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THE IMPACT OF THE MONOID HOMOMORPHISM ON THE STRUCTURE OF SKEW GENERALIZED POWER SERIES RINGS

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Abstract

Let R be a ring, $(S, \leq)$ be a strictly ordered monoid and $\omega : S \to End(R)$ be a monoid homomorphism. In this paper, we study the properties of monoid homomorphism ω and its impact on the structure of skew generalized power series ring $R[[S, \omega]]$. We show that: if $\omega^{(1)} \sim \omega^{(2)}$, then $R_1[[S, \omega^{(1)}]] \cong R_2[[S, \omega^{(2)}]]$, and $R_1 \oplus R_2[[S, \omega^{(1)} \oplus \omega^{(2)}]] \cong R_1[[S, \omega^{(1)}]] \oplus R_2[[S, \omega^{(2)}]]$.

1. Introduction

In 2007, Mazurek and Ziembowski [1] constructed a new ring which is the generalization of generalized power series rings (GPSR) $R[[S]]$ that was

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constructed by Ribenboim [2] by using a monoid homomorphism $\omega : S \to End(R)$ to change the convolution product on GPSR $R[[S]]$. Furthermore, this new ring is known as *skew generalized power series ring* (SGPSR) denoted by $R[[S, \omega]]$ or $R[[S, \omega, \leq]]$. Now we will give the definition and some examples of SGPSR $R[[S, \omega]]$.

Regarding ordered sets, ordered monoids, artinian and narrow set, we will follow the terminology used in [2-6]. Now, we recall the construction of SGPSR [1]. Let $(S, \leq)$ be a strictly ordered monoid, R be a commutative ring with an identity element and $\omega : S \to End(R)$ be a monoid homomorphism. For any $s \in S$ let ω_s denote the image of s under ω, i.e., $\omega(s) = \omega_s$.

Define $R^S = \{f \mid f : S \to R\}$ and $R[[S, \omega]] = \{f \in R^S \mid supp(f)$ is artinian and narrow$\}$, where $supp(f) = \{s \in S \mid f(s) \neq 0\}$.

Under pointwise addition and skew convolution, multiplication defined by

$$(fg)(s) = \sum_{(x, y) \in \chi_s(f, g)} f(x)\omega_x(g(y)), \tag{1}$$

for all $f, g \in R[[S, \omega]]$, where

$$\chi_s(f, g) = \{(x, y) \in supp(f) \times supp(g) \mid xy = s\}$$

is finite, $R[[S, \omega]]$ is a ring which is known as *skew generalized power series ring* (SGPSR).

Some special cases of SGPSR $R[[S, \omega]]$ are given by the following example.

Example 1.1. Let R be a ring, id_R be an identity map in $End(R)$, N_0 be a set of positive integers, $\mathbb{Z}$ be a set of integers and $(S, \leq)$ be a strictly ordered monoid.

(1) If $S = N_0$ with usual addition, trivial order $\leq$ and $\omega_s = id_R$, for all $s \in S$, then SGPSR $R[[S, \omega]]$ is polynomial ring $R[X]$.

(2) If $S = \mathbb{Z}$ with usual addition, trivial order $\leq$ and $\omega_s = id_R$, for all $s \in S$, then SGPSR $R[[S, \omega]]$ is Laurent polynomial ring $R[X, X^{-1}]$.

(3) If $S = N_0$ with usual addition, trivial order $\leq$ and $\omega_0 = \sigma$, for some endomorphism ring $\sigma \in End(R)$, then SGPSR $R[[S, \omega]]$ is skew polynomial ring $R[X; \sigma]$.

(4) If $S = N_0$ with usual addition, usual order $\leq$ and $\omega_s = id_R$, for all $s \in S$, then SGPSR $R[[S, \omega]]$ is power series ring $R[[X]]$.

(5) If $S = \mathbb{Z}$ with usual addition, usual order $\leq$ and $\omega_s = id_R$, for all $s \in S$, then SGPSR $R[[S, \omega]]$ is Laurent series ring $R[[X, X^{-1}]]$.

(6) If $S = N_0$ with usual addition, usual order $\leq$ and $\omega_0 = \sigma$, for some endomorphism ring $\sigma \in End(R)$, then SGPSR $R[[S, \omega]]$ is skew power series ring $R[[X; \sigma]]$.

(7) If $\omega_s = id_R$, for all $s \in S$, then SGPSR $R[[S, \omega]]$ is generalized power series ring $[[R^{(S, \leq)}]] = R[[S]]$.

