

The Combination of Several RCBDs

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Abstract: The application of Experimental Design nowadays is very extensive in many research areas, especially in Engineering, Agriculture, Education and Life sciences. In many experimental designs sometimes the researchers want to compare parameters from some design of experiments. This paper will discuss the approach to combine several RCBDs (Randomized Complete Block Designs) for the fixed effect model. The combined model is non full column rank and has constraint on its parameters. The approach used in this paper is MRM (Model Reduction Method) to transform the constrained model into unconstrained model. Then the analysis of interest will be based on the unconstrained model.

Key words: RCBDs, MRM(Model Reduction Method), orthogonal Matrix, Likelihood Ratio Test, testing hypothesis.

INTRODUCTION

The experimental design as a tool to find the information of interest in research has been extensively used in many areas of research such as in Engineering, Agriculture, Medical sciences and in Education. The application of the experimental design in Engineering can be found for example in Mustofa, et al (2008), Fine (2006), Montgomery and Runger (1994). The analysis of combined of several model has become an interesting areas of research, for example see Peterson (1994), has discussed the combination of analysis of several experimental design applied in the areas of agriculture. The combination of some linear regression model by using the general linear model can be found in Theil (1971). The combination of some linear regression model by using dummy variable also can be found in many regression book such as Neter et al (1990), Gujarati (1970). In this paper will discuss the approach to analyze of the combination of several RCBDs model under fixed effect model.

The combined model is non full column rank and has constraint on its parameters. The approach used in this paper is MRM (Model Reduction Method) given in Hocking (1985) to transform the constrained model into unconstrained model. Then the analysis of interest will be based on the unconstrained model. To estimate and test the hypothesis of interest, will be considered two cases of the variance covariance structure.

RCBD analysis methods:

It is well known that the RCBD model is

$$Y_{ij} = \mu + \beta_i + \tau_j + \varepsilon_{ij} \quad (1)$$

where Y_{ij} is the observation from the i th block and j th treatment, ε_{ij} is the error from the i th block and j th column and assume that it is iid $n(0, \sigma^2)$. Under the fixed effect model, it is assume that the parameter has a constraint

$$\sum_{i=1}^b \beta_i = 0 \text{ and } \sum_{j=1}^t \tau_j = 0. \quad (2)$$

The model (1) and its constraint (2) can be written as

$$Y = X\theta + \varepsilon \quad (3)$$

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Subject to $G\theta=0$

where Y is $n \times 1$ vector of observation,

$$X = [1_n, I_b \otimes 1_t, 1_b \otimes I_t], \theta = (\mu, \beta_1, \beta_2, \dots, \beta_b, \tau_1, \tau_2, \dots, \tau_t)',$$

ε has normally distribution with mean 0 and variance σ^2 . and

$$G = \begin{bmatrix} 0 & 1'_b & 0'_t \\ 0 & 0'_b & 1'_t \end{bmatrix}$$

and the kronecker product of matrix A $s \times t$ and B $r \times u$ denoted by $A \otimes B$ is $sr \times tu$ matrix formed by multiplying each element a_{ij} by entire matrix B (Moser, 1996, Theil, 1971). Model (3) is not full column rank and has constraint on its parameters. To deal with this type of problem there are some approach can be used, for example see (Magnus, 1988; Hocking, 1985). Hocking, (1985) proposed of Model Reduction Method (MRM) to transform the non full rank and constrained model into unconstrained full rank model. The idea is as follow: Suppose that the linear model

$$Y = X\theta + \varepsilon \tag{4}$$

Subject to $G\theta=g$

where Y is n -vector of observation, X is $n \times p$ design matrix of rank $\leq p$, θ is p -vector of unknown parameters, ε is n -vector of error with $\varepsilon \sim N(0, I)$, where \sim is read "is distributed as" and $N(\mu, V)$ denotes the multivariate normal distribution with mean vector μ and covariance matrix V , and G is $q \times p$ matrix of rank q . Assume that θ and G are partition so that the constraint are written as

$$G_1\theta_1 + G_2\theta_2 = g \tag{5}$$

Where G_1 is $q \times p_1$ of rank q . Solving for θ_1 yields

$$\theta_1 = G_1^{-1}g + G_1^{-1}G_2\theta_2 \tag{6}$$

Partition X as the same way as $X = [X_1 \ X_2]$ and substituting into model (4), we obtain

$$Y_r = X_r\theta_r + e \tag{7}$$

where $Y_r = Y - X_1G_1^{-1}g$, $X_r = [X_2 - X_1G_1^{-1}G_2]$ and $\theta_r = \theta_2$. The model (7) is called unconstrained model

Hocking, (1985).

