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Research Paper

On the Characteristic Function of the Four-Parameter Generalized Beta of the Second Kind (GB2) Distribution and Its Approximation to the Singh-Maddala, Dagum, and Fisk Distributions

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Abstract
Researchers have thoroughly investigated generalized distributions due to their inherent flexibility, which allows them to include several well-known distributions as special cases. Among these, the four-parameter Generalized Beta of the Second Kind (GB2) distribution stands out as one of the most versatile frameworks in probability theory. Despite its broad applications, the GB2 distribution's characteristic function, a critical tool in probability and statistical analysis, lacks a closed-form solution in the existing literature. This study pursues two primary objectives: first, to derive the characteristic function and the k^{th} moment of the GB2 distribution, and second, to demonstrate how the GB2 distribution can serve as a close approximation to the Singh-Maddala, Dagum, and Fisk distributions using its characteristic function and k^{th} moment. These derivations and approximations rely on gamma and beta functions, supplemented by the Maclaurin series expansion.

Keywords
Generalized Beta of the Second Kind, Characteristic Function, Moment, Gamma Function, Beta Function, Mclaurin Series

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1. INTRODUCTION

Numerous authors have recommended generalized distributions for their adaptability, as they encompass numerous well-known distributions as specific instances. Notably, the Generalized Beta distribution of the Second Kind (GB2) with four parameters stands out for its remarkable flexibility and suitability for probabilistic modeling. First introduced by McDonald (1984), the GB2 distribution plays a crucial role in probability and statistics, providing substantial versatility for modeling a wide range of data types, from those with moderate skewness to those exhibiting extreme kurtosis. This distribution's ability to represent various other forms, along with its broad applicability, makes it a valuable tool across fields such as economics, finance, insurance, environmental studies, and healthcare.

The GB2 distribution is widely used to model income distribution and is a prominent tool for modeling income distributions, noted for its ability to capture heavy tails effectively. Its accuracy in representing income data is well-recognized (Jenkins, 2009, 2011; McDonald and Xu, 1995). In the insurance domain, Cummins et al. (1990) explored the GB2 distribution's capability in modeling loss processes, emphasizing its effectiveness in addressing the skewness and heavy tails standard in insurance losses-making it particularly useful in actuarial applications. Graf and Nedyalkova (2014) recommended the GB2 distribution for modelling income and poverty metrics, while Feng et al. (2006) used the GB2 model to resolve temporal inconsistencies in high-income data from the US Population Survey. Parker (1999) suggested that income distributions generated by behavioral optimization models align with the GB2 form. Warsono (2010) linked the GB2 distribution to the generalization of the log-logistic distribution through the reparameterization of the moment generating function. Furthermore, Warsono et al. (2017) explored the connection of the GB2 to the generalized F distributions. Warsono et al. (2018) employed a method of maximum likelihood to examine the properties of this distribution across various sample sizes.

The GB2 distribution includes several other distributions as exceptional cases, such as the Dagum (Burr Type III), Singh-Maddala (Burr Type XII), and Fisk probability distributions, among others (Chotikapanich et al., 2018; Cummins et al., 1990; Jenkins, 2011; Kleiber, 2008; McDonald, 1984; McDonald and Xu, 1995; McDonald et al., 2013; Slottje, 2010). The Singh-Maddala distribution is extensively used to analyze diverse datasets, including those related to income, expenditures, actuarial data, environmental assessments, and reliability studies. Singh and Maddala (1976) demonstrate that this distribution offers a superior fit in modeling income data compared to other models, such as the lognormal and Pareto distributions. Han and Cheng (2019) indicate that a truncated form of the Singh-Maddala model provides a more accurate depiction of income distribution and inequality, especially in the presence of high-income outliers that significantly influence metrics like the Gini coefficient. The Dagum distribution, as discussed by Kleiber (2008), Domma and Perri (2009), Pérez and Alaiz (2011), and Dey et al. (2017), is commonly used in reliability studies, particularly in the development of test plans for products and materials. It helps to determine optimal test termination times, taking into account sample sizes and risk factors (Dey et al., 2017). Kurniasari et al. (2017) analyzed several properties of the Dagum distribution, including its moments, cumulants, kurtosis, and characteristic function. Meanwhile, the Fisk distribution is frequently applied in insurance data analysis (Klugman et al., 2012), with Ahsanullah and Alzaatreh (2018) examining its fundamental properties.

The characteristic function of a distribution is a fundamental concept in both probability and statistics. Unlike the moment-generating function, it has a universal definition due to the bounded nature of the exponential function when applied to an imaginary argument (Kendall and Stuart, 1968). This function serves as a powerful and adaptable tool, playing a central role in distribution theory, limit theorems, stability analysis, and various practical statistical applications (Lukacs, 1963). However, there is a lack of closed-form expressions for the characteristic function of the GB2 distribution in existing literature. The GB2 distribution features a complex probability density function with multiple shape parameters and the beta function, complicating the derivation of its characteristic function. Consequently, the exact form of the characteristic function for the GB2 distribution remains complex and is typically not available in a closed form.

The aim of this paper is twofold. Firstly, it seeks to derive the characteristic function and the kth moment of the GB2 distribution through closed-form expressions, incorporating constants, variables, and a limited set of fundamental functions connected by arithmetic operations and function composition. Secondly, by utilizing the characteristic function, the paper demonstrates how the GB2 distribution can be approximated to specific instances of the Singh-Maddala, Dagum, and Fisk distributions.

