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# Forging Connections between Computational Mathematics and Computational Geometry

## Papers from the 3rd International Conference on Computational Mathematics and Computational Geometry

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# Foreword

This volume of conference proceedings contains a collection of research papers presented at the 3rd Annual International Conference on Computational Mathematics, Computational Geometry & Statistics (CMCGS 2014) organized by Global Science and Technology Forum, held in Singapore on 3–4 February 2014.

The CMCGS 2014 conference is an international event for the presentation, interaction, and dissemination of new advances relevant to computational mathematics, computational geometry, and statistics research. As member of the Board of Governors, GSTF, I would like to express my sincere thanks to all those who have contributed to the success of CMCGS 2014.

A special thanks to all our speakers, authors, and delegates for making CMCGS 2014 a successful platform for the industry, fostering growth, learning, networking, and inspiration. We sincerely hope you find the conference proceedings enriching and thought-provoking.



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# Preface

We are pleased to welcome you to the 3rd Annual International Conference on Computational Mathematics, Computational Geometry & Statistics (CMCGS 2014) organized by Global Science and Technology Forum, held in Singapore on 3–4 February 2014.

The CMCGS 2014 conference continuously aims to foster the growth of research in mathematics, geometry, statistics, and its benefits to the community at large. The research papers published in the proceedings are comprehensive in that it contains a wealth of information that is extremely useful to academics and professionals working in this and related fields.

It is my pleasure to announce the participation of leading academics and researchers in their respective areas of focus from various countries at this event. The Conference Proceedings and the presentations made at CMCGS 2014 are the end result of a tremendous amount of innovative work and a highly selective review process. We have received research papers from distinguished participating academics from various countries. There will be "BEST PAPER AWARDS" for authors and students, to recognize outstanding contributions and research publications.

We thank all authors for their participation and we are happy that they have chosen CMCGS 2014 as the platform to present their work. Credit also goes to the Program Committee members and review panel members for their contribution in reviewing and evaluating the submissions and for making CMCGS 2014 a success.

Anton Ravindran



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# Contents





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# Generalized Variance Estimations of Normal-Poisson Models

**Célestin C. Kokonendji and Khoirin Nisa**

**Abstract** This chapter presents three estimations of generalized variance (i.e., determinant of covariance matrix) of normal-Poisson models: maximum likelihood (ML) estimator, uniformly minimum variance unbiased (UMVU) estimator, and Bayesian estimator. First, the definition and some properties of normal-Poisson models are established. Then ML, UMVU, and Bayesian estimators for generalized variance are derived. Finally, a simulation study is carried out to assess the performance of the estimators based on their mean square error (MSE).

**Keywords** Covariance matrix • Determinant • Normal stable Tweedie • Maximum likelihood • UMVU • Bayesian estimator

## Introduction

In multivariate analysis, generalized variance (i.e., determinant of covariance matrix) has important roles in the descriptive analysis and inferences. It is the measure of dispersion within multivariate data which explains the variability and the spread of observations. Its estimation usually based on the determinant of the sample covariance matrix. Many studies related to the generalized variance estimation have been done by some researchers; see, e.g., [1–3] under normality and non-normality hypotheses.

A normal-Poisson model is composed by distributions of random vector $\mathbf{X} = (X_1, X_2, \ldots, X_k)^{\mathrm{T}}$ with $k > 1$, where $X_j$ is a univariate Poisson variable, and $(X_1, \ldots, X_{j-1}, X_{j+1}, \ldots, X_k)$ given $X_j$ are $k$-1 real independent Gaussian variables with variance $X_j$. It is a particular part of normal stable Tweedie (NST) models [4] with $p = 1$ where $p$ is the power variance parameter of distributions within the Tweedie family. This model was introduced in [4] for the particular case of normal-Poisson with $j = 1$. Also, normal-Poisson is the only NST model which has a discrete component, and it is correlated to the continuous normal parts.

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In literature, there is also a model known as Poisson-Gaussian [5–7] which is completely different from normal-Poisson. For any value of $j$, a normal-Poisson$_j$ model has only one Poisson component and $k$-1 normal (Gaussian) components, while a Poisson-Gaussian$_j$ model has $j$ Poisson components and $k$-$j$ Gaussian components. Poisson-Gaussian is also a particular case of simple quadratic natural exponential family (NEF) [5] with variance function $\mathbf{V}_F(\mathbf{m}) = \mathbf{Diag}_k(m_1, \ldots, m_j, 1, \ldots, 1)$, where $\mathbf{m} = (m_1, \ldots, m_k)$ is the mean vector and its generalized variance function is $\det\mathbf{V}_F(\mathbf{m}) = m_1, \ldots, m_j$. The estimations of generalized variance of Poisson-Gaussian can be seen in [8, 9].

Motivated by generalized variance estimations of Poisson-Gaussian, we present our study on multivariate normal-Poisson models and the estimations of their generalized variance using ML, UMVU, and Bayesian estimators.

## Normal-Poisson Models

In this section, we establish the definition of normal-Poisson$_j$ models as generalization of normal-Poisson$_1$ model which was introduced in [4], and then we give some properties.

**Definition 2.1** For a $k$-dimensional normal-Poisson random vector $\mathbf{X} = (X_1, X_2, \ldots, X_k)^{\mathrm{T}}$ with $k > 1$, it must hold that

1. $X_j$ follows a univariate Poisson distribution.
2. $\left(X_1, \ldots, X_{j-1}, X_{j+1}, \ldots, X_k\right) =: \mathbf{X}_j^c \Big| X_j$ are independent normal variables with mean 0 and variance $X_j$, i.e., $\mathbf{X}_j^c \Big| X_j \sim$ i.i.d. $N\left(0, X_j\right)$.

In order to satisfy the second condition, we need $X_j > 0$, but in practice it is possible to have $x_j = 0$ in the Poisson sample. In this case, the corresponding normal components are degenerated as $\delta_0$ which makes their values become 0s.