In model (3) by assumption that the block and treatment are connected, then the rank of the design matrix X in (3) is $b+t-1$. As can be shown in the following lemma (see, Mustofa, 1995).

Lemma 1:

By assumption of connectedness, the rank of X in (3) is $b+t-1$.

Proof.:

Because the first and the last column of design matrix $X = [1_n, I_b \otimes 1_t, 1_b \otimes I_t]$ is linear combination of the other columns, then the rank of $X \leq b+t-1$. By assumption of connectedness,

$$\begin{aligned} \beta_i - \beta_{i'} & \text{ is estimable } & i \neq i' \\ \tau_j - \tau_{j'} & \text{ is estimable } & j \neq j' \end{aligned}$$

let

$$H\theta = \begin{bmatrix} \mu + \beta_1 + \tau_1 \\ \beta_1 - \beta_2 \\ \beta_1 - \beta_3 \\ \vdots \\ \beta_1 - \beta_b \\ \tau_1 - \tau_2 \\ \tau_1 - \tau_3 \\ \vdots \\ \tau_1 - \tau_t \end{bmatrix}$$

where

$$H = \begin{bmatrix} 1 & 1 & 0'_{b-1} & 1 & 0'_{t-1} \\ 0_{b-1} & 1_{b-1} & -I_{b-1} & 0_{b-1} & 0_{(b-1) \times (t-1)} \\ 0_{t-1} & 0_{t-1} & 0_{(t-1) \times (b-1)} & 1_{t-1} & -I_{t-1} \end{bmatrix},$$

Rank(H) = b+t-1. Since Hθ is estimable, then there is a matrix A such that H=AX. Therefore, b+t-1=rank(H) = rank (AX) ≤ rank(X) ≤ b+t-1. So rank

(X)=b+t-1.

Application of Model Reduction Methods in RCBD:

Consider model (3), $X = [1_n, I_b \otimes 1_t, 1_b \otimes I_t]$

$[1_n \ E \ F]$, where $E = I_b \otimes 1_t$, $F = 1_b \otimes I_t$. To test the hypothesis that Ho: $H\beta=0$ against the alternative hypothesis Ha: $H\beta \neq 0$. We wrote the model (3)

$$Y = [E, F, 1_n] \begin{bmatrix} \beta \\ \tau \\ \mu \end{bmatrix} + \varepsilon \tag{8}$$

Subject to $\sum_{i=1}^b \beta_i = 0, \sum_{j=1}^t \tau_j = 0$

To transform the model (8), the non full rank constrained model into unconstrained full rank model by

MRM(Model Reduction Method), first we transform the parameter $(\beta' \ \tau' \ \mu)'$ into

$(\beta_1 \ \tau_1 \ \beta'_{(1)} \ \tau'_{(1)} \ \mu)'$ by

permutation matrix T, T is an orthogonal such that $T'T = I_{b+t+1}$. We have

$$\theta_1^* = T\theta^*$$

Where $\theta^* = (\beta' \ \tau' \ \mu)'$ and

$$\theta_1^* = (\beta_1 \ \tau_1 \ \beta'_{(1)} \ \tau'_{(1)} \ \mu)'$$

where $\beta'_{(1)} = (\beta_2, \beta_3, \dots, \beta_b)$,

$$\tau'_{(1)} = (\tau_1, \tau_2, \dots, \tau_t).$$

Model (8) can be written as

$$Y = [E, F, 1_n] T' T \begin{bmatrix} \beta \\ \tau \\ \mu \end{bmatrix} + \varepsilon \tag{9}$$

$$\text{Subject to } \sum_{i=1}^b \beta_i = 0, \quad \sum_{j=1}^t \tau_j = 0$$

or

$$Y = X_1 \theta_1^* + \varepsilon \tag{10}$$

$$\text{Subject to } G_1 \theta_1^* = 0.$$

$$\text{Where } X_1 = [E, F, 1_n] T'$$

$$\text{and } G_1 = \begin{bmatrix} 1 & 0 & 1'_{b-1} & 0'_{t-1} & 0 \\ 0 & 1 & 0'_{b-1} & 1'_{t-1} & 0 \end{bmatrix}.$$

