighbor not in C to make it dominant. Therefore, part (b) is proved. □

Corollary 1. *If n is odd then $\chi_L(C_n) = 3$. Otherwise, $\chi_L(C_n) = 4$.*

Proof. Let $V(C_n) = \{v_1, v_2, \cdots, v_n\}$ and $E(C_n) = \{v_1v_2, v_2v_3, \cdots, v_nv_1\}$. Since $n \geq 3$ then $\chi_L(G) \geq 3$. For odd n, define a locating-chromatic coloring c on C_n such that: $c(v_n) = 1$, $c(v_{n-1}) = 2$, $c(v_{n-2}) = 3$, and for $1 \leq i \leq n-3$, $c(v_i) = 3$ if i is odd and 1 if i is even. Therefore, $\chi_L(C_n) = 3$ if n is odd.

For even n, define let $c(v_n) = 1$, $c(v_{n-1}) = 2$, $c(v_{n-2}) = 3$, $c(v_{n-3}) = 4$, and for $1 \leq i \leq n-4$, $c(v_i) = 4$ if i is odd and 1 if i is even. Therefore, $\chi_L(C_n) \leq 4$ if n is even. However, by Lemma 3(b), the locating-chromatic number of an even cycle cannot be 3. □

MAXIMAL GRAPHS AND CHARACTERIZATION

Let $\mathscr{F}$ be the set of all graphs containing cycles with locating chromatic number 3. In this section, we will find all maximal graphs in $\mathscr{F}$. From these maximal graphs we then characterize all graphs in $\mathscr{F}$.

From now on, let $F_1 \in \mathscr{F}$ and C is a smallest odd cycle in F_1. Let x, y, z be the dominant vertices in F_1. Assume that $c(x) = 1$, $c(y) = 2$ and $c(z) = 3$. By Lemma 2, there are three clear paths connecting vertices x and y, y and z, and z and x using all vertices of C in F. Let the three paths be ${}_xP_y = (x, u_1, u_2, \cdots, u_{r-1}, u_r = y)$, ${}_yP_z = (y, v_1, v_2, \cdots, v_{s-1}, v_s = z)$, and ${}_zP_x = (z, w_1, w_2, \cdots, w_{t-1}, w_t = x)$, with r, s, t are odd. Since ${}_xP_y$ is a clear path, $c(x) = 1$, and $c(y) = 2$ then $c(u_i) = 2$ for odd i and 1 for even i (provided $r > 1$). Similarly, all internal vertices of ${}_yP_z$ have colors 3 and 2 alternatingly, and all internal vertices of ${}_zP_x$ have colors 1 and 3 alternatingly.

Lemma 4. *If $r = s = t = 1$ then $d(x) \leq 4$, $d(y) \leq 4$, and $d(z) \leq 4$.*

Proof. For a contradiction assume the degree of vertex x in F_1: $d(x) \geq 5$. Since $\chi_L(F_1)$ is 3, two neighbors of x other than y and z will have the same color code, a contradiction. Similarly it also holds for vertices y and z. □

Lemma 5. *If $r = s = 1$ and $t > 1$ then:*

a. $d(x) \leq 3$, $d(y) \leq 4$, $d(z) \leq 3$, $d(w_{\lfloor \frac{t}{2} \rfloor}) \leq 3$, $d(w_{\lfloor \frac{t+1}{2} \rfloor}) \leq 3$, *and* $d(w_i) = 2$ *for other i.*

b. There is no path in F_1 *connecting* $a \in \{x,y,z\}$ *to* $b \in \{w_{\lfloor \frac{t}{2} \rfloor}, w_{\lfloor \frac{t+1}{2} \rfloor}\}$ *without involving other vertices in C.*

c. There is no path in F_1 *connecting x to z without involving other vertices in C.*

Proof. To show $d(y) \leq 4$ we use the same argument as in Lemma 4. Now, if $d(x) \geq 4$ then the color codes of w_{t-1} and a will be the same, where a is a neighbor of x other than y and w_{t-1}, a contradiction. Therefore, $d(x) \leq 3$. A similar argument can be applied to $d(z)$, $d(w_{\lfloor \frac{t}{2} \rfloor})$ and $d(w_{\lfloor \frac{t+1}{2} \rfloor})$. Now, we will show that $d(w_i) = 2$ for any other i. Assume $d(w_i) \geq 3$ for some i. Let b be the neighbor of w_i which is not in C. Then, $c(b) = 1$ or 3. This implies that the color codes of b and one of $\{w_{i-1}, w_{i+1}\}$ will be the same, a contradiction. So, $d(w_i) = 2$ for any other i.