The NEF $F_t = F(\mu_t)$ of a $k$-dimensional normal-Poisson random vector $\mathbf{X}$ is generated by

$$\mu_t(d\mathbf{x}) = \frac{t^{x_j}\left(x_j!\right)^{-1}}{\left(2\pi x_j\right)^{(k-1)/2}} \exp\left(-t - \frac{1}{2x_j}\sum_{\ell \neq j} x_\ell^2\right) I_{x_j \in \mathbb{N}\setminus\{0\}} \delta_{x_j}\left(dx_j\right) \prod_{\ell \neq j} dx_\ell,$$

for a fixed power of convolution $t > 0$, where $I_A$ is the indicator function of the set A and $\delta_{x_j}$ is the Dirac measure at $x_j$. Since $t > 0$, then $\mu_t := \mu^{*t}$ is an infinitely divisible measure.

The cumulant function which is the log of the Laplace transform of $\mu_t$, i.e., $\mathbf{K}_{\mu_t}(\boldsymbol{\theta}) = log\int_{\mathbb{R}^k} exp\left(\boldsymbol{\theta}^T\mathbf{x}\right)\mu_t(d\mathbf{x})$, is given by

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$$\mathbf{K}_{\mu_t}(\boldsymbol{\theta}) = t exp\left(\theta_j + \frac{1}{2}\sum_{\ell \neq j} \theta_\ell^2\right). \quad (1)$$

The function $\mathbf{K}_{\mu_t}(\boldsymbol{\theta})$ in (1) is finite for all $\boldsymbol{\theta}$ in the canonical domain:

$$\boldsymbol{\Theta}(\mu_t) = \left\{\boldsymbol{\theta} \in R^k; \boldsymbol{\theta}^{\mathrm{T}}\tilde{\boldsymbol{\theta}}_j^c := \theta_j + \sum_{\ell \neq j} \theta_\ell^2/2 < 0\right\}$$

with

$$\boldsymbol{\theta} = (\theta_1, \cdots, \theta_k)^T \quad \text{and} \quad \tilde{\boldsymbol{\theta}}_j^c := \left(\theta_1, \ldots, \theta_{j-1}, \theta_j = 1, \theta_{j+1}, \ldots, \theta_k\right)^T. \quad (2)$$

The probability distribution of normal-Poisson$_j$ is

$$P(\boldsymbol{\theta}; t)(d\mathbf{x}) = \exp\left\{\boldsymbol{\theta}^T\mathbf{x} - \mathbf{K}_{\mu_t}(\boldsymbol{\theta})\right\}\mu_t(d\mathbf{x})$$

which is a member of NEF $F(\mu_t) = \{P(\boldsymbol{\theta}; t); \boldsymbol{\theta} \in \boldsymbol{\Theta}(\mu_t)\}$.

From (1), we can calculate the first derivative of the cumulant function that produces a $k$-vector as the mean vector of $F_{\mu_t}$ and also its second derivative which is a $k \times k$ matrix that represents the covariance matrix. Using notations in (2), we obtain

$$\mathbf{K}'_{\mu_t}(\boldsymbol{\theta}) = \mathbf{K}_{\mu_t}(\boldsymbol{\theta}) \cdot \tilde{\boldsymbol{\theta}}_j^c \quad \text{and} \quad \mathbf{K}''_{\mu_t}(\boldsymbol{\theta}) = \mathbf{K}_{\mu_t}(\boldsymbol{\theta})\left\lfloor \tilde{\boldsymbol{\theta}}_j^c \tilde{\boldsymbol{\theta}}_j^{cT} + \mathbf{I}_k^{0_j} \right\rfloor$$

with $\mathbf{I}_k^{0_j} = \mathbf{Diag}_k\left(1, \ldots, 1, 0_j, 1, \ldots, 1\right)$.

The cumulant function presented in (1) and its derivatives are functions of the canonical parameter $\boldsymbol{\theta}$. For practical calculation, we need to use the mean parameterization:

$$P(\mathbf{m}; F_t) := P(\boldsymbol{\theta}(\mathbf{m}); \mu_t)$$

with $\boldsymbol{\theta}(\mathbf{m})$ is the solution in $\boldsymbol{\theta}$ of equation $\mathbf{m} = \mathbf{K}'_{\mu_t}(\boldsymbol{\theta})$.

The variance function of a normal-Poisson$_j$ model which is the variance-covariance matrix in term of mean parameterization is obtained through the second derivative of the cumulant function, i.e., $\mathbf{V}_{F_t}(\mathbf{m}) = \mathbf{K}''_{\mu_t}[\boldsymbol{\theta}(\mathbf{m})]$. Then we have

$$\mathbf{V}_{F_t}(\mathbf{m}) = \frac{1}{m_j}\mathbf{m}\mathbf{m}^{\mathrm{T}} + \mathbf{Diag}_k\left(m_j, \ldots, m_j, 0_j, m_j, \ldots, m_j\right) \quad (3)$$

with $m_j > 0$ and $m_\ell \in R,\ \ell \neq j$.

For $j = 1$, the covariance matrix of $\mathbf{X}$ can be expressed as below

$$V_{F_t}(m) = \left[\begin{array}{c|cccccc} m_1 & m_2 & \cdots & m_j & \cdots & m_k \\ \hline m_2 & m_1^{-1}m_2^2 + m_1 & \cdots & m_1^{-1}m_2m_j & \cdots & m_1^{-1}m_2m_k \\ \vdots & \vdots & \ddots & \vdots & & \vdots \\ m_j & m_1^{-1}m_jm_2 & \cdots & m_1^{-1}m_j^2 + m_1 & \cdots & m_1^{-1}m_jm_k \\ \vdots & \vdots & & \vdots & \ddots & \vdots \\ m_k & m_1^{-1}m_km_2 & \cdots & m_1^{-1}m_km_j & \cdots & m_1^{-1}m_k^2 + m_1 \end{array}\right].$$

Indeed, for the covariance matrix above, one can use the following particular Schur representation of the determinant

$$\det\begin{pmatrix} \gamma & \mathbf{a}^T \\ \mathbf{a} & \mathbf{A} \end{pmatrix} = \gamma \det\left(\mathbf{A} - \gamma^{-1}\mathbf{a}\mathbf{a}^T\right) \tag{4}$$

with the non-null scalar $\gamma = m_1$, the vector $\mathbf{a} = (m_2, \cdots, m_k)^T$, and the $(k-1) \times (k-1)$ matrix $\mathbf{A} = \gamma^{-1}\mathbf{a}\mathbf{a}^T + m_1\mathbf{I}_{k-1}$, where $\mathbf{I}_j = \mathbf{Diag}_j\,(1, \cdots, 1)$ is the $j \times j$ unit matrix.