0_n is $n \times 1$ vector null. Now, partition G_1 and X_1 as $G_1 = [G_{11} \ G_{12}]$, $X_1 = [X_{11} \ X_{12}]$.

By MRM, we have the unconstrained model

$$Y = X_{1r} \theta_{1r} + \varepsilon \tag{11}$$

Where

$$X_{1r} = A_1 D_1, A_1 = [E \ F_{(t)}], F_{(t)}$$
 is the matrix F without the last column of F , the matrix

$$D_1 = [I_{b+t-1} \ C_1] T' \begin{bmatrix} G_{12} \\ I_{b+t-1} \end{bmatrix},$$

$$C_1 = \begin{bmatrix} 1_b & 1_b \\ -1_{t-1} & 0_{t-1} \end{bmatrix} \text{ and } \theta_{1r} = (\beta'_{(1)} \ \tau'_{(1)} \ \mu)'$$

It can be shown that D_1 is nonsingular, so the rank of the design matrix X_{1r} is $b+t-1$, which is full column rank. The model (11) is satisfied all the assumption of Gauss Markov model see Graybill,1976; Theil, 1971; Moser, 1996. Therefore, the estimation of parameter, estimation of confidence interval, ratio of parameters and testing hypothesis can be derived from this model (11).

The ideas of the transformation model from the constrained and non full rank model into unconstrained model by MRM will be applied to the combination of several RCBDs model.

The Analysis of the Combination of Several RCBDs:

Supposed that there are k RCBDs model,

$$Y_{ij1} = \mu_1 + \beta_{i1} + \tau_{j1} + \varepsilon_{ij1} \tag{12}$$

$$Y_{ij2} = \mu_2 + \beta_{i2} + \tau_{j2} + \varepsilon_{ij2}$$

⋮

$$Y_{ijk} = \mu_k + \beta_{ik} + \tau_{jk} + \varepsilon_{ijk}$$

where Y_{iji} is the observation from the i th block, j th treatment in the l th design, μ_i is the grand mean in the l th design, β_{ij} is the effect of the i th block in the l th design, τ_{ij} is the effect of the j th treatment in the l th design and ε_{iji} is an error from the i th block, j th treatment and the l th design, $i=1,2,\dots,b$; $j=1,2, \dots, t$; $l=1,2, \dots, k$. Model (11) can be written as

$$Y = \text{diag}[X_1 \ X_2 \ \dots X_k] \Theta + \Psi \tag{13}$$

where $X_l = [1_n \ E \ F]$ as given in model (8), $\Theta = (\theta'_1 \ \theta'_2 \ \dots \ \theta'_l)'$ where $\theta_l = (\mu_l \ \beta'_l \ \tau'_l)'$ and

$$\beta'_l = (\beta_{1l} \ \beta_{2l} \ \dots \ \beta_{bl}), \ \tau'_l = (\tau_{1l} \ \tau_{2l} \ \dots \ \tau_{tl}),$$

$$\Psi = (\varepsilon'_1, \varepsilon'_2, \dots, \varepsilon'_k)', \ \varepsilon'_l = (\varepsilon_{11l} \ \varepsilon_{12l} \ \varepsilon_{13l} \ \dots \ \varepsilon_{btl}) \text{ and assume that } \text{Var}(\varepsilon_l) = \sigma_l^2 I_n, \ n=bt, \text{ and}$$

$$\text{Var}(\Psi) = \text{diag}(\sigma_1^2 I_n, \sigma_2^2 I_n, \dots, \sigma_k^2 I_n)$$

and the constraint on its parameters are

$$\sum_{i=1}^b \beta_{il} = 0 \ \forall l, \quad \sum_{j=1}^t \tau_{jl} = 0 \ \forall l. \tag{14}$$