2. Main Results

In this section, we give the definition and some properties of monoid homomorphism ω and its impact on the structure of SGPSR $R[[S, \omega]]$. First, we give the definition of equivalency of two monoid homomorphism.

Definition 2.1. Let R_1 and R_2 be rings, $(S, \leq)$ be a strictly ordered monoid, and $\omega^{(1)} : S \to End(R_1)$ and $\omega^{(2)} : S \to End(R_2)$ be monoid homomorphisms. Then $\omega^{(1)}$ and $\omega^{(2)}$ are said to be *equivalent* if there exists an isomorphism $\varphi : R_1 \to R_2$ such that $\omega_s^{(2)} = \varphi\omega_s^{(1)}\varphi^{-1}$ for all $s \in S$. In this case, we write $\omega^{(1)} \sim \omega^{(2)}$.

Example 2.2. Let $S = \mathbb{N}_0$, $R_1 = \mathbb{Q} \times \mathbb{Q} = \mathbb{Q}^2$ and $R_2 = \mathbb{Z} \times \mathbb{Z} = \mathbb{Z}^2$. With operation

$$(x, y) + (m, n) = (x + m, y + n) \text{ and } (x, y)(m, n) = (xm, yn),$$

R_1 and R_2 become rings and S becomes a strictly ordered commutative monoid with pointwise addition and usual order. For any $s \in S$, $(p, q) \in R_1$ and $(x, y) \in R_2$, we define monoid homomorphism

$$\omega^{(1)} : S \to End(R_1),$$

where $\omega_s^{(1)}(p, q) = (0, q)$, and

$$\omega^{(2)} : S \to End(R_2),$$

where $\omega_s^{(2)}(x, y) = (x, 0)$.

Next, we define a map

$$\varphi : R_1 \to R_2$$

with $\varphi(p, q) = (q, p)$ for all $(p, q) \in R_1$.

Since for any $(p, q), (m, n) \in R_1$, imply

$$\begin{aligned} \varphi((p, q) + (m, n)) &= \varphi((p + m, q + n)) \\ &= (q + n, p + m) \\ &= (q, p) + (n, m) \\ &= \varphi((p, q)) + \varphi((m, n)) \end{aligned}$$

and

$$\begin{aligned} \varphi((p, q)(m, n)) &= \varphi((pm, qn)) \\ &= (qn, pm) \\ &= (q, p)(n, m) \\ &= \varphi((p, q))\varphi((m, n)), \end{aligned}$$

φ is a ring homomorphism.

Furthermore, if $\varphi((p, q)) = \varphi((m, n))$, then $(q, p) = (n, m)$, which is $q = n$ and $p = m$. In other words, we have $(p, q) = (m, n)$. Hence, φ is an injective homomorphism. For any $(x, y) \in R_2$, there exists $(p, q) \in R_1$ with $p = y$ and $q = x$ such that $\varphi((p, q)) = (q, p) = (x, y)$. Then, φ is a surjective homomorphism. In other words, $\varphi : R_1 \to R_2$ is a ring isomorphism.

Moreover, since

$$\begin{aligned}
\omega_s^{(2)}\varphi((p, q)) &= \omega_s^{(2)}(\varphi((p, q))) \\
&= \omega_s^{(2)}((q, p)) \\
&= (q, 0) \\
&= \varphi((0, q)) \\
&= \varphi(\omega_s^{(1)}((p, q))) \\
&= \varphi\omega_s^{(1)}((p, q)),
\end{aligned}$$

$\omega^{(1)} \sim \omega^{(2)}$.

Based on Definition 2.1, the impact of equivalency of two monoid homomorphisms on the structure of SGPSR $R[[S, \omega]]$ is given by the following proposition.

Proposition 2.3. *Let* R_1 *and* R_2 *be rings,* $(S, \leq)$ *be a strictly ordered monoid, and* $\omega^{(1)} : S \to End(R_1)$ *and* $\omega^{(2)} : S \to End(R_2)$ *be monoid homomorphisms. If* $\omega^{(1)} \sim \omega^{(2)}$, *then* $R_1[[S, \omega^{(1)}]] \cong R_2[[S, \omega^{(2)}]]$.

Proof. Suppose $\omega^{(1)} \sim \omega^{(2)}$. Then by Definition 2.1, there exists an isomorphism $\varphi : R_1 \to R_2$ such that $\omega_s^{(2)} = \varphi\omega_s^{(1)}\varphi^{-1}$ for all $s \in S$. Next, we define a map

$$\psi : R_1[[S, \omega^{(1)}]] \to R_2[[S, \omega^{(2)}]],$$

where $\psi(f) = \bar{f} = \varphi \circ f$ for all $f \in R_1[[S, \omega^{(1)}]]$.