2. EXPERIMENTAL SECTION

2.1 Probability Distribution Functions

As stated by McDonald (1984), a four-parameter GB2 distribution characterizes a continuous and random variable X, with its probability density function (pdf) given by the expression in Equation (1):

$$f(x;\alpha,\beta,\rho,\delta)=\frac{\alpha x^{\alpha\rho-1}}{\beta^{\alpha\rho}B(\rho,\delta)\left(1+(\frac{x}{\beta})^{\alpha}\right)^{\rho+\delta}}$$
$$0<x, 0<\alpha, 0<\beta, 0<\rho, 0<\delta \tag{1}$$

The symbol $B(\rho,\delta)$ refers to the complete beta function, while $\Gamma(\rho)$ represents denotes the gamma function. The parameter $\beta>0$ acts as the scale parameter, whereas $\rho>0$, $\delta>0$ and $a>0$ are the shape parameters. More specifically, α characterizes the overall shape, ρ controls the left tail, and δ denotes the right tail.

The pdf for the Singh-Maddala distribution is described by Equation (2):

$$f(x;\alpha,\beta,\delta)=\frac{\alpha\delta x^{\alpha-1}}{\beta^{\alpha}\left(1+(\frac{x}{\beta})^{\alpha}\right)^{1+\delta}}$$
$$0<x, 0<\alpha, 0<\beta 0<\delta \tag{2}$$

The pdf for the Dagum distribution is defined by Equation (3):

$$f(x;\alpha,\beta,\rho)=\frac{\alpha\rho x^{\alpha-1}}{\beta^{\alpha\rho}\left(1+(\frac{x}{\beta})^{\alpha}\right)^{\rho+1}}$$
$$0<x, 0<\alpha, 0<\beta 0<\rho, \tag{3}$$

Similarly, the pdf for the Fisk distribution is represented by Equation (4):

$$f(x;\alpha,\beta)=\frac{\alpha x^{\alpha-1}}{\beta^{\alpha}\left(1+(\frac{x}{\beta})^{\alpha}\right)^{2}}$$
$$0<x, 0<\alpha, 0<\beta \tag{4}$$

2.2 Characteristic Function

The characteristic function of a distribution is essential in probability theory and statistics, offering significant insights into the distribution's properties and behavior in the frequency domain. This function is handy for analyzing moments, convolutions, and the tail behavior of distributions. For the GB2 distribution, the characteristic function is key to understanding and utilizing its properties. In theory, it is preferable to obtain a closed-form expression for the characteristic function of any distribution, whenever feasible. The characteristic function for the GB2 distribution is given by Equation (5):

$$\varphi_X(t)=E(e^{itx})=\int_0^{\infty}e^{itx}f(x;\alpha,\beta,\rho,\delta)dx \tag{5}$$

where $i=\sqrt{-1}$ is a complex number.

2.3 Moment

The characteristic function is pivotal in deriving the moments of a distribution, which are fundamental for capturing its shape, spread, and other critical attributes. Moments provide valuable insights into key distribution characteristics, such as the first moment for a expected value, the second central moment for a variance, the third standardized moment for skewness, and the fourth standardized moment for kurtosis. In this research, moments are generated using the characteristic function by computing the partial derivatives with respect to t, and then evaluating them at $t = 0$. Suppose $\varphi_x(t)$ is a characteristic function of a random variable X, then moments of X are calculated through Equation (6):

$$E(X^k) = (-i)^k \frac{d^k}{dt^k}\varphi_X(0) \tag{6}$$

2.4 McLaurin Series, Beta Function, Gamma Function, and Stirling Formula

This study employs the Maclaurin series expansion of the exponential function to derive the characteristic function, as given in Equation (7):

$$e^{tx} = \sum_{n=0}^{\infty} \frac{(tx)^n}{n!} \tag{7}$$

The gamma function, as described by Abramowitz et al. (1988), is expressed in Equation (8):

$$\Gamma(\rho) = \int_0^{\infty} x^{-\rho} e^{-x} dx \tag{8}$$

For this analysis, two fundamental properties of the beta function, as outlined by Abramowitz et al. (1988), are applied. These properties are expressed in Equations (9) and (10):

$$B(\rho, \delta) = \frac{\Gamma(\rho)\Gamma(\delta)}{\Gamma(\rho+\delta)} \tag{9}$$

$$B(\rho, \delta) = \int_0^{\infty} \frac{(z)^{\rho-1}}{(1+z)^{\rho\delta}} \tag{10}$$

2.5 Graphical Method

This study provides graphical illustrations of the pdf and characteristic function for different parameter values of the GB2, Singh-Maddala, Dagum, and Fisk distributions. The graphs are designed to demonstrate the core properties of these distributions and aid in their approximation.

3. RESULT AND DISCUSSION

3.1 Basic Features of the GB2 Distribution

McDonald (1984), McDonald et al. (2013), and Chotikapanich et al. (2018) highlight that the Singh–Maddala and Dagum distributions represent specific instances of the GB2 distribution, derived by setting $\rho = 1, \delta = 1$, respectively. When both ρ and δ are equal to 1, the Fisk distribution becomes a special case of the GB2 distribution.

Figure 1 presents the probability density functions of the GB2, Singh-Maddala, Dagum, and Fisk distributions for various parameter values, as detailed by Chotikapanich et al. (2018). The figure demonstrates that, for the chosen parameter values, the Dagum distribution closely approximates the GB2 distribution. Furthermore, the Singh-Maddala and Fisk distributions show similarities to the GB2 distribution, characterized by asymmetry and positive skewness, indicating a tendency toward higher values of the random variable. Consequently, the Dagum, Singh-Maddala, and Fisk distributions can be effectively represented by the GB2 distribution in graphical analysis.