Consequently, the determinant of the covariance matrix $\mathbf{V}_{F_t}\,(\mathbf{m})$ for $j = 1$ is

$$\det\mathbf{V}_{F_t}\,(\mathbf{m}) = m_1^k$$

Then, it is trivial to show that for $j \in \{1, \ldots, k\}$, the generalized variance of normal-Poisson$_j$ model is given by

$$\det\mathbf{V}_{F_t}\,(\mathbf{m}) = m_j^k \tag{5}$$

with $m_j > 0, m_\ell \in R,\ \ell \neq j$. (5) expresses that the generalized variance of normal-Poisson models depends mainly on the mean of the Poisson component (and the dimension space $k > 1$) .

Among NST models, normal-gamma which is also known as gamma-Gaussian is the only model that has been characterized completely; see [5] or [10] for characterization by variance function and [11] for characterization by generalized variance function. For normal-Poisson models, here we give our result regarding to characterization by variance function and generalized variance. We state the results in the following theorems without proof.

**Theorem 2.1** Let $k \in \{2, 3, \ldots\}$ and $t > 0$. If an NEF $F_t$ satisfies (3), then, up to affinity, $F_t$ is of normal-Poisson model.

**Theorem 2.2** Let $F_t = F(\mu_t)$ be an infinitely divisible NEF on $\mathrm{R}^k$ such that

1. The canonical domain $\mathbf{\Theta}(\mu) = \mathrm{R}^k$
2. $\det\mathbf{K}''_\mu\,(\boldsymbol{\theta}) = t\exp\left(k \cdot \boldsymbol{\theta}^T \tilde{\boldsymbol{\theta}}_j^c\right)$

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for $\boldsymbol{\theta}$ and $\tilde{\boldsymbol{\theta}}_j^c$ given in (2). Then, up to affinity and power convolution, $F_t$ is of normal-Poisson model.

All the technical details of proofs will be given in our article which is in preparation. In fact, the proof of Theorem 2.1 obtained by algebraic calculations and by using some properties of NEF is described in Proposition 2.1 below. An idea to proof Theorem 2.2 can be obtained using the infinite divisibility property of normal-Poisson for which this proof is the solution to the particular Monge–Ampère equation [12]: $\det \mathbf{K}''_\mu(\boldsymbol{\theta}) = t exp\left(k \cdot \boldsymbol{\theta}^T \tilde{\boldsymbol{\theta}}_j^c\right)$. Gikhman and Skorokhod [13] showed that if $\mu$ is an infinitely divisible measure, then there exist a symmetric nonnegative definite $d \times d$ matrix $\boldsymbol{\Sigma}$ with rank $k$-1 and a positive measure $\nu$ on $R^k$ such that

$$\mathbf{K}''_\mu(\boldsymbol{\theta}) = \boldsymbol{\Sigma} + \int_{\mathsf{R}^k} \mathbf{x}\mathbf{x}^T \exp\left(\boldsymbol{\theta}^T \mathbf{x}\right) \nu(d\mathbf{x}).$$

The Lévy–Khintchine formula of infinite divisibility distribution is also applied.

**Proposition 2.1** Let $\mu$ and $\tilde{\mu}$ be two $\sigma$-finite positive measures on $R^k$ such that $F = F(\mu)$, $\tilde{F} = F(\tilde{\mu})$, and $\mathbf{m} \in \mathbf{M_F}$.

1. If there exists $(\mathbf{d},c) \in \mathrm{R}^k \mathrm{xR}$ such that $\tilde{\mu}(d\mathbf{x}) = exp\left\{\mathbf{d}^T\mathbf{x}\right\rangle + c\left.\right\} \mu(d\mathbf{x})$, then $F = \tilde{F} : \boldsymbol{\Theta}(\tilde{\mu}) = \boldsymbol{\Theta}(\mu) - \mathbf{d}$ and $K_{\tilde{\mu}}(\boldsymbol{\theta}) = K_\mu(\boldsymbol{\theta} + \mathbf{d}) + c$, for $\overline{\mathbf{m}} = \mathbf{m} \in \mathbf{M_F}$, $\mathbf{V}_{\tilde{F}}(\overline{\mathbf{m}}) = \mathbf{V}_F(\mathbf{m})$, and $\det \mathbf{V}_{\tilde{F}}(\overline{\mathbf{m}}) = \det \mathbf{V_F}(\mathbf{m})$.
2. If $\tilde{\mu} = \phi_* \mu$ is the image measure of $\mu$ by the affine transformation $\phi(\mathbf{x}) = \mathbf{A}\mathbf{x} + \mathbf{b}$, where $\mathbf{A}$ is a $k \times k$ nondegenerate matrix and $\mathbf{b} \in \mathrm{R}^k$, then $\boldsymbol{\Theta}(\tilde{\mu}) = \mathbf{A}^T \boldsymbol{\Theta}(\mu)$ and $K_{\tilde{\mu}}(\boldsymbol{\theta}) = K_\mu\left(\mathbf{A}^T\boldsymbol{\theta}\right) + \mathbf{b}^T\boldsymbol{\theta}$; for $\overline{\mathbf{m}} = \mathbf{A}\mathbf{m} + \mathbf{b} \in \phi(\mathbf{M_F})$, $\mathbf{V}_{\tilde{F}}(\overline{\mathbf{m}}) = \mathbf{A}\mathbf{V}_F\left(\phi^{-1}(\overline{\mathbf{m}})\right)\mathbf{A}^T$, and $\det \mathbf{V}_{\tilde{F}}(\overline{\mathbf{m}}) = (\det \mathbf{A})^2 \det \mathbf{V}_F(\mathbf{m})$.
3. If $\tilde{\mu} = \mu^{*t}$ is the $t$-th convolution power of $\mu$ for $t > 0$, then $\boldsymbol{\Theta}(\tilde{\mu}) = \boldsymbol{\Theta}(\mu)$ and $K_{\tilde{\mu}}(\boldsymbol{\theta}) = tK_\mu(\boldsymbol{\theta})$; for $\overline{\mathbf{m}} = t\mathbf{m} \in t\mathbf{M}_F$, $\mathbf{V}_{\tilde{F}}(\overline{\mathbf{m}}) = t\mathbf{V}_F\Big(\phi t^{-1}(\overline{\mathbf{m}})$, and $\det \mathbf{V}_{\tilde{F}}(\overline{\mathbf{m}}) = t^k det \mathbf{V}_F(\mathbf{m})$.