To show (b), now assume there is a path in F_1 connecting x to $b \in \{w_{\lfloor \frac{t}{2} \rfloor}, w_{\lfloor \frac{t+1}{2} \rfloor}\}$ without involving other vertices in C. Let denote this path by $L_1 = (x, l_1, \cdots, l_p, b)$ with $l_1 \neq w_{t-1}$. Therefore, the color of l_1 is either 1 or 3. On the other hand, we have another path L_2 connecting x and b involving only vertices in C (as a subpath of ${}_zP_x$). But now, two neighbors of x in these paths, namely l_1 and w_{t-1}, will have the same color code, a contradiction. Similarly, there is no path in F_1 connecting z to $b \in \{w_{\lfloor \frac{t}{2} \rfloor}, w_{\lfloor \frac{t+1}{2} \rfloor}\}$ without involving other vertices in C. Of course, there is no path F_1 connecting y to $b \in \{w_{\lfloor \frac{t}{2} \rfloor}, w_{\lfloor \frac{t+1}{2} \rfloor}\}$ without involving other vertices in C, since this path (if exists) will be a path with the internal vertices of colors 2 or (either 1 or 3 depending b). This implies that b becoming an extra dominant vertex, a contradiction.

To show (c), again assume there exists such a path. Then, this path is a clear path. Therefore there are two clear independent paths connecting x and z. This will implies that two neighbors of x in these paths have the same color code. □

Lemma 6. *If $r = 1$, $s > 1$, and $t > 1$ then:*

a. $d(x) \leq 3$, $d(y) \leq 3$, $d(z) = 2$, $d(v_{\lfloor \frac{s}{2} \rfloor}) \leq 3$, $d(v_{\lfloor \frac{s+1}{2} \rfloor}) \leq 3$, $d(w_{\lfloor \frac{t}{2} \rfloor}) \leq 3$, $d(w_{\lfloor \frac{t+1}{2} \rfloor}) \leq 3$ *and* $d(v_i) = d(w_i) = 2$ *for any other i.*

b. There is no path in F_1 *connecting* $a \in \{x,y\}$ *to* $b \in \{v_{\lfloor \frac{s}{2} \rfloor}, v_{\lfloor \frac{s+1}{2} \rfloor}, w_{\lfloor \frac{t}{2} \rfloor}, w_{\lfloor \frac{t+1}{2} \rfloor}\}$ *without involving other vertices in C.*

c. There is no path in F_1 *connecting* $c \in \{v_{\lfloor \frac{s}{2} \rfloor}, v_{\lfloor \frac{s+1}{2} \rfloor}\}$ *to* $d \in \{w_{\lfloor \frac{t}{2} \rfloor}, w_{\lfloor \frac{t+1}{2} \rfloor}\}$ *without involving vertices in C.*

Proof. To show $d(x) \leq 3$, $d(y) \leq 3$, $d(v_{\lfloor \frac{s}{2} \rfloor}) \leq 3$, $d(v_{\lfloor \frac{s+1}{2} \rfloor}) \leq 3$, $d(w_{\lfloor \frac{t}{2} \rfloor}) \leq 3$, $d(w_{\lfloor \frac{t+1}{2} \rfloor}) \leq 3$ and $d(v_i) = d(w_i) = 2$ for any other i, we use the same argument as in Lemma 5. Now, assume $d(z) \geq 3$. Let b be the third neighbor of z. Then, $c(b) = 2$ or 1 and the color codes of b and one of $\{v_{s-1}, w_1\}$ will be the same, a contradiction. To show (b) we use the same argument as in Lemma 5.