3.2 Characteristic Function of the GB2 Distribution

Theorem 1. *Let X be a random variable of four-parameter GB2(α, β, ρ, δ) distribution, then the characteristic function of X is expressed by Equation (11):*

$$\varphi_X(t) = \sum_{n=0}^{\infty} \frac{(it)^n}{n!}(\beta)^n \frac{\Gamma\left(\rho+\frac{n}{\alpha}\right)\Gamma\left(\delta-\frac{n}{\alpha}\right)}{\Gamma(\rho)\Gamma(\delta)} \tag{11}$$

Proof: The characteristic function $\varphi_X(t)$ is the expected value of e^{itx}, expressed as:

$$\begin{aligned}\varphi_X(t) =& E\left(e^{itx}\right) = \int_0^{\infty} e^{itx} f(x;\alpha,\beta,\rho,\delta)\,dx \\ &= \int_0^{\infty} e^{itx} \frac{\alpha x^{\alpha\rho-1}}{\beta^{\alpha\rho}B(\rho,\delta)\left(1+\left(\frac{x}{\beta}\right)^{\alpha}\right)^{\rho+\delta}} dx\end{aligned}$$

Simplifying further as Equation (12):

$$\varphi_X(t) = \frac{1}{\beta^{\alpha\rho}B(\rho,\delta)} \int_0^{\infty} e^{itx} \frac{\alpha x^{\alpha\rho-1}}{\left(1+\left(\frac{x}{\beta}\right)^{\alpha}\right)^{\rho+\delta}} dx \tag{12}$$

Now, let $u = \left(\frac{x}{\beta}\right)^{\alpha}$, which implies that $x = u^{1/\alpha}\beta$ and $dx = \frac{1}{\alpha}u^{1/\alpha-1}\beta\, du$. As a result, Equation (12) can be rewritten as Equation (13):

$$\varphi_X(t) = \frac{1}{\beta^{\alpha\rho}B(\rho,\delta)} \int_0^{\infty} e^{it\beta u^{1/\alpha}} \frac{\alpha (u^{1/\alpha}\beta)^{\alpha\rho-1}}{(1+u)^{\rho+\delta}} \frac{1}{\alpha} u^{1/\alpha-1}\beta\, du \tag{13}$$

Using a well-known property of the Maclaurin series, Equation (13) is transformed into Equation (14):

$$\begin{aligned}\varphi_X(t) &= \frac{\beta^{\alpha\rho}}{\beta^{\alpha\rho}B(\rho,\delta)} \int_0^{\infty} \sum_{n=0}^{\infty} \frac{(it\beta)^n}{n!} u^{n/\alpha} \frac{(u^{1/\alpha})^{\alpha\rho-1}}{(1+u)^{\rho+\delta}} \\ &\quad u^{1/\alpha-1}\, du \\ &= \frac{1}{B(\rho,\delta)} \int_0^{\infty} \sum_{n=0}^{\infty} \frac{(it\beta)^n}{n!} u^{\frac{n}{\alpha}+\rho-1} \frac{1}{(1+u)^{\rho+\delta}}\, du \\ &= \frac{1}{B(\rho,\delta)} \sum_{n=0}^{\infty} \frac{(it\beta)^n}{n!} \int_0^{\infty} \frac{u^{\frac{n}{\alpha}+\rho-1}}{(1+u)^{\rho+\delta}}\, du \end{aligned} \tag{14}$$