Proposition 2.1 shows that the generalized variance function $\det \mathbf{V}_F(\mathbf{m})$ of $F$ is invariant for any element of its generating measure (Part 1) and for the affine transformation $\phi(\mathbf{x}) = \mathbf{A}\mathbf{x} + \mathbf{b}$ such that $\det \mathbf{A} = \pm 1$, particularly for a translation $\mathbf{x} \to \mathbf{x} + \mathbf{b}$ (Part 2).

A reformulation of Theorem 2.2, by changing the canonical parameterization into mean parameterization, is stated in the following theorem.

**Theorem 2.3** Let $F_t = F(\mu_t)$ be an infinitely divisible NEF on $\mathrm{R}^k$ such that

1. $m_j > 0$ and $m_\ell \in \mathrm{R}$ with $\ell \neq j$
2. $\det \mathbf{V}_F(\mathbf{m}) = m_j^k$.

Then $F_t$ is of normal-Poisson type.

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Theorem 2.3 is equivalent to Theorem 2.2. The former is used for the estimation of generalized variance, and the latter is used for characterization by generalized variance.

## Generalized Variance Estimations

Here we present three methods for generalized variance estimations of normal-Poisson models $P(\mathbf{m}; Ft) \in F_t = F(\mu_t)$, and then we report the result of our simulation study.

Consider $\mathbf{X}_1, \cdots, \mathbf{X}_n$ be random vectors i.i.d. from $P(\mathbf{m}; F_t)$ of normal-Poisson models, and we denote $\overline{\mathbf{X}} = (\mathbf{X}_1 + \cdots + \mathbf{X}_n)/n = \left(\overline{X}_1, \cdots, \overline{X}_k\right)^T$ as the sample mean with positive $j$-th component $\overline{X}_j$. The followings are ML, UMVU, and Bayesian generalized variance estimators.

### *Maximum Likelihood Estimator*

**Proposition 3.1** The ML estimator of $\det\mathbf{V}_{F_t}(\mathbf{m}) = m_j^k$ is given by

$$T_{n,t} = \det\mathbf{V}_{F_t}\left(\overline{\mathbf{X}}\right) = \left(\overline{X}_j\right)^k. \tag{6}$$

*Proof* The ML estimator above is easily obtained by replacing $m_j$ in (5) with its ML estimator $\overline{X}_j$. □

### *Uniformly Minimum Variance Unbiased Estimator*

**Proposition 3.2** The UMVU estimator of $\det\mathbf{V}_{F_t}(\mathbf{m}) = m_j^k$ is given by

$$U_{n,t} = n^{-k+1}\overline{X}_j\left(n\overline{X}_j - 1\right)\ldots\left(n\overline{X}_j - k + 1\right), \quad \text{if } n\overline{X}_j \geq k. \tag{7}$$

*Proof* This UMVU estimator is obtained using intrinsic moment formula of univariate Poisson distribution as follows:

$$\mathrm{E}\left[X(X-1)\ldots(X-k+1)\right] = m_j^k.$$

Letting $Y = n\overline{X}_j$ gives the result that (7) is the UMVU estimator of (5), because, by the completeness of NEFs, the unbiased estimation is unique. So, we deduced the desired result. ■

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A deep discussion about ML and UMVU methods on generalized variance estimations can be seen in [9] for NEF and [4] for NST models.

## *Bayesian Estimator*

**Proposition 3.3** Under assumption of prior gamma distribution of $m_j$ with parameter $\alpha > 0$ and $\beta > 0$, the Bayesian estimator of $\det \mathbf{V}_{F_t}(\mathbf{m}) = m_j^k$ is given by

$$B_{n,t,\alpha,\beta} = \left(\frac{\alpha + n\overline{X}_j}{\beta + n}\right)^k. \tag{8}$$

*Proof* Let $X_{1j}, \cdots, X_{nj}$ given $m_j$ are Poisson($m_j$) with probability mass function

$$P\left(X_{ij} = x_{ij} \middle| m_j\right) = \frac{m_j^{x_{ij}}}{x_{ij}!} e^{-m_j} = p\left(x_{ij} \middle| m_j\right).$$

Assuming that $m_j$ follows gamma$(\alpha, \beta)$, then the prior probability distribution function of $m_j$ is given by

$$f\left(m_j; \alpha, \beta\right) = \frac{\beta^\alpha}{\Gamma(\alpha)} m_j^{\alpha-1} e^{-\beta m_j} \text{ for } m_j > 0 \text{ and } \alpha, \beta > 0$$

where $\Gamma(\alpha)$ is the gamma function: $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx$. Using the Bayes theorem, the posterior distribution of $m_j$ given an observation sequence can be expressed as

$$\begin{aligned} f\left(m_j \middle| x_{ij}; \alpha, \beta\right) &= \frac{p\left(x_{ij} \middle| m_j\right) f\left(m_j; \alpha, \beta\right)}{\int_{m_j > 0} p\left(x_{ij} \middle| m_j\right) f\left(m_j; \alpha, \beta\right) dm_j} \\ &= \frac{(\beta + 1)^{\alpha + x_{ij}}}{\Gamma\left(\alpha + x_{ij}\right)} m_j^{\alpha + x_{ij} - 1} e^{-(\beta+1)m_j} \end{aligned}$$

which is a gamma density with parameters $\alpha' = x_{ij} + \alpha$ and $\beta' = 1 + \beta$. Then with random sample $X_{1j}, \ldots, X_{nj}$, the posterior will be gamma $\left(\alpha + n\overline{X}_j, \beta + n\right)$. The Bayesian estimator of $m_j$ is given by the mean of the posterior distribution, i.e., $\widehat{m}_b = \frac{\alpha + n\overline{X}_j}{\beta + n}$, and then this concludes the proof. ■