Model (13) and its constraint (14) can be written as follow:

$$Y = \Gamma \Theta + \Psi \tag{15}$$

Subject to $\Omega \Theta = 0$

Where Y is $k(bt) \times 1$ vector of observation, and

$$\Gamma = \text{diag}[X_1 \ X_2 \ \dots X_k],$$

$$\Omega = \text{diag}[G_1 \ G_2 \ \dots \ G_k],$$

$$G_l = \begin{bmatrix} 0 & 1'_b & 0'_t \\ 0 & 0'_b & 1'_t \end{bmatrix}, \quad l=1,2, \dots, k.$$

Model (15) is non full rank model and has constraint on its parameters. Under the assumption of connectedness and apply the idea of Lemma 1, it can be shown that the design matrix Γ in (15) is $k(b+t-1)$. To transform the non full rank constrained model into full column rank and unconstrained model, we use the procedure MRM to the above model as follow.

Define the transform matrix A , A is an orthogonal matrix and

$$\Lambda = \text{diag}[T_1 \ T_2 \ \dots \ T_k]$$

where T_i is an orthogonal matrix $T_i T_i' = I_{b+t-1}$, and

$$\Lambda' \Lambda = \text{diag}[I_{b+t-1}, I_{b+t-1}, \dots, I_{b+t-1}]$$

or $\Lambda' \Lambda = I_k \otimes I_{b+t-1}$.

Model (15) can be written as

$$Y = \Gamma^* \Theta^* + \Psi \tag{16}$$

Subject to $\Omega^* \Theta^* = 0$

$$\Gamma^* = \text{diag}[(E \ F \ 1_n), (E \ F \ 1_n), \dots, (E \ F \ 1_n)]$$

$$\Theta^* = (\beta'_1 \ \tau'_1 \ \mu_1 \ \beta'_2 \ \tau'_2 \ \mu_2 \ \dots \ \beta'_k \ \tau'_k \ \mu_k), \text{ and}$$

$$\Omega^* = \text{diag}[G_1^* \ G_2^* \ \dots \ G_k^*],$$

$$G_l^* = \begin{bmatrix} 1'_b & 0'_l & 0 \\ 0'_b & 1'_l & 0 \end{bmatrix}, \ l = 1, 2, \dots, k.$$

Now we transform the model,

$$Y = \Gamma^* A' A \Theta^* + \Psi$$

or

$$Y = \Gamma_l^* \Theta_l^* + \Psi \tag{17}$$

Subject to $\Omega_1^* \Theta_1^* = 0$

where

$$\Gamma_1^* = \text{diag}[(E \ F \ 1_n)T'_1, (E \ F \ 1_n)T'_1, \dots, (E \ F \ 1_n)T'_1]$$

and

$$\Theta_1^* = [\beta_{11} \ \tau_{11} \ \beta'_{(11)} \ \tau'_{(11)} \ \mu_1, \beta_{12} \ \tau_{12} \ \beta'_{(12)} \ \tau'_{(12)} \ \mu_2, \dots, \beta_{1k} \ \tau_{1k} \ \beta'_{(1k)} \ \tau'_{(1k)} \ \mu_k]'$$

and

$$\beta'_{(11)} = (\beta_{21} \ \beta_{31} \ \dots, \beta_{b1}),$$

$$\beta'_{(12)} = (\beta_{22} \ \beta_{32} \ \dots, \beta_{b2}),$$

⋮

$$\beta'_{(1k)} = (\beta_{2k} \ \beta_{3k} \ \dots, \beta_{bk}),$$

$$\tau'_{(11)} = (\tau_{21} \ \tau_{31} \ \dots, \tau_{t1}),$$

$$\tau'_{(12)} = (\tau_{22} \ \tau_{32} \ \dots, \tau_{t2}),$$

⋮

$$\tau'_{(1k)} = (\tau_{2k} \ \tau_{3k} \ \dots, \tau_{tk})$$

Next, we find second transformation matrix A^* such that,

$$\Lambda^* \Theta_1^* = [\beta_{11} \ \tau_{11} \ \beta_{12} \ \tau_{12} \ \dots, \beta_{1k} \ \tau_{1k} \ \beta'_{(11)} \ \tau'_{(11)}],$$

$$\left[\beta'_{(12)} \quad \tau'_{(12)} \dots \beta'_{(1k)} \quad \tau'_{(1k)} \quad \mu_1 \quad \mu_2 \dots \mu_k \right]'$$