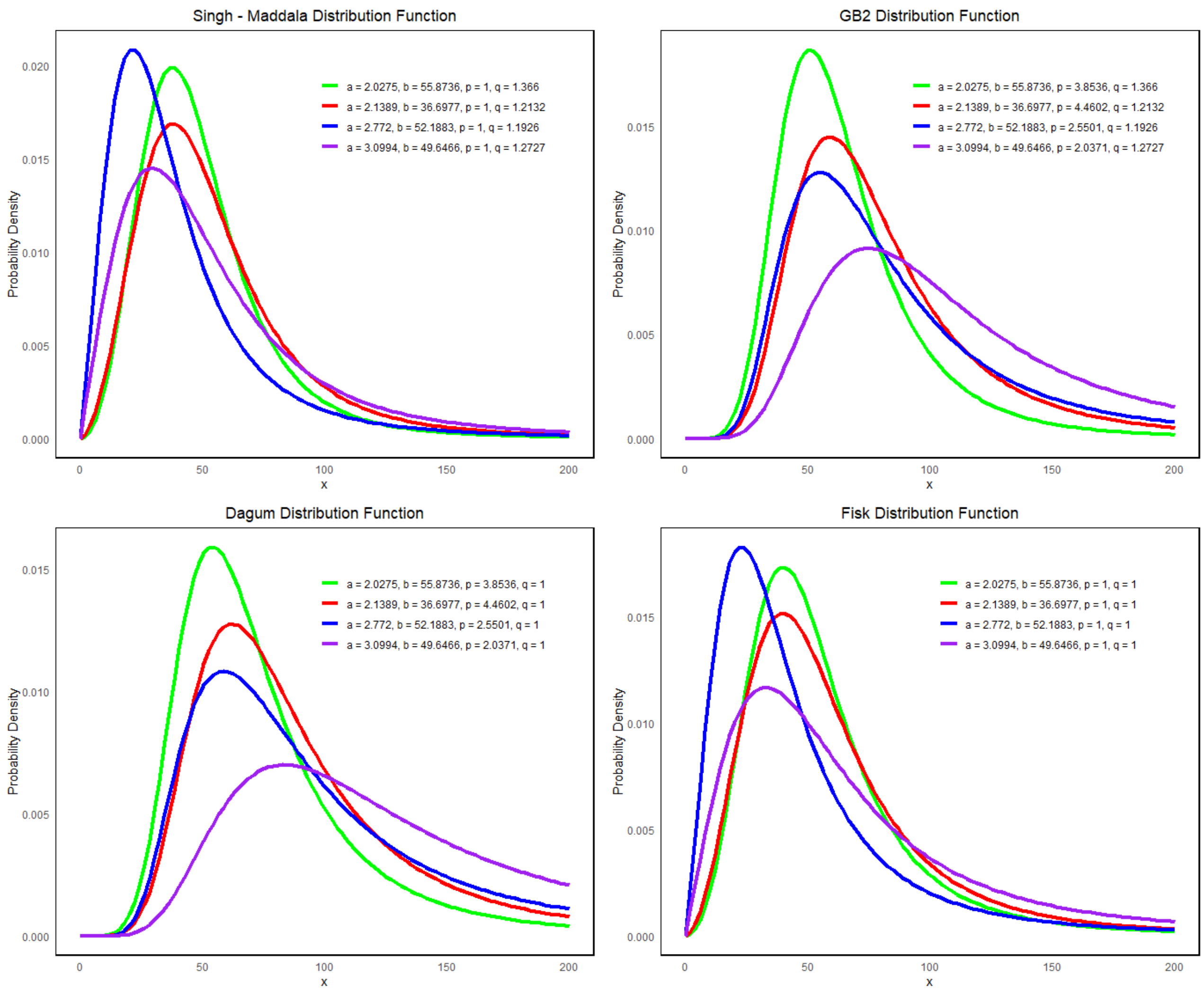

Figure 1. Probability Density Functions of the GB2, Singh-Maddala, Dagum, and Fisk Distributions, with Varying Parameter Values for Each Distribution

The integral term in Equation (14) can be expressed in terms of a beta function, as demonstrated in Equation (10). Consequently, Equation (14) is restructured as Equation (15):

$$\varphi_X(t) = \frac{1}{B(\rho,\delta)} \sum_{n=0}^{\infty} \frac{(it\beta)^n}{n!} B\left(\rho + \frac{n}{\alpha}, \delta - \frac{n}{\alpha}\right)$$
$$= \sum_{n=0}^{\infty} \frac{(it\beta)^n}{n!} \frac{B\left(\rho + \frac{n}{\alpha}, \delta - \frac{n}{\alpha}\right)}{B(\rho,\delta)} \quad (15)$$

The characteristic function, as seen in Equation (15), is then rewritten in terms of the gamma function:

$$\varphi_X(t) = \sum_{n=0}^{\infty} \frac{(it)^n}{n!} (\beta)^n \frac{\Gamma\left(\rho + \frac{n}{\alpha}\right)\Gamma\left(\delta - \frac{n}{\alpha}\right)}{\Gamma(\rho)\Gamma(\delta)} \quad (16)$$

Thus, the theorem presented above has been proven.

Figure 2 exemplifies the characteristic functions of the GB2, Singh-Maddala, Dagum, and Fisk distributions. The x-axis corresponds to the real components of the parameter pairs a and b, while the y-axis represents the imaginary components, which are expressed as $\frac{(it\beta)^n}{n!} \frac{B(\rho+\frac{n}{\alpha},\delta-\frac{n}{\alpha})}{B(\rho,\delta)}$. The z-axis is used to display the values of the characteristic functions.

The proposition highlights a characteristic property of the GB2 distribution's characteristic function.

Proposition 1: *Given $\varphi_X(X)$ represents the characteristic function of the GB2 distribution, we have $\varphi_X(0) = 1$*

Proof: The characteristic function of X, denoted as $\varphi_X(0)$, is expressed as:

$$\varphi_X(t) = \sum_{n=0}^{\infty} \frac{(it\beta)^n}{n!} \frac{B\left(\rho + \frac{n}{\alpha}, \delta - \frac{n}{\alpha}\right)}{B(\rho,\delta)}$$

which expand as:

$$= 1 + \frac{(it\beta)^1}{1!} \frac{B\left(\rho + \frac{1}{\alpha}, \delta - \frac{1}{\alpha}\right)}{B(\rho,\delta)} + \frac{(it\beta)^2}{2!} \frac{B\left(\rho + \frac{2}{\alpha}, \delta - \frac{2}{\alpha}\right)}{B(\rho,\delta)} +$$