The choice of $\alpha$ and $\beta$ depends on the information of $m_j$. Notice that for any positive value $c \in (0, \infty)$, if $\alpha = c\overline{X}_j$ and $\beta = c$, then the Bayesian estimator is the same as ML estimator. In practice, the parameter of prior distribution of

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$m_j$ must be known or can be assumed confidently before the generalized variance estimation. One can see, e.g., [14–16] for more details about Bayesian inference on $m_j$ (univariate Poisson parameter).

## Simulation Study

In order to look at the performances of ML, UMVU, and Bayesian estimators of the generalized variance, we have done a Monte Carlo simulation using R software [17]. We have generated $k = 2, 4, 6, 8$ dimensional data from multivariate normal-Poisson distribution $F(\mu_t)$ with $m_j = 1$. Fixing $j = 1$, we set several sample sizes $n$ varied from 5 until 300, and we generated 1,000 samples for each sample size. For calculating the Bayesian estimator, in this simulation we assume that the parameters of prior distribution depend on sample mean of Poisson component, $\overline{X}_j$, and the dimension $k$. Then we set three different prior distributions: gamma $\left(\overline{X}_j, k\right)$, gamma $\left(\overline{X}_j, k/2\right)$, and gamma $\left(\overline{X}_j, k/3\right)$.

We report the results of the generalized variance estimations using the three methods in Table 1. From these values, we calculated the mean square error (MSE) of each method over 1,000 data sets using this following formula

$$MSE\left(\widehat{GV}\right) = \frac{1}{1,000}\sum_{i=1}^{1,000}\left(\widehat{GV}_i - m_j^k\right)^2$$

where $\widehat{GV}$ is the estimate of $m_j^k$ using each method.

From the values in Table 1, we can observe different performances of ML estimator ($T_{n,t}$), UMVU estimator ($U_{n,t}$), and Bayesian estimator ($B_{n,t,\alpha,\beta}$) of the generalized variance. The values of $T_{n,t}$ and $B_{n,t,\alpha,\beta}$ converge, while the values of $U_{n,t}$ do not, but $U_{n,t}$ which is the unbiased estimator always approximate the parameter ($m_1^k = 1$) and closer to the parameter than $T_{n,t}$ and $B_{n,t,\alpha,\beta}$ for small sample sizes $n \leq 25$. For all methods, the standard error of the estimates decreases when the sample size increases. The Bayesian estimator with gamma $\left(\overline{X}_j, k/2\right)$ prior distribution, i.e., $B_{n,t,\overline{X}_j,k/2}$, is exactly the same as $T_{n,t}$ for $k = 2$. This is because in this case, the Bayesian and ML estimators of $m_1$ are the same (i.e., $c = 1$).

The goodness of Bayesian estimator depends on the parameter of prior distribution, $\alpha$ and $\beta$. From our simulation, the result shows that smaller parameter $\beta$ gives greater standard error to the estimations in small sample sizes, and the accuracy of $B_{n,t,\alpha,\beta}$ with respect to $\beta$ varies with dimensions $k$. However, they are all asymptotically unbiased.

There are more important performance characterizations for an estimator than just being unbiased. The MSE is perhaps the most important of them. It captures the

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**Table 1** The expected values (with standard error) of $T_{n,t}$, $U_{n,t}$, and $B_{n,t,\alpha,\beta}$ with $m_1 = 1$ and $k \in \{2, 4, 6, 8\}$ (target values $m_1^k = 1$)