So model (16) can be written as

$$Y = \Gamma_2^* \Theta_2^* + \Psi \tag{18}$$

Subject to $\Omega_2^* \Theta_2^* = 0$

where $\Gamma_2^* = \Gamma_1^* \Lambda^*$, and

$$\Omega_2^* = [I_{2k} \quad \Delta \quad 0'_k \otimes 1_{2k}] ,$$

$$\Delta = \text{diag} \left[\begin{bmatrix} 1'_b & 0'_t \\ 0'_b & 1'_t \end{bmatrix}, \begin{bmatrix} 1'_b & 0'_t \\ 0'_b & 1'_t \end{bmatrix}, \dots, \begin{bmatrix} 1'_b & 0'_t \\ 0'_b & 1'_t \end{bmatrix} \right]$$

$$\Theta_2^* = \Lambda^* \Theta_1^* .$$

Now partition Ω_2^* and Γ_2^* as $\Omega_2^* = [\Omega_{21}^* \quad \Omega_{22}^*]$ and $\Gamma_2^* = [\Gamma_{21}^* \quad \Gamma_{22}^*]$ is the first $2k$ column of Ω_2^* and the Ω_{22}^* is the rest of the column of Ω_2^* . $\Omega_{21}^* = I_{2k}$. By applying model reduction methods, then we have the unconstraint model

$$Y_r = \Gamma_r \Theta_r + \Psi \tag{19}$$

where

$$Y_r = Y - \Gamma_{21}^* \Omega_{21}^{*-1} g , \text{ since } g = 0, \text{ then}$$

$$Y_r = Y ,$$

$$\Gamma_r = \Gamma_{22}^* - \Gamma_{21}^* \Omega_{21}^{*-1} \Omega_{22}^* , \text{ since } \Omega_{21}^* = I_{2k} ,$$

then we have

$$\Gamma_r = \Gamma_{22}^* - \Gamma_{21}^* \Omega_{22}^* .$$

and

$$\Theta_r = \left[\beta'_{(11)} \quad \tau'_{(11)} \quad \beta'_{(12)} \quad \tau'_{(12)} \dots \beta'_{(1k)} \quad \tau'_{(1k)} \quad \mu_1 \quad \mu_2 \dots \mu_k \right]$$

Design matrix Γ_r has rank of size $k(b+t-1)$. Therefore, model (19) is full column rank model and unconstraint model. To analyze the model, namely to estimation and testing the parameter of the model (19) we can use the standard methods.

Estimation and Testing Hypothesis:

Under the assumption that the variance of each model in (12) : $\sigma_1^2, \sigma_2^2 \dots, \sigma_k^2$ are known and equal, then the variance and covariance matrix

$$\text{Var}(\Psi) = \Sigma = \sigma^2 I_{kn} , \tag{20}$$

and assume that the distribution of Ψ is multivariate normal with mean zero vector and variance and covariance matrix satisfied (20).

Then the estimation of

$$\hat{\Theta}_r = (\Gamma'_r \Gamma_r)^{-1} \Gamma'_r Y \tag{21}$$

and

$$\hat{\sigma}^2 = \frac{1}{kn - k(b+t-1)} Y' (I_{kn} - \Gamma'_r \Gamma_r^-) Y \tag{22}$$

where the A stand for generalized inverse of a matrix A (see, Graybill (1969, 1976)). The estimation given in (21) and (22) has the optimal property, Uniformly Minimum Variance Unbiased Estimation (UMVUE), see Graybill (1976), Theil (1971).