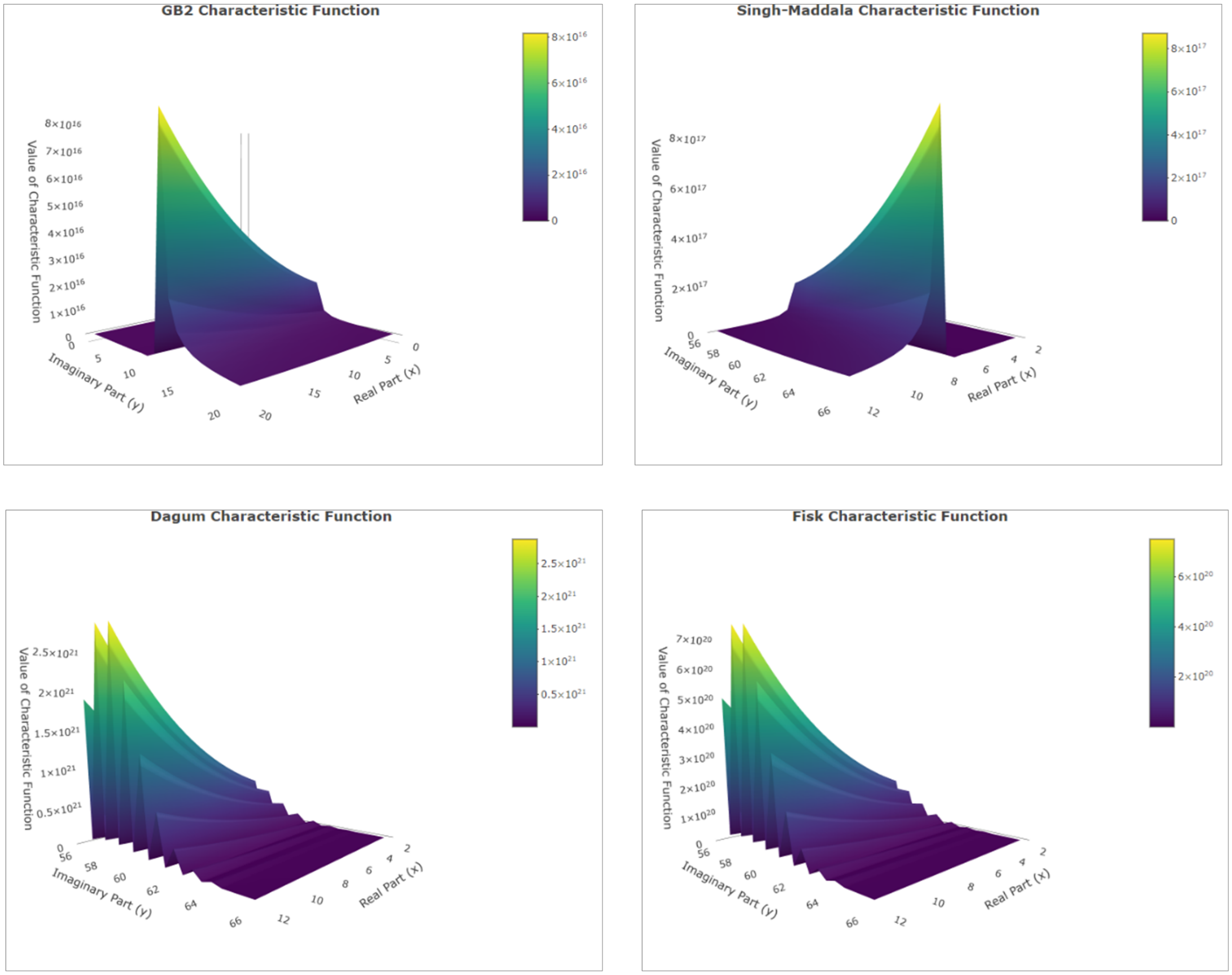

Figure 2. Characteristic Function Graphs for GB2, Singh-Maddala, Dagum, and Fisk Distributions

$$\frac{(it\beta)^3}{3!}\frac{B\left(\rho+\frac{3}{\alpha},\delta-\frac{3}{\alpha}\right)}{B(\rho,\delta)}+\cdots$$

At $t = 0$, the characteristic function simplifies to:

$$\varphi_X(t=0) = 1+0+0+0+\cdots = 1$$

3.3 Moment of the GB2 Distribution Based on Characteristic Function

The characteristic function is crucial for obtaining moments, which are fundamental in characterizing the shape, dispersion, and other vital features of a distribution. Moments are integral to statistical analysis, especially in applied settings, as they enable a thorough assessment of a distribution's primary properties. These moments yield vital statistical measures, such as the mean and variance of the distribution.

The k^{th} moment of a random variable X that follows the GB2 distribution can be determined using the theorem below:

Theorem 2. *The kth moment of the random variable X of the GB2 distribution is given by Equation* (17):

$$E(X^k) = \beta^k\frac{B(\rho+\frac{k}{\alpha},\delta-\frac{k}{\alpha})}{B(\rho,\alpha)} \tag{17}$$

Proof:
The derivative expressions of the characteristic function, as shown in Equation (11), indicate the following results: $\varphi'_X(0) = iE(X)$, $\varphi''_X(0) = -E(X^2)$, and $\varphi'''_X(0) = -iE(X^3)$, with analogous patterns for higher-order derivatives. Therefore, the first moment of the GB2 distribution can be derived by differentiating the characteristic function, as shown.