For all $s \in S$ and $f, g \in R_1[[S, \omega^{(1)}]]$, we have

$$\begin{aligned}
\varphi \circ (f + g)(s) &= \varphi((f + g)(s)) \\
&= \varphi(f(s) + g(s)) \\
&= \varphi(f(s)) + \varphi(g(s)) \\
&= (\varphi \circ f)(s) + (\varphi \circ g)(s)
\end{aligned}$$

and

$$\begin{aligned}
(\varphi \circ (fg))(s) &= \varphi((fg)(s)) \\
&= \varphi\left(\sum_{s=xy} f(x)\omega_x^{(1)}(g(y))\right) \\
&= \sum_{s=xy} \varphi(f(x)\omega_x^{(1)}(g(y))) \\
&= \sum_{s=xy} \varphi(f(x))\varphi(\omega_x^{(1)}(g(y))) \\
&= \sum_{s=xy} (\varphi \circ f)(x)(\varphi \circ \omega_x^{(1)})(g(y)) \\
&= \sum_{s=xy} (\varphi \circ f)(x)(\omega_x^{(2)} \circ \varphi)(g(y)) \\
&= \sum_{s=xy} (\varphi \circ f)(x)\omega_x^{(2)}(\varphi(g(y))) \\
&= \sum_{s=xy} (\varphi \circ f)(x)\omega_x^{(2)}((\varphi \circ g)(y)) \\
&= ((\varphi \circ f)(\varphi \circ g))(s).
\end{aligned}$$

Since $supp(\bar{f}) \subseteq supp(f)$, $\bar{f} \in R_2[[S, \omega^{(2)}]]$. Then, we have

$$\begin{aligned}\psi(f+g) &= \overline{f+g} \\ &= \varphi \circ (f+g) \\ &= (\varphi \circ f) + (\varphi \circ g) \\ &= \bar{f} + \bar{g} \\ &= \psi(f) + \psi(g)\end{aligned}$$

and

$$\begin{aligned}\psi(fg) &= \overline{fg} \\ &= \varphi \circ (fg) \\ &= (\varphi \circ f)(\varphi \circ g) \\ &= \bar{f}\bar{g} \\ &= \psi(f)\psi(g),\end{aligned}$$

for all $f, g \in R_1[[S, \omega^{(1)}]]$. In other words, the map $\psi : R_1[[S, \omega^{(1)}]] \to R_2[[S, \omega^{(2)}]]$ is a ring homomorphism.

Now, we will show that ψ is injective. Let $f \in Ker(\psi)$. Then $\psi(f) = 0$. Then, for all $s \in S$, we have $(\varphi \circ f)(s) = 0(s)$. In other words, $\varphi(f(s)) = 0$. Since φ is a ring isomorphism, $f(s) = 0$, for all $s \in S$. Then $Ker(\psi) = 0$, so ψ is injective.

Furthermore, we will show that ψ is surjective. For all $g \in R_2[[S, \omega^{(2)}]]$, there exists $h = \varphi^{-1} \circ g \in R_1[[S, \omega^{(1)}]]$ such that $\psi(h) = \bar{h} = \varphi \circ h = \varphi \circ \varphi^{-1} g = g$. Then ψ is surjective. So $R_1[[S, \omega^{(1)}]] \cong R_2[[S, \omega^{(2)}]]$. □

Now we will give the definition of direct sum of two monoid homomorphisms.

Definition 2.4. Let R_1 and R_2 be rings, $(S, \leq)$ be a strictly ordered monoid, and $\omega^{(1)} : S \to End(R_1)$ and $\omega^{(2)} : S \to End(R_2)$ be monoid homomorphisms. Then the *direct sum of* $\omega^{(1)}$ *and* $\omega^{(2)}$ is defined by

$$\omega^{(1)} \oplus \omega^{(2)} : S \to End(R_1 \oplus R_2),$$

where

$$(\omega^{(1)} \oplus \omega^{(2)})_s(r_1, r_2) = (\omega_s^{(1)}(r_1), \omega_s^{(2)}(r_2)),$$

for all $s \in S$ and $(r_1, r_2) \in R_1 \oplus R_2$.