To show (c), assume there is a path in F_1 connecting $c \in \{v_{\lfloor \frac{s}{2} \rfloor}, v_{\lfloor \frac{s+1}{2} \rfloor}\}$ to $d \in \{w_{\lfloor \frac{t}{2} \rfloor}, w_{\lfloor \frac{t+1}{2} \rfloor}\}$ without involving vertices in C. Let denote this path by $P_1 = (c, l_1, \cdots, l_p, d)$. On the other hand, there is another path connecting c to d by using only vertices in C. Thus, there are two independent paths connecting c to d. Therefore, two neighbors of c in these paths have the same color code or there is an extra dominant vertex in this path, a contradiction. □

Lemma 7. *If $r > 1$, $s > 1$, and $t > 1$ then:*

a. $d(x) = d(y) = d(z) = 2$, $d(u_{\lfloor \frac{r}{2} \rfloor}) \leq 3$, $d(u_{\lfloor \frac{r+1}{2} \rfloor}) \leq 3$, $d(v_{\lfloor \frac{s}{2} \rfloor}) \leq 3$, $d(v_{\lfloor \frac{s+1}{2} \rfloor}) \leq 3$, $d(w_{\lfloor \frac{t}{2} \rfloor}) \leq 3$, $d(w_{\lfloor \frac{t+1}{2} \rfloor}) \leq 3$ *and* $d(u_i) = d(v_i) = d(w_i) = 2$ *for any other i.*

b. There is no path in F_1 *connecting* $a \in \{u_{\lfloor \frac{r}{2} \rfloor}, u_{\lfloor \frac{r+1}{2} \rfloor}\}$ *to* $b \in \{v_{\lfloor \frac{s}{2} \rfloor}, v_{\lfloor \frac{s+1}{2} \rfloor}, w_{\lfloor \frac{t}{2} \rfloor}, w_{\lfloor \frac{t+1}{2} \rfloor}\}$ *without involving vertices in C.*

c. There is no path in F_1 *connecting* $c \in \{v_{\lfloor \frac{s}{2} \rfloor}, v_{\lfloor \frac{s+1}{2} \rfloor}\}$ *to* $d \in w_{\lfloor \frac{t}{2} \rfloor}, w_{\lfloor \frac{t+1}{2} \rfloor}\}$ *without involving vertices in C.*

Proof. To show (a), (b), and (c) we use a similar argument as in Lemmas 5 and 6. □

Lemma 8. *Let* $F_1 \in \mathscr{F}$ *and C is a smallest odd cycle in* F_1*. If* $a \in V(F_1) \backslash C$ *then the degree* $d(a) \leq 3$*.*

Proof. Let $a \in V(F_1) \backslash C$. Since F_1 is connected, then there exists a path P connecting a to a vertex of C. Let b be the first vertex in C is traversed by path P. For a contradiction assume $d(a) \geq 4$. Let a_1, a_2, a_3, a_4 be the neighbors of a. Assume $c(a) = 1$. Since F contains an odd cycle then by Lemma 3 all dominant vertices will be in C. Therefore the colors of all neighbors of a must be the same, say 2. Now, let $d(a, S_3) = t$. Then, $d(a_i, S_3)$ is $t-1$, t, or $t+1$, for $i = 1,2,3,4$. Therefore, there are two vertices a_i will have the same color code, a contradiction. Thus, $d(a) \leq 3$. □

Lemma 9. *Let* $F_1 \in \mathscr{F}$ *and C is a smallest odd cycle in* F_1*. Let x and y be dominant vertices in C with the length* *of the clear path between them is* 1. *Then, all paths (if any) in F_1 connecting x to y without involving other vertices in C must have different odd length. Furthermore, all such paths induce a ladder in F_1 maximally.*

Proof. Let P in F_1 be any path connecting x to y without involving other vertices in C. Consequently, P is a clear path. By Lemma 2, P has odd length. Therefore, the length of any path connecting x to y without involving other vertices in C is odd. Now, assume we have two different such paths with the same length: $P_1 = (x, l_1, \cdots, l_i, \cdots, l_p, y)$ and $P_2 = (x, m_1, \cdots, m_i, \cdots, m_p, y)$. But, then we have $c_\Pi(l_i) = c_\Pi(m_i)$ for any i. This means that $l_i = m_i$ for any i, a contradiction.