$$\frac{d(\varphi(t))}{d(t)} = \frac{d}{d(t)}\left(\sum_{n=0}^{\infty}\frac{(it\beta)^n}{n!}\frac{B\left(\rho+\frac{n}{\alpha},\delta-\frac{n}{\alpha}\right)}{B(\rho,\delta)}\right)$$

$$\frac{d(\varphi(t))}{d(t)} = \frac{d}{d(t)}\Bigg(1+\frac{(it\beta)^1}{1!}\frac{B\left(\rho+\frac{1}{\alpha},\delta-\frac{1}{\alpha}\right)}{B(\rho,\delta)}+$$
$$\frac{(it\beta)^2}{2!}\frac{B\left(\rho+\frac{2}{\alpha},\delta-\frac{2}{\alpha}\right)}{B(\rho,\delta)}+\cdots\Bigg)$$

$$\frac{d(\varphi(t))}{d(t)} = 0 + (i\beta)\frac{B\left(\rho+\frac{1}{\alpha},\delta-\frac{1}{\alpha}\right)}{B(\rho,\delta)} + \frac{2(i\beta)^2 t}{2!}\frac{B\left(\rho+\frac{2}{\alpha},\delta-\frac{2}{\alpha}\right)}{B(\rho,\delta)} + \cdots$$

$$\left.\frac{d(\varphi(t))}{d(t)}\right|_{t=0} = 0 + (i\beta)\frac{B\left(\rho+\frac{1}{\alpha},\delta-\frac{1}{\alpha}\right)}{B(\rho,\delta)} + 0 + 0 + \cdots = (i\beta)\frac{B\left(\rho+\frac{1}{\alpha},\delta-\frac{1}{\alpha}\right)}{B(\rho,\delta)}$$

Therefore, the value of $\varphi_X^{'}(0)$ is given by $iE(X)$, which is equal to

$$(i\beta)\frac{B\left(\rho+\frac{1}{\alpha},\delta-\frac{1}{\alpha}\right)}{B(\rho,\delta)},$$

and the first moment is written by Equation (18):

$$E(X) = \beta\frac{B\left(\rho+\frac{1}{\alpha},\delta-\frac{1}{\alpha}\right)}{B(\rho,\delta)} \tag{18}$$

The second moment of the GB2 distribution is obtained by following these steps.

$$\frac{d(\varphi(t))}{d(t)} = (i\beta)\frac{B\left(\rho+\frac{1}{\alpha},\delta-\frac{1}{\alpha}\right)}{B(\rho,\delta)} + \frac{2(i\beta)^2 t}{2!}\frac{B\left(\rho+\frac{2}{\alpha},\delta-\frac{2}{\alpha}\right)}{B(\rho,\delta)} + \cdots$$

$$\frac{d^2(\varphi(t))}{d(t)^2} = \frac{d}{d(t)^2}\left((i\beta)\frac{B\left(\rho+\frac{1}{\alpha},\delta-\frac{1}{\alpha}\right)}{B(\rho,\delta)} + \frac{2(i\beta)^2 t}{2!}\frac{B\left(\rho+\frac{2}{\alpha},\delta-\frac{2}{\alpha}\right)}{B(\rho,\delta)} + \cdots\right)$$

$$\frac{d^2(\varphi(t))}{d(t)^2} = 0 + \frac{2(i\beta)^2}{2!}\frac{B\left(\rho+\frac{2}{\alpha},\delta-\frac{2}{\alpha}\right)}{B(\rho,\delta)} + \cdots$$

$$\left.\frac{d^2(\varphi(t))}{d(t)^2}\right|_{t=0} = 0 + \frac{2(i\beta)^2}{2!}\frac{B\left(\rho+\frac{2}{\alpha},\delta-\frac{2}{\alpha}\right)}{B(\rho,\delta)} + 0 + \cdots = (i\beta)^2\frac{B\left(\rho+\frac{2}{\alpha},\delta-\frac{2}{\alpha}\right)}{B(\rho,\delta)}$$

The second moment of the GB2 distribution can thus be expressed as shown in Equation (19):

$$E(X^2) = -\varphi_X^{''}(0) = -(i\beta)^2\frac{B\left(\rho+\frac{2}{\alpha},\delta-\frac{2}{\alpha}\right)}{B(\rho,\delta)} = \beta^2\frac{B\left(\rho+\frac{2}{\alpha},\delta-\frac{2}{\alpha}\right)}{B(\rho,\delta)} \tag{19}$$

Therefore,

$$\varphi_X^{''}(0) = -E(X^2) = (i\beta)^2\frac{B\left(\rho+\frac{2}{\alpha},\delta-\frac{2}{\alpha}\right)}{B(\rho,\delta)}$$

$$E(X^2) = \beta^2\frac{\Gamma\left(\rho+\frac{2}{\alpha}\right)\Gamma\left(\delta-\frac{2}{\alpha}\right)}{\Gamma(\rho)\Gamma(\delta)}$$

The k^{th} moment of the GB2 distribution can ultimately be determined as given in Equation (17):

$$\left.\frac{d^k(\varphi(t))}{d(t)^k}\right|_{t=0} = (i\beta)^k\frac{B\left(\rho+\frac{k}{\alpha},\delta-\frac{k}{\alpha}\right)}{B(\rho,\delta)} = (i\beta)^k\frac{\Gamma\left(\rho+\frac{k}{\alpha}\right)\Gamma\left(\delta-\frac{k}{\alpha}\right)}{\Gamma(\rho)\Gamma(\delta)}$$

$$\left.\frac{d^k(\varphi(t))}{d(t)^k}\right|_{t=0} = i^k\beta^k\frac{\Gamma\left(\rho+\frac{k}{\alpha}\right)\Gamma\left(\delta-\frac{k}{\alpha}\right)}{\Gamma(\rho)\Gamma(\delta)}$$

$$E(X^k) = (-i)^k\varphi_X^{(k)}(0) = (-i)^k i^k\beta^k\frac{B\left(\rho+\frac{k}{\alpha},\delta-\frac{k}{\alpha}\right)}{B(\rho,\delta)}$$

$$E(X^k) = \beta^k\frac{B\left(\rho+\frac{k}{\alpha},\delta-\frac{k}{\alpha}\right)}{B(\rho,\delta)} = \beta^k\frac{\Gamma\left(\rho+\frac{k}{\alpha}\right)\Gamma\left(\delta-\frac{k}{\alpha}\right)}{\Gamma(\rho)\Gamma(\delta)}$$

Thus, Theorem 1 has been formulated. The closed-form formula for the k^{th} moment of the GB2 distribution, obtained from its characteristic function as outlined earlier and shown in Equation (17), fully aligns with the moments identified by McDonald (1984), Jenkins (2011), Graf and Nedyalkova (2014), and Chotikapanich et al. (2018). Furthermore, the accuracy of these moment equations is further validated by the expected value definition provided in Theorem 3.