| ***k* = 2*n*** | $T_{n,t}$ | $U_{n,t}$ | $B_{n,t,\overline{X}_j,k}$ | $B_{n,t,\overline{X}_j,k/2}$ | $B_{n,t,\overline{X}_j,k/3}$ |
|---|---|---|---|---|---|
| k + 1 | 1.2790 (1.3826) | 0.9533 (1.2050) | 0.8186 (0.8849) | 1.2790 (1.3826) | 1.5221 (1.6454) |
| k + 5 | 1.1333 (0.8532) | 0.9915 (0.8000) | 0.8955 (0.6742) | 1.1333 (0.8532) | 1.2340 (0.9290) |
| k + 10 | 1.1121 (0.6295) | 1.0276 (0.6056) | 0.9589 (0.5428) | 1.1121 (0.6295) | 1.1714 (0.6631) |
| 25 | 1.0357 (0.4256) | 0.9959 (0.4175) | 0.9604 (0.3946) | 1.0357 (0.4256) | 1.0628 (0.4367) |
| 60 | 1.0090 (0.2526) | 0.9924 (0.2505) | 0.9767 (0.2445) | 1.0090 (0.2526) | 1.0201 (0.2553) |
| 100 | 1.0086 (0.1988) | 0.9986 (0.1979) | 0.9890 (0.1950) | 1.0086 (0.1988) | 1.0153 (0.2002) |
| 300 | 0.9995 (0.1141) | 0.9962 (0.1140) | 0.9929 (0.1134) | 0.9995 (0.1141) | 1.0017 (0.1144) |
| ***k* = 4*n*** | $T_{n,t}$ | $U_{n,t}$ | $B_{n,t,\overline{X}_j,k}$ | $B_{n,t,\overline{X}_j,k/2}$ | $B_{n,t,\overline{X}_j,k/3}$ |
| k + 1 | 2.3823 (4.6248) | 0.9460 (2.5689) | 0.4706 (0.9135) | 1.2859 (2.4964) | 1.9190 (3.7254) |
| k + 5 | 1.6824 (2.4576) | 0.9531 (1.6995) | 0.5890 (0.8605) | 1.1491 (1.6786) | 1.4756 (2.1555) |
| k + 10 | 1.4664 (1.6345) | 1.0027 (1.2456) | 0.7072 (0.7882) | 1.1328 (1.2626) | 1.3430 (1.4969) |
| 25 | 1.2711 (1.0895) | 1.0169 (0.9327) | 0.8212 (0.7039) | 1.0930 (0.9368) | 1.2079 (1.0353) |
| 60 | 1.0978 (0.5682) | 0.9961 (0.5288) | 0.9060 (0.4689) | 1.0287 (0.5324) | 1.0741 (0.5559) |
| 100 | 1.0589 (0.4209) | 0.9983 (0.4028) | 0.9419 (0.3744) | 1.0180 (0.4046) | 1.0451 (0.4154) |
| 300 | 1.0273 (0.2305) | 1.0071 (0.2271) | 0.9874 (0.2215) | 1.0138 (0.2275) | 1.0228 (0.2295) |
| ***k* = 6*n*** | $T_{n,t}$ | $U_{n,t}$ | $B_{n,t,\overline{X}_j,k}$ | $B_{n,t,\overline{X}_j,k/2}$ | $B_{n,t,\overline{X}_j,k/3}$ |
| k + 1 | 4.7738 (13.9827) | 0.9995 (4.7073) | 0.2593 (0.7594) | 1.2514 (3.6655) | 2.3548 (6.8972) |
| k + 5 | 2.9818 (6.2595) | 0.9958 (2.7565) | 0.3689 (0.7743) | 1.1825 (2.4823) | 1.8446 (3.8723) |
| k + 10 | 2.2232 (4.0454) | 1.0124 (2.2131) | 0.4733 (0.8612) | 1.1406 (2.0756) | 1.5778 (2.8709) |
| 25 | 1.6399 (2.2478) | 0.9555 (1.4833) | 0.5708 (0.7824) | 1.0513 (1.4410) | 1.3076 (1.7923) |
| 60 | 1.2479 (0.9978) | 0.9827 (0.8226) | 0.7778 (0.6220) | 1.0283 (0.8222) | 1.1319 (0.9051) |
| 100 | 1.1830 (0.7646) | 1.0235 (0.6800) | 0.8853 (0.5722) | 1.0517 (0.6798) | 1.1151 (0.7207) |
| 300 | 1.0530 (0.3758) | 1.0022 (0.3608) | 0.9539 (0.3404) | 1.0119 (0.3612) | 1.0322 (0.3684) |
| ***k* = 8*n*** | $T_{n,t}$ | $U_{n,t}$ | $B_{n,t,\overline{X}_j,k}$ | $B_{n,t,\overline{X}_j,k/2}$ | $B_{n,t,\overline{X}_j,k/3}$ |
| k + 1 | 8.5935 (31.9230) | 0.8677 (5.4574) | 0.1232 (0.4576) | 1.0535 (3.9134) | 2.5038 (9.3010) |
| k + 5 | 4.7573 (12.5015) | 0.8468 (3.0478) | 0.1856 (0.4878) | 1.0065 (2.6448) | 1.9345 (5.0836) |
| k + 10 | 3.6816 (9.0892) | 1.0394 (3.2258) | 0.2994 (0.7392) | 1.1394 (2.8130) | 1.8789 (4.6387) |
| 25 | 2.9055 (6.3150) | 1.1341 (2.9623) | 0.4314 (0.9377) | 1.2129 (2.6362) | 1.7675 (3.8416) |
| 60 | 1.6201 (1.8804) | 1.0511 (1.3062) | 0.6794 (0.7885) | 1.1035 (1.2807) | 1.3059 (1.5156) |
| 100 | 1.2890 (1.0907) | 0.9850 (0.8667) | 0.7541 (0.6381) | 1.0199 (0.8630) | 1.1308 (0.9569) |
| 300 | 1.1056 (0.5378) | 1.0086 (0.4968) | 0.9199 (0.4474) | 1.0213 (0.4967) | 1.0578 (0.5145) |

bias and the variance of the estimator. For this reason, we compare the quality of the estimators using MSE in Table 2 which are presented graphically in Figs. 1, 2, 3, and 4. From these figures, we conclude that all estimators become more similar when the sample size increases. For small sample sizes, $B_{n,t,\overline{X}_j,k}$ always has the smallest MSE, while $T_{n,t}$ always has the greatest MSE (except for $k = 2$). For $n \leq 25$, $U_{n,t}$ is preferable than $T_{n,t}$. In this situation, the difference between $U_{n,t}$ and $T_{n,t}$ increases when the dimension increases and also the difference between $T_{n,t}$ and $B_{n,t,\alpha,\beta}$.

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**Table 2** The mean square error of $T_{n,t}$, $U_{n,t}$, and $B_{n,t,\alpha,\beta}$ of Table 1