Example 2.5. Let monoid S, rings R_1 and R_2, $\omega^{(1)}$ and $\omega^{(2)}$ be given as in Example 2.2. Then, we can define the *direct sum of* $\omega^{(1)}$ *and* $\omega^{(2)}$ by

$$\omega^{(1)} \oplus \omega^{(2)} : S \to End(R_1 \oplus R_2),$$

where

$$\begin{aligned}(\omega^{(1)} \oplus \omega^{(2)})_s((p, q), (x, y)) &= (\omega_s^{(1)}((p, q)), \omega_s^{(2)}((x, y))) \\ &= ((0, q), (x, 0)),\end{aligned}$$

for all $s \in S$ and $((p, q), (x, y)) \in R_1 \oplus R_2$.

The following lemma shows that the direct sum $\omega^{(1)} \oplus \omega^{(2)}$ that defined in Definition 2.4 is a monoid homomorphism.

Lemma 2.6. *Let* R_1 *and* R_2 *be rings*, $(S, \leq)$ *be a strictly ordered monoid*, *and* $\omega^{(1)} : S \to End(R_1)$ *and* $\omega^{(2)} : S \to End(R_2)$ *be monoid homomorphisms*. *Then the direct sum*

$$\omega^{(1)} \oplus \omega^{(2)} : S \to End(R_1 \oplus R_2)$$

is a monoid homomorphism.

Proof. For any $s, t \in S$ and $(r_1, r_2) \in R_1 \oplus R_2$, we have

$$\begin{aligned}(\omega^{(1)} \oplus \omega^{(2)})_{st}(r_1, r_2) &= (\omega^{(1)}_{st}(r_1), \omega^{(2)}_{st}(r_2)) \\ &= ((\omega^{(1)}_s \omega^{(1)}_t)(r_1), (\omega^{(2)}_s \omega^{(2)}_t)(r_2)) \\ &= (\omega^{(1)}_s(\omega^{(1)}_t(r_1)), \omega^{(2)}_s(\omega^{(2)}_t(r_2))) \\ &= (\omega^{(1)} \oplus \omega^{(2)})_s(\omega^{(1)}_t(r_1), \omega^{(2)}_t(r_2)) \\ &= ((\omega^{(1)} \oplus \omega^{(2)})_s(\omega^{(1)} \oplus \omega^{(2)})_t)(r_1 r_2).\end{aligned}$$

Hence, we obtain

$$(\omega^{(1)} \oplus \omega^{(2)})(st) = (\omega^{(1)} \oplus \omega^{(2)})(s)(\omega^{(1)} \oplus \omega^{(2)})(t).$$

So the direct sum $\omega^{(1)} \oplus \omega^{(2)}$ is monoid homomorphism. □

Now, based on Definition 2.4 and Lemma 2.6 we get the following proposition.

Proposition 2.7. *Let* R_1 *and* R_2 *be rings*, $(S, \leq)$ *be a strictly ordered monoid*, *and* $\omega^{(1)} : S \to End(R_1)$ *and* $\omega^{(2)} : S \to End(R_2)$ *be monoid homomorphisms*. *Then*

$$(R_1 \oplus R_2)[[S, \omega^{(1)} \oplus \omega^{(2)}]] \cong R_1[[S, \omega^{(1)}]] \oplus R_2[[S, \omega^{(2)}]].$$

Proof. Let $i_1 : R_1 \to R_1 \oplus R_2$ and $i_2 : R_2 \to R_1 \oplus R_2$ be natural injections, and let $p_1 : R_1 \oplus R_2 \to R_1$ and $p_2 : R_1 \oplus R_2 \to R_2$ be natural projections. Then we have

$$\omega^{(1)}_s = p_1(\omega^{(1)} \oplus \omega^{(2)})_s i_1$$

and

$$\omega_s^{(2)} = p_2(\omega^{(1)} \oplus \omega^{(2)})_s i_2,$$

as seen in the following diagram:

$$\begin{array}{ccccccc} R_1 & \xrightarrow{i_1} & R_1 \oplus R_2 & \xleftarrow{i_2} & R_2 \\ \downarrow \omega_s^{(1)} & & \downarrow (\omega^{(1)} \oplus \omega^{(2)})_s & & \downarrow \omega_s^{(2)} \\ R_1 & \xleftarrow{p_1} & R_1 \oplus R_2 & \xrightarrow{p_2} & R_2 \end{array}$$

Then we obtain

$$\begin{aligned} \omega_s^{(1)} p_1 &= p_1(\omega^{(1)} \oplus \omega^{(2)})_s i_1 p_1 \\ &= p_1(\omega^{(1)} \oplus \omega^{(2)})_s id_{R_1} \\ &= p_1(\omega^{(1)} \oplus \omega^{(2)})_s \end{aligned}$$

and

$$\begin{aligned} \omega_s^{(2)} p_2 &= p_2(\omega^{(1)} \oplus \omega^{(2)})_s i_2 p_2 \\ &= p_2(\omega^{(1)} \oplus \omega^{(2)})_s id_{R_2} \\ &= p_2(\omega^{(1)} \oplus \omega^{(2)})_s. \end{aligned}$$