Theorem 3: *Let $E(X)$ be an expected value of the GB2, then $E(X)$ can be calculated using Equation (20) and proven by Equation (21):*

$$E(X) = \beta\frac{\Gamma\left(\rho+\frac{1}{\alpha}\right)\Gamma\left(\delta-\frac{1}{\alpha}\right)}{\Gamma(\rho)\Gamma(\delta)} \tag{20}$$

Proof:

$$\begin{aligned} E(X) &= \int_0^\infty x f(x;\alpha,\beta,\rho,\delta)\,dx \\ &= \int_0^\infty x\frac{\alpha x^{\alpha\rho-1}}{\beta^{\alpha\rho}B(\rho,\delta)\left(1+\left(\frac{x}{\beta}\right)^\alpha\right)^{\rho+\delta}}\,dx \\ &= \frac{\alpha}{\beta^{\alpha\rho}B(\rho,\delta)}\int_0^\infty\frac{x^{\alpha\rho}}{\left(1+\left(\frac{x}{\beta}\right)^\alpha\right)^{\rho+\delta}}\,dx \end{aligned} \tag{21}$$

The expression for $E(X)$ in Equation (18) can be reformulated as presented in Equation (22) by defining $u = \left(\frac{x}{\beta}\right)^{\alpha}$:

$$E(X) = \frac{\alpha}{\beta^{\alpha\rho}B(\rho,\delta)}\int_0^{\infty}\left(\left(u^{1/\alpha}\beta\right)^{\alpha\rho}\frac{1}{(1+u)^{\rho+\delta}}\frac{1}{\alpha}u^{1/\alpha-1}\beta\,du\right)$$
$$= \frac{\beta}{B(\rho,\delta)}\int_0^{\infty}\frac{u^{\rho+1/\alpha-1}}{(1+u)^{\rho+\delta}}\,du \tag{22}$$

Equation (22) can then be reformulated to take the form of a beta function, as illustrated in Equation (10) below:

$$E(X) = \frac{\beta}{B(\rho,\delta)}B\left(\rho+\frac{1}{\alpha}-1+1,\delta-\frac{1}{\alpha}+1-1\right)$$
$$= \frac{\beta B\left(\rho+\frac{1}{\alpha},\delta-\frac{1}{\alpha}\right)}{B(\rho,\delta)}$$
$$E(X) = \beta\frac{\Gamma\left(\rho+\frac{1}{\alpha}\right)\Gamma\left(\delta-\frac{1}{\alpha}\right)}{\Gamma(\rho)\Gamma(\delta)}$$

The proof of Theorem 3, concerning the expected value of the GB2 distribution, has been established. Using a similar approach, Equation (22) is shown to lead to Equation (23).

$$E(X^2) = \frac{\alpha}{\beta^{\alpha\rho}B(\rho,\delta)}\int_0^{\infty}\frac{x^2x^{\alpha\rho-1}}{(1+(x/\beta)^{\alpha})^{\rho+\delta}}\,dx$$
$$E(X^2) = \frac{\alpha}{\beta^{\alpha\rho}B(\rho,\delta)}\int_0^{\infty}\frac{x^{\alpha\rho+1}}{(1+(x/\beta)^{\alpha})^{\rho+\delta}}\,dx \tag{23}$$

By redefining $u = \left(\frac{x}{\beta}\right)^{\alpha}$, Equation (23) for $E(X^2)$ can be rewritten as presented in Equation (24):

$$E(X^2) = \frac{\alpha}{\beta^{\alpha\rho}B(\rho,\delta)}\int_0^{\infty}\left(\left(u^{1/\alpha}\beta\right)^{\alpha\rho+1}\frac{1}{(1+u)^{\rho+\delta}}\frac{1}{\alpha}u^{1/\alpha-1}\beta\,du\right)$$
$$= \frac{\beta^2}{B(\rho,\delta)}\int_0^{\infty}\frac{(u^{1/\alpha})^{\alpha\rho+1}}{(1+u)^{\rho+\delta}}u^{1/\alpha-1}\,du$$
$$= \frac{\beta^2}{B(\rho,\delta)}\int_0^{\infty}\frac{u^{\rho+1/\alpha}}{(1+u)^{\rho+\delta}}u^{1/\alpha-1}\,du$$
$$= \frac{\beta^2}{B(\rho,\delta)}\int_0^{\infty}\frac{u^{\rho+2/\alpha-1}}{(1+u)^{\rho+\delta}}\,du \tag{24}$$

The integral term in Equation (24) can be represented through the use of beta and gamma functions, as shown below:

$$E(X^2) = \frac{\beta^2}{B(\rho,\delta)}B\left(\rho+\frac{2}{\alpha}-1+1,\delta-\frac{2}{\alpha}+1-1\right)$$
$$E(X^2) = \frac{\beta^2 B\left(\rho+\frac{2}{\alpha},\delta-\frac{2}{\alpha}\right)}{B(\rho,\delta)} = \beta^2\frac{\Gamma\left(\rho+\frac{2}{\alpha}\right)\Gamma\left(\delta-\frac{2}{\alpha}\right)}{\Gamma(\rho)\Gamma(\delta)}$$

The k-expectation of the GB2 distribution can ultimately be represented by following the previously defined stages:

$$E(X^k) = \beta^k\frac{\Gamma\left(\rho+\frac{k}{\alpha}\right)\Gamma\left(\delta-\frac{k}{\alpha}\right)}{\Gamma(\rho)\Gamma(\delta)}$$

The moments of the GB2 distribution, derived from its characteristic function and the expected value definition, exhibit a similar pattern.