| $\boldsymbol{k=2n}$ | $MSE(T_{n,t})$ | $MSE(U_{n,t})$ | $MSE(B_{n,t,\overline{X}_j,k})$ | $MSE(B_{n,t,\overline{X}_j,k/2})$ | $MSE(B_{n,t,\overline{X}_j,k/3})$ |
|---|---|---|---|---|---|
| k + 1 | 1.9894 | 1.4542 | 0.8159 | 1.9894 | 2.9800 |
| k + 5 | 0.7458 | 0.6401 | 0.4654 | 0.7458 | 0.9179 |
| k + 10 | 0.4088 | 0.3675 | 0.2963 | 0.4088 | 0.4690 |
| 25 | 0.1824 | 0.1743 | 0.1573 | 0.1824 | 0.1947 |
| 60 | 0.0639 | 0.0628 | 0.0603 | 0.0639 | 0.0656 |
| 100 | 0.0396 | 0.0391 | 0.0381 | 0.0396 | 0.0403 |
| 300 | 0.0130 | 0.0130 | 0.0129 | 0.0130 | 0.0131 |
| $\boldsymbol{k=4n}$ | $MSE(T_{n,t})$ | $MSE(U_{n,t})$ | $MSE(B_{n,t,\overline{X}_j,k})$ | $MSE(B_{n,t,\overline{X}_j,k/2})$ | $MSE(B_{n,t,\overline{X}_j,k/3})$ |
| k + 1 | 23.2999 | 6.6019 | 1.1149 | 6.3136 | 14.7231 |
| k + 5 | 6.5055 | 2.8904 | 0.9093 | 2.8398 | 4.8724 |
| k + 10 | 2.8891 | 1.5514 | 0.7071 | 1.6118 | 2.3585 |
| 25 | 1.2604 | 0.8702 | 0.5274 | 0.8862 | 1.1151 |
| 60 | 0.3324 | 0.2797 | 0.2287 | 0.2843 | 0.3146 |
| 100 | 0.1806 | 0.1622 | 0.1435 | 0.1640 | 0.1746 |
| 300 | 0.0539 | 0.0516 | 0.0492 | 0.0519 | 0.0532 |
| $\boldsymbol{k=6n}$ | $MSE(T_{n,t})$ | $MSE(U_{n,t})$ | $MSE(B_{n,t,\overline{X}_j,k})$ | $MSE(B_{n,t,\overline{X}_j,k/2})$ | $MSE(B_{n,t,\overline{X}_j,k/3})$ |
| k + 1 | 209.7568 | 22.1589 | 1.1254 | 13.4989 | 49.4073 |
| k + 5 | 43.1085 | 7.5980 | 0.9979 | 6.1952 | 15.7078 |
| k + 10 | 17.8618 | 4.8981 | 1.0191 | 4.3278 | 8.5761 |
| 25 | 5.4622 | 2.2020 | 0.7964 | 2.0790 | 3.3071 |
| 60 | 1.0571 | 0.6769 | 0.4362 | 0.6769 | 0.8366 |
| 100 | 0.6181 | 0.4629 | 0.3406 | 0.4647 | 0.5327 |
| 300 | 0.1440 | 0.1302 | 0.1180 | 0.1306 | 0.1368 |
| $\boldsymbol{k=8n}$ | $MSE(T_{n,t})$ | $MSE(U_{n,t})$ | $MSE(B_{n,t,\overline{X}_j,k})$ | $MSE(B_{n,t,\overline{X}_j,k/2})$ | $MSE(B_{n,t,\overline{X}_j,k/3})$ |
| k + 1 | 1,076.7380 | 29.8009 | 0.9782 | 15.3177 | 88.7698 |
| k + 5 | 170.4059 | 9.3124 | 0.9012 | 6.9951 | 26.7168 |
| k + 10 | 89.8046 | 10.4076 | 1.0373 | 7.9326 | 22.2895 |
| 25 | 43.5105 | 8.7931 | 1.2025 | 6.9949 | 15.3466 |
| 60 | 3.9204 | 1.7088 | 0.7246 | 1.6509 | 2.3907 |
| 100 | 1.2732 | 0.7515 | 0.4676 | 0.7452 | 0.9327 |
| 300 | 0.3003 | 0.2469 | 0.2066 | 0.2472 | 0.2681 |

In this simulation, $B_{n,t,\overline{X}_j,k}$ is the best estimator because of its smallest MSE, but in general we cannot say that Bayesian estimator is much better than ML and UMVU estimators since it depends on the prior distribution parameters. In fact, one would prefer $U_{n,t}$ as it is the unbiased estimator with the minimum variance. However, if in practice we know the information about prior distribution of $m_j$, we can get a better estimate (in the sense of having a lower MSE) than $U_{n,t}$ by using $B_{n,t,\alpha,\beta}$.

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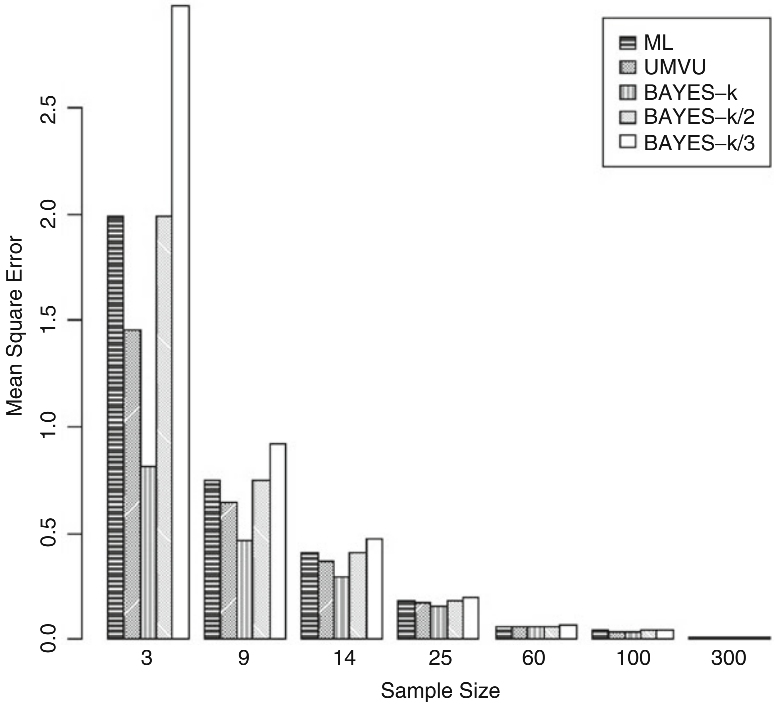


**Fig. 1** MSE plot of $T_{n,t}$, $U_{n,t}$, $B_{n,t,\overline{x}_j,k}$, $B_{n,t,\overline{x}_j,k/2}$, and $B_{n,t,\overline{x}_j,k/3}$ for $k = 2$

## Conclusion

In this chapter, we have established the definition and properties of normal-Poisson$_j$ models as a generalization of normal-Poisson$_1$ and showed that the generalized variance of normal-Poisson models depends mainly on the mean of the Poisson component. The estimations of generalized variance using ML, UMVU, and Bayesian estimators show that UMVU produces a better estimation than ML estimator, while compared to Bayesian estimator, UMVU is worse for some choice of prior distribution parameters, but it can be much better for other cases. However, all methods are consistent estimators, and they become more similar when the sample size increases.