Based on model (19) we can test some parameters of interest. For instant, we can test that the k th model given in (12) are equal. Under model (19), the hypothesis can be written as

$$\beta_{i2} = \beta_{i3} = \dots = \beta_{ik} = \beta_i^* \quad \forall i = 1, 2, \dots, b.$$

Ho:

$$\tau_{j2} = \tau_{j3} = \dots = \tau_{jk} = \tau_j^* \quad \forall j = 1, 2, \dots, t$$

β_i^* and τ_j^* are known constants.

Which is equivalent to test the hypothesis that

$$\text{Ho: } H\Theta_r = h \tag{23}$$

Where

$$H = \begin{bmatrix} H^* & 0_{k(b+t-2) \times k} \end{bmatrix}$$

$$H^* = \text{diag} \left[\begin{bmatrix} I_{b-1} & 0 \\ 0 & I_{t-1} \end{bmatrix}, \begin{bmatrix} I_{b-1} & 0 \\ 0 & I_{t-1} \end{bmatrix}, \dots, \begin{bmatrix} I_{b-1} & 0 \\ 0 & I_{t-1} \end{bmatrix} \right]$$

$\text{rank}(H^*) = k(b+t-2)$.

$$h' = (\beta_2^*, \beta_3^*, \dots, \beta_b^*, \tau_2^*, \tau_3^*, \dots, \tau_t^*, \dots, \beta_2^*, \beta_3^*, \dots, \beta_b^*, \tau_2^*, \dots, \tau_t^*)$$

h is known vector $k(b+t-2) \times 1$.

The likelihood ratio test is given by

$$\lambda_1 = \frac{(H\hat{\Theta}_r - h)' [H(\Gamma'_r \Gamma_r)^{-1} H']^{-1} (H\hat{\Theta}_r - h)}{Y'(I - \Gamma'_r \Gamma_r^-) Y} \times \left(\frac{kn - k(b+t-1)}{k(b+t-2)} \right)$$

Under the null hypothesis, λ_1 has F distribution with degrees of freedom $kn - k(b+t-1)$ and $k(b+t-2)$.

If the variances of each model (12) : $\sigma_1^2, \sigma_2^2, \dots, \sigma_k^2$ are known but unequal, then the variance covariance matrix of the combined model is given by

$$\text{Var}(\Psi) = \text{diag}(\sigma_1^2 I_n, \sigma_2^2 I_n, \dots, \sigma_k^2 I_n),$$

or

$$\text{Var}(\Psi) = \Sigma \otimes I_n,$$

where $\Sigma = \text{diag}(\sigma_1^2, \sigma_2^2, \dots, \sigma_k^2)$.

The approach to deal with this type of problem, we can use generalized linear model (Mustofa, 1995; Bhapkar, 1976; Graybill, 1976; Rao, 1973; Theil, 1971; Arnold, 1980). Then the estimation of parameter vector Θ_r is given by

$$\hat{\Theta}_r = \left(\Gamma_r' (\Sigma^{-1} \otimes I_n) \Gamma_r \right)^{-1} \Gamma_r' (\Sigma^{-1} \otimes I_n) Y \quad (24)$$

With the covariance matrix

$$\text{Var}(\Theta_r) = \left[\Gamma_r' (\Sigma^{-1} \otimes I_n) \Gamma_r \right]^{-1} \quad (25)$$

Under this assumption, to test the hypothesis given in (23), the likelihood ratio test statistics is

$$\lambda_2 = \frac{(H\hat{\Theta}_r - h)' \left[H (\Gamma_r' (\Sigma^{-1} \otimes I_n) \Gamma_r)^{-1} H' \right]^{-1} (H\hat{\Theta}_r - h)}{(Y - \Gamma_r \hat{\Theta}_r)' (I - \Gamma_r \Gamma_r^{-1}) (Y - \Gamma_r \hat{\Theta}_r)} \times \left(\frac{kn - k(b+t-1)}{k(b+t-2)} \right) \quad (26)$$

Under the null hypothesis, λ_2 has an F distribution with $kn - k(b+t-1)$ and $k(b+t-2)$ degrees of freedom.

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