Now, consider all such paths connecting x to y without involving other vertices in C. Thus, these paths are clear paths, and all vertices in these paths have colors either $c(x)$ or $c(y)$. So, the subgraph induced by these paths will have locating-chromatic number 2. Thus, its chromatic number will be also 2. By Lemma 8 the maximum degree of every vertex in $V(F_1)\backslash C$ is 3. As a result, These paths maximally will form a ladder (bipartite subgraph) of F_1. □

Lemma 10. *Let $F_1 \in \mathscr{F}$ and C is a smallest odd cycle in F_1. Let x and y be dominant vertices in C with the length of the clear path between them is $r(> 1)$. Then, all paths (if any) in F_1 connecting $u_{\lfloor \frac{r}{2} \rfloor}$ to $u_{\lfloor \frac{r+1}{2} \rfloor}$ without involving vertices in C must have different odd length. Furthermore, all such paths induce a ladder in F_1 maximally.*

Proof. Let P in F_1 be a path connecting $u_{\lfloor \frac{r}{2} \rfloor}$ to $u_{\lfloor \frac{r+1}{2} \rfloor}$ without involving vertices in C. Since all internal vertices of P only can be colored by 1 and 2, alternatingly, then P is clear path. By Lemma 2, P has odd length. To show that all such paths have different length and they form a ladder maximally, we use a similar argument as in Lemma 9. □

Theorem 1. *Let C be a fixed odd cycle. Then, there are exactly four types of maximal graphs as depicted in Figure 1 with locating-chromatic number 3 containing C as a smallest odd cycle.*

Proof. Let $F_1 \in \mathscr{F}$ and C is a smallest odd cycle in F_1. Then, by Lemma 3(a), C contains three dominant vertices of F_1. Let x, y, z be the three dominant vertices of F_1. Assume that $c(x) = 1$, $c(y) = 2$ and $c(z) = 3$. By Lemma 2, there are three clear paths using only vertices in C with odd length, namely: ${}_xP_y = (x, u_1, u_2, \cdots, u_{r-1}, u_r = y)$, ${}_yP_z = (y, v_1, v_2, \cdots, v_{s-1}, v_s = z)$, and ${}_zP_x = (z, w_1, w_2, \cdots, w_{t-1}, w_t = x)$, with r, s, t are odd.

Now, consider if length of C is 3. In this case, $r = s = t = 1$. By Lemma 4, we have that $d(x) \leq 4$, $d(y) \leq 4$, $d(z) \leq 4$. Lemmas 8 and 9 implies that if there are other clear paths connecting every two dominant vertices and without involving other vertices in C then such paths (if all exists) will induce three ladders maximally with the common vertices x, y and z. Therefore, for this type we obtain the graph depicted in Figure 1(i) as the maximal graph with locating chromatic number 3 and having a triangle as a smallest cycle.

Now, consider the length of C is greater than 3. Then, we have the following three cases.

Case 1. $r = s = 1$ and $t > 1$.
By Lemmas 5, 9 and 10, F_1 will be maximal if it has three ladders with two ladders having one common vertex as depicted in Figure 1(ii).

Case 2. $r = 1$, $s > 1$, and $t > 1$.
By Lemmas 6, 9 and 10, F_1 will be maximal if it has three independent ladders as depicted in Figure 1(iii).

Case 3. $r > 1$, $s > 1$, and $t > 1$.
By Lemmas 7 and 10, F_1 will be maximal if it has three independent ladders as depicted in Figure 1(iv). □

Theorem 2. *Let F be any graph having a smallest odd cycle C. F has locating-chromatic number 3 if and only if F is a subgraph of one of the graphs in Figure 1 which every vertex $a \notin C$ of degree 3 must be lie in a path connecting two different vertices in C.*

Proof. Let F be a graph having C as a smallest cycle.

($\leftarrow$) Let F be a subgraph of one of the graphs in Figure 1 in which every vertex $a \notin C$ of degree 3 must be lie in a path connecting two different vertices in C. Then F contains C and 'almost' independent subgraphs of three ladders (that may have one common vertex in dominant vertex for each pair). Such subgraphs have the property that every vertex of degree 3 in this induced subgraph lie in a path connecting two different vertices in C. Such path must be connecting either two dominant vertices or two mid vertices in a clear path involving vertices only in C.

Let x, y, z be the dominant vertices in C. Color $c(x) = 1, c(y) = 2, c(z) = 3$, and color the internal vertices of the clear paths involving vertices in C: ${}_xP_y$, ${}_yP_z$, ${}_zP_x$ by 1,2, or 2,3, or 3,1, alternatively and respectively. Colors each subgraph of the ladder attaching to the vertices a and b in C by two colors $c(a)$ and $c(b)$. By this coloring, we have a locating 3-coloring in F. Since F contains an odd cycle then $\chi_L(F) \geq 3$. Therefore, F has locating-chromatic number 3.