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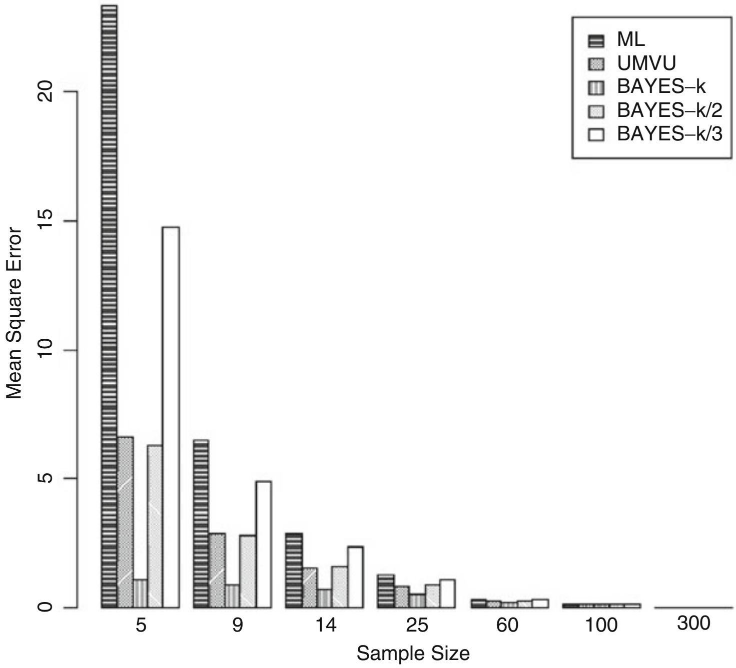


**Fig. 2** MSE plot of $T_{n,t}$, $U_{n,t}$, $B_{n,t,\overline{x}_j,k}$, $B_{n,t,\overline{x}_j,k/2}$, and $B_{n,t,\overline{x}_j,k/3}$ for $k = 4$

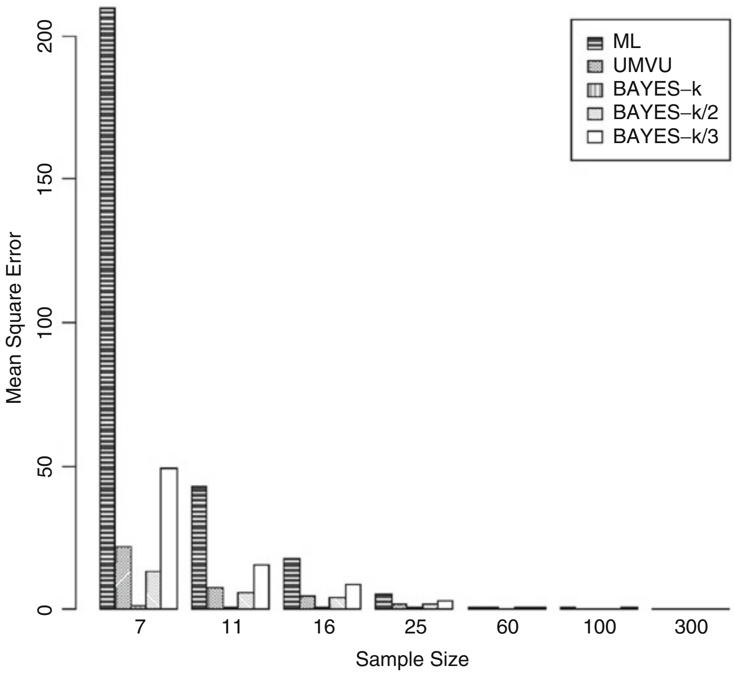


**Fig. 3** MSE plot of $T_{n,t}$, $U_{n,t}$, $B_{n,t,\overline{x}_j,k}$, $B_{n,t,\overline{x}_j,k/2}$, and $B_{n,t,\overline{x}_j,k}$ for $k = 6$

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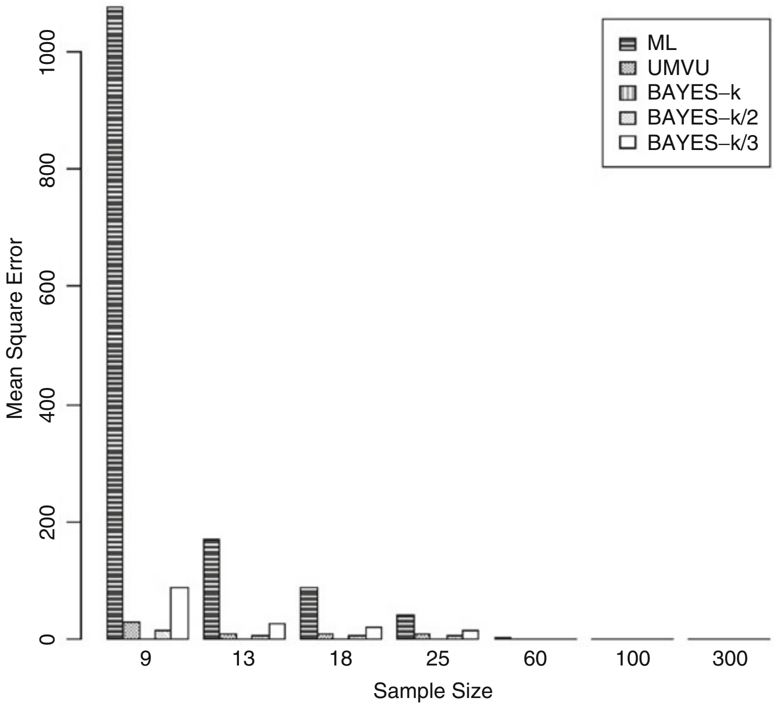


**Fig. 4** MSE plot of $T_{n,t}$, $U_{n,t}$, $B_{n,t,\overline{x}_j,k}$, $B_{n,t,\overline{x}_j,k/2}$, and $B_{n,t,\overline{x}_j,k/3}$ for $k = 8$

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