Now, for any $f \in (R_1 \oplus R_2)[[S, \omega^{(1)} \oplus \omega^{(2)}]]$, we define a map

$$\psi : (R_1 \oplus R_2)[[S, \omega^{(1)} \oplus \omega^{(2)}]] \to R_1[[S, \omega^{(1)}]] \oplus R_2[[S, \omega^{(2)}]]$$

by $\psi(f) = (f_1, f_2)$, where $f_1 = p_1 \circ f$ and $f_2 = p_2 \circ f$.

For $i = 1, 2,$ we will show $p_i \circ (f + g) = (p_i \circ f) + (p_i \circ g)$ and $p_i \circ (fg) = (p_i \circ f)(p_i \circ g)$. For any $s \in S$, $f, g \in (R_1 \oplus R_2)[[S, \omega^{(1)} \oplus \omega^{(2)}]]$ and $i = 1, 2,$ we have

$$\begin{aligned}(p_i \circ (f+g))(s) &= p_i((f+g)(s))\\ &= p_i(f(s)+g(s))\\ &= p_i(f(s))+p_i(g(s))\\ &= (p_i \circ f)(s)+(p_i \circ g)(s)\end{aligned}$$

and

$$\begin{aligned}(p_i \circ (fg))(s) &= p_i((fg)(s))\\ &= p_i\left(\sum_{s=xy} f(x)(\omega^{(1)} \oplus \omega^{(2)})_s(g(y))\right)\\ &= \sum_{s=xy} p_i f(x) p_i(\omega^{(1)} \oplus \omega^{(2)})_s(g(y))\\ &= \sum_{s=xy} p_i f(x)\omega_s^{(1)} p_i(g(y))\\ &= \sum_{s=xy} (p_i \circ f)(x)\omega_s^{(1)}((p_i \circ g)(y))\\ &= ((p_i \circ f)(p_i \circ g))(s).\end{aligned}$$

Since for any $f, g \in (R_1 \oplus R_2)[[S, \omega^{(1)} \oplus \omega^{(2)}]]$, we have

$$\begin{aligned}\psi(f+g) &= ((f+g)_1, (f+g)_2)\\ &= (p_1 \circ (f+g), p_2 \circ (f+g))\\ &= ((p_1 \circ f)+(p_1 \circ g), (p_2 \circ f)+(p_2 \circ g))\\ &= (f_1+g_1, f_2+g_2)\\ &= (f_1, f_2)+(g_1, g_2)\\ &= \psi(f)+\psi(g)\end{aligned}$$

and

$$\begin{aligned}\psi(fg) &= ((fg)_1, (fg)_2)\\ &= (p_1 \circ (fg), \ p_2 \circ (fg))\\ &= ((p_1 \circ f)(p_1 \circ g), (p_2 \circ f)(p_2 \circ g))\\ &= (f_1 g_1, \ f_2 g_2)\\ &= (f_1, \ f_2)(g_1, \ g_2)\\ &= \psi(f)\psi(g),\end{aligned}$$

ψ is a ring homomorphism.

Now, we will show ψ is injective. Let $f \in Ker(\psi)$. Then we will show $f = 0$. Since $f \in Ker(\psi)$, $\psi(f) = (0, 0)$. So, for any $s \in S$ and $i = 1, 2$, we have $(p_i \circ f)(s) = 0(s)$. In other words, $p_i(f(s)) = 0$. Since p_i is a natural projection, $f(s) = 0$ for all $s \in S$. So $Ker(\psi) = 0$ or ψ is injective. Furthermore, we will show ψ is surjective. For all $(f_1, f_2) \in R_1[[S, \omega^{(1)}]] \oplus R_2[[S, \omega^{(2)}]]$, there exists

$$f = \sum_{k=1}^{2} i_k \circ f_k \in R_1 \oplus R_2[[S, \omega^{(1)} \oplus \omega^{(2)}]]$$

such that $\psi(f) = (f_1, f_2)$. So, ψ is surjective. Then, ψ is a ring isomorphism. So $R_1 \oplus R_2[[S, \omega^{(1)} \oplus \omega^{(2)}]] \cong R_1[[S, \omega^{(1)}]] \oplus R_2[[S, \omega^{(2)}]]$. □

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