($\rightarrow$) If F has locating-chromatic number 3 and having C as a smallest odd cycle then by Lemmas 4-10 we know

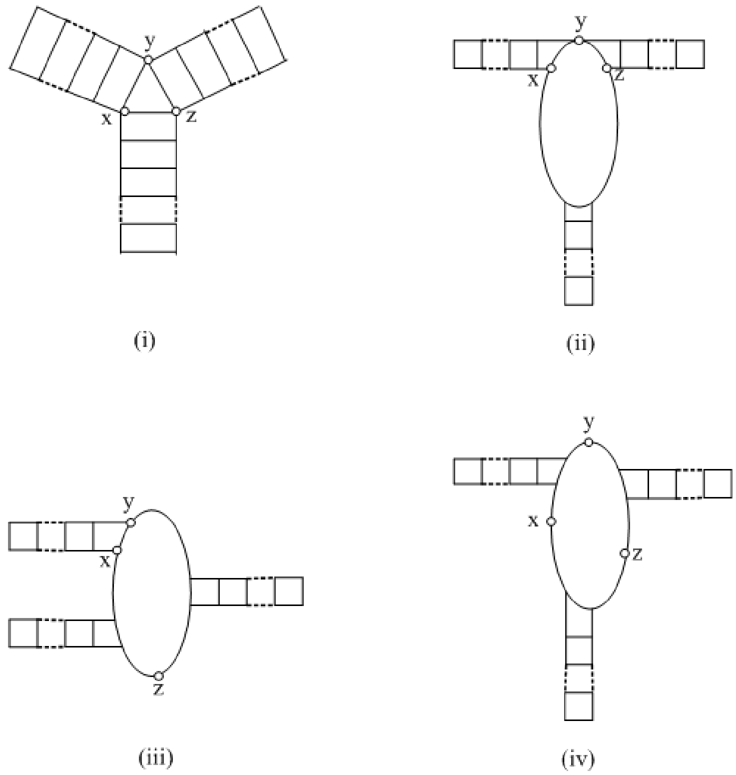

FIGURE 1. The four types of maximal graphs in $\mathscr{F}$ containing an odd cycle.

the restrictions of all the degrees of vertices of C in F. We also know that the conditions of all vertices outside C and how they are connected to vertices in C. If there is a vertex $a \in F \backslash C$ of degree 3 and this vertex is not lie in any path connecting two vertices in C then this vertex must be connected to a cycle C by a path ${}_bP_a$ with initial vertex b in C (since F is connected). Let a_1, a_2, a_3 be the neighbors of a in F, where a_1 is in such path ${}_bP_a$. Of course $c(a_2) = c(a_3)$, since otherwise a will be an extra dominant vertex in F (recall that C has already 3 dominant vertices). Thus, we obtain the color codes of a_2 and a_3 will be the same, a contradiction. Therefore, if any vertex of degree 3 in $F \backslash C$ will be lie in a path connecting two (adjacent) vertices in C. Therefore, by these lemmas and Theorem 1 we conclude that F must be a subgraph of one the graphs in Figure 1 in which every vertex $a \notin C$ of degree 3 must be lie in a path connecting two different vertices in C. □

Next, we will characterize all graphs containing only even cycles with locating-chromatic number 3. For this purpose, from now on, let $F_2 \in \mathscr{F}$ and having only even cycles. By Lemma 3(b), F_2 will have at most three dominant vertices in which two of them are two adjacent vertices in some cycle. Let C be the maximal even cycle containing the two dominant vertices. Let x, y, z be the dominant vertices (if there are three) in F_2 and x, y lie in C. Assume $c(x) = 1$ and $c(y) = 2$. Therefore, the color of vertices in C must be 1 and 2, alternatingly. By Lemma 3(b), each dominant vertex in C must have a neighbor not in C. Let $ux, vy \in E(F_2)$ where $u, v \notin C$. Since x, y are dominant vertices, we must have $c(u) = c(v) = 3$. Then, we have the following lemma.

Lemma 11. *Let C be the maximal even cycle in F_2 that have two dominant vertices x and y. Let $V(C) = \{x = a_1, a_2, \ldots, a_{h-1}, a_h = y\}$. Then:*

a. *$d(x) = d(y) = 3$, $d(a_i) \leq 3$ for any other i. Furthermore for the third neighbor of a_i, if $a_i w \in E(F_2)$ and $i \notin \{\frac{h}{2}, \frac{h}{2} + 1\}$ then $w = a_{h+1-i}$.*
b. *There is no path in F_2 connecting $b \in \{u, v\}$ to a_i for $i \notin \{1, h\}$ without involving other vertices in C.*
c. *If F_2 has the third dominant vertex z then one of the following statements is true:*
   - *If $z = v$ then $d(u) \leq 2$ and $d(v) \leq 3$.*
   - *If $z = u$ then $d(u) \leq 3$ and $d(v) \leq 2$.*

- *If $z \notin \{u,v\}$ then $d(z) \le 2$, $d(u) \le 2$, $d(v) \le 2$, and there is a path connecting z to either u or v.*

d. *If $w \in V(F_2)\backslash C$ then there is a path connecting from w to either $u, v, a_{\frac{h}{2}}$ or $a_{\frac{h}{2}+1}$, and all internal vertices of this path have degree 2.*

Proof. To show $d(x) = d(y) = 3$. Assume $d(x) \ge 4$. Then, there are two neighbors of x having the same color code, a contradiction. A similar argument is needed to show $d(y) = 3$. Now, assume $d(a_i) \ge 4$ for some i. If $c(a_i) = 1$ then each neighbor of a_i will have color 2. If $d(a_i, S_3) = t$ then $d(w, S_3) = t-1, t$, or $t+1$, where w is any neighbor of a_i. Then, there are two neighbors of a_i having the same color code, a contradiction. Now, if $d(a_i) = 3$ for $i \notin \{\frac{h}{2}, \frac{h}{2}+1\}$ then its third neighbor must be a_{h+1-i}, otherwise its two neighbors will have the same color code, a contradiction.

To show (b), assume there is a path L_1 in F_2 connecting b to a_i for some $i \notin \{1, h\}$ without involving other vertices in C. Then, this path together with the path L_2 connecting b to a_i using vertices in C will form an even cycle. Since $c(b) = 3$ and the colors of internal vertices of L_2 are 1 and 2 alternatively, then they will give four dominant vertices, a contradiction.

To show (c), let z be the third dominant vertex of F_2. If $z = v$ then clearly $d(v) \le 3$. Assume $d(u) \ge 3$. Since u is not dominant, then three of its neighbors must have the same color, then two of neighbors will have the same color code, a contradiction. A similar argument can be applied for the case of $u \in F_2$ being a dominant vertex.

Now, if $z \notin \{u, v\}$. By Lemma 11(a), we cannot have a path connecting z to a_i for $i \notin \{\frac{h}{2}, \frac{h}{2}+1\}$ without passing either u or v. There is also no such path connecting z to a_i for $i \in \{\frac{h}{2}, \frac{h}{2}+1\}$, since otherwise it will create an extra dominant vertex. Therefore, there must be a path connecting z to either u or v. To avoid the same color code, all internal vertices in this path have degree 2 and $d(z) \le 2$.

The statement of (d) is a consequence of statements (a) and (c). □

Lemma 12. *Let C be the maximal even cycle in F_2 that have two dominant vertices x and y. Then, all paths (if any) in F_2 connecting x to y with involving only vertices in C have different odd length. Furthermore, all such paths (if any) in F_2 induce a ladder maximally.*

Proof. Let P in F_2 be a path connecting x to y with involving only vertices in C. Consequently, P is a clear path. By Lemma 2, P must have odd length. Now, assume we have two different clear paths with the same length $P_1 = (x = a_1, a_2, \cdots, a_i, \cdots, a_{h-1}, a_h = y)$ and $P_2 = (x = b_1, b_2, \cdots, b_i, \cdots, b_{h-1}, b_h = y)$. Therefore, $c_\Pi(a_i) = c_\Pi(b_i)$ for any i. This means that $a_i = b_i$ for any i, a contradiction.

Now, consider all such paths connecting x to y involving only vertices in C. Thus, these paths are clear paths, and all vertices in these paths have colors either 1 or 2. So, the subgraph induced by these paths will have locating-chromatic number 2. Thus, its chromatic number will be also 2. By Lemma 11(a) the maximum degree of every vertex in C is 3. As a result, These paths maximally will form a ladder (bipartite subgraph) of F_2. □

Theorem 3. *Let C be a fixed even cycle. Then, the graph in Figure 2 is the maximal graph $F_2 \in \mathscr{F}$ containing only even cycles and C as a largest even cycle.*

Proof. Let C be a largest even cycle in $F_2 \in \mathscr{F}$, where $V(C) = \{x = a_1, a_2, \ldots, a_{h-1}, a_h = y\}$. Then, by Lemma 3(b) C has two dominant vertices, x, y and let $xu, yv \in E(F_2)$, where $u, v \notin C$. Let $c(x) = 1, c(y) = 2$. Then, $c(a_i) = 1$ and 2, alternatingly. By Lemma 12, all clear paths (if any) connecting vertices x and y will induce a ladder maximally. Considering the facts from Lemma 11, we can conclude that the graph in Figure 2 is the maximal graph F_2 containing only even cycles and C is a largest cycle in F_2. □

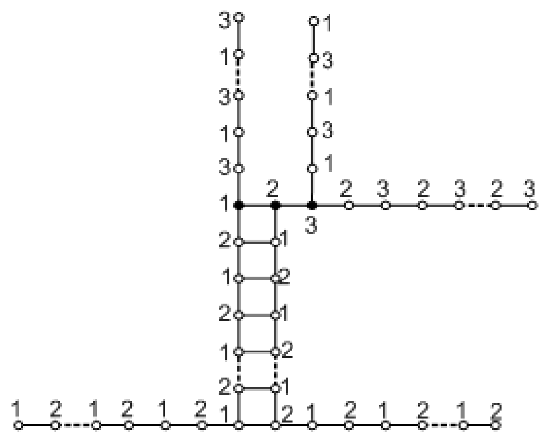

FIGURE 2. The maximal graph in $\mathscr{F}$ containing even cycles only.

Theorem 4. *Let F be a graph having only even cycles and C as a largest even cycle. F has locating-chromatic number 3 if and only if F is a subgraph of the graph in Figure 2 with vertices u and v in F.*

Proof. Let F be a graph having only even cycles and C as a largest even cycle.

($\leftarrow$) If F is a subgraph of one of the graphs in Figure 2 with vertices u and v in F then F contains C and some paths connecting to either $u, v, a_{\frac{h}{2}}$ or $a_{\frac{h}{2}+1}$. It is obvious that F can be colored 3 (the smallest). Therefore,

$\chi_L(F) = 3$.

($\rightarrow$) If F has locating-chromatic number 3 and all cycles in F are even and C is the smallest even cycle then by Lemmas 11 and 12, we know the restrictions of all the degrees of vertices of C and vertices u and v in F. We also know that the conditions of all other vertices in F and how they are connected to vertices in C and/or u and v. Therefore, by these lemmas and Theorem 3 we conclude that F must be a subgraph of the graph in Figure 2 with u and v in F. □

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REFERENCES

1. Asmiati, H. Assiyatun, E. T. Baskoro, Locating-Chromatic Number of Amalgamation of Stars, *ITB J.Sci.* **43A**, 1-8 (2011).
2. Asmiati, E.T. Baskoro, H. Assiyatun, D. Suprijanto, R. Simanjuntak, S. Uttunggadewa, The Locating-Chromatic Number of Firecracker Graphs, *Far East Journal of mathematical Sciences*, 63:1 (2012), 11-23.
3. E. T. Baskoro, I. A. Purwasih, The locating-chromatic number for corona product of graphs, submitted (2011).
4. E. T. Baskoro, Asmiati, Characterizing all trees with locating-chromatic number 3, submitted (2011).
5. A. Behtoei, B. Omoomi, On the locating chromatic number of kneser graphs, *Disrete Applied Mathematics* **159**, 2214-2221 (2011).
6. A. Behtoei, B. Omoomi, On the locating chromatic number of the cartesian product of graphs, *to appear in Ars. Combin.* (2011).
7. A. Behtoei, B. Omoomi, The locating chromatic number of the join of graphs, submitted (2011).
8. G. Chartrand, D. Erwin, M. A. Henning, P.J. Slater, P. Zang , The locating-chromatic number of a graph, *Bulls. Inst. Combin. Appl.* **36**, 89-101 (2002).
9. G. Chartrand, D. Erwin, M. A. Henning, P. J. Slater, P. Zang , Graph of order n with locating-chromatic number $n-1$, *Discrete Mathematics* **269**, 65-79 (2003).
10. G. Chartrand, E. Salehi, P. Zhang, On the partition dimension of graph, *Congr. Numer.* **130**, 157-168 (1998).
11. G. Chartrand, E. Salehi, P. Zhang, The partition dimension of a graph, *Aequationes Math.* **59**, 45-54 (2000).