Moreover, the formulas below present the expressions for calculating the mean and variance of the GB2 distribution:

$$E(X) = \beta\frac{\Gamma\left(\rho+\frac{1}{\alpha}\right)\Gamma\left(\delta-\frac{1}{\alpha}\right)}{\Gamma(\rho)\Gamma(\delta)}$$

$$\mathrm{Var}(X) = \beta^2\left(\frac{\Gamma\left(\rho+\frac{2}{\alpha}\right)\Gamma\left(\delta-\frac{2}{\alpha}\right)}{\Gamma(\rho)\Gamma(\delta)} - \left(\frac{\Gamma\left(\rho+\frac{1}{\alpha}\right)\Gamma\left(\delta-\frac{1}{\alpha}\right)}{\Gamma(\rho)\Gamma(\delta)}\right)^2\right)$$

3.4 Approximation of the GB2 Distribution to Singh-Maddala Distribution

Jenkins (2011) states that the Singh-Maddala distribution is a particular case of the GB2 distribution when $\rho = 1$. When $\rho = 1$, the characteristic function of the GB2 distribution, as given in Equation (15), can be rewritten as shown in Equation (25):

$$\varphi_X(t) = \sum_{n=0}^{\infty}\frac{(it\beta)^n}{n!}\frac{B(1+n/\alpha,\delta-n/\alpha)}{B(1,\delta)}$$
$$= \sum_{n=0}^{\infty}\frac{(it\beta)^n}{n!}B(1+n/\alpha,\delta-n/\alpha)\frac{\Gamma(1+\delta)}{\Gamma(1)\Gamma(\delta)} \tag{25}$$

As a result, the characteristic function of the Singh-Maddala distribution can be written as:

$$\varphi_X(t) = \delta\sum_{n=0}^{\infty}\frac{(it\beta)^n}{n!}B(1+n/\alpha,\delta-n/\alpha)$$

The explicit formula for the characteristic function of the Singh-Maddala distribution, given in Equation (25), is consistent with the characteristic function derived from its definition, as described in Theorem 4.

Theorem 4. *Let X denote a random variable following the Singh-Maddala distribution. The characteristic function of X is given by Equation (26):*

$$\varphi_X(t) = \delta\sum_{n=0}^{\infty}\frac{(it\beta)^n}{n!}B(1+n/\alpha,\delta-n/\alpha) \tag{26}$$

Proof:
The characteristic function $\varphi_X(t)$ represents the expected value of e^{itx}, which can be written as:

$$\varphi_X(t) = E(e^{itx}) = \int_0^{\infty}e^{itx}f(x;\alpha,\beta,\delta)\,dx$$

$$= \int_0^\infty \frac{e^{itx}\,(\alpha\delta x^{\alpha-1})}{\beta^\alpha(1+(x/\beta)^\alpha)^{1+\delta}}\,dx$$

By defining $u = (x/\beta)^\alpha$, Equation (25) for $\varphi_X(t)$ can be reformulated as follows:

$$= \int_0^\infty e^{it\beta u^{1/\alpha}} \frac{\alpha\delta(u^{1/\alpha}\beta)^{\alpha-1}}{\beta^\alpha(1+u)^{1+\delta}} \frac{1}{\alpha} u^{1/\alpha-1}\beta\,du$$

$$= \frac{\delta\beta^{\alpha-1}\beta}{\beta^\alpha} \int_0^\infty \frac{e^{it\beta u^{1/\alpha}}(u^{1/\alpha})^{\alpha-1}}{(1+u)^{1+\delta}} u^{1/\alpha-1}\,du$$

$$= \delta \int_0^\infty \sum_{n=0}^\infty \frac{(it\beta)^n}{n!} u^{n/\alpha} \frac{1}{(1+u)^{1+\delta}}\,du$$

$$= \delta \int_0^\infty \sum_{n=0}^\infty \frac{(it\beta)^n}{n!} \frac{u^{n/\alpha}}{(1+u)^{1+\delta}}\,du$$

$$= \delta \sum_{n=0}^\infty \frac{(it\beta)^n}{n!} \int_0^\infty \frac{u^{n/\alpha}}{(1+u)^{1+\delta}}\,du$$

$$= \delta \sum_{n=0}^\infty \frac{(it\beta)^n}{n!} \int_0^\infty \frac{u^{1+n/\alpha-1}}{(1+u)^{(1+n/\alpha-1)+(\delta-n/\alpha+1)}}\,du \quad (27)$$

By replacing an integral component of Equation (27) with a beta function, the characteristic function of the Singh-Maddala distribution can be expressed as:

$$\varphi_X(t) = \delta \sum_{n=0}^\infty \frac{(it\beta)^n}{n!} B(1+n/\alpha, \delta - n/\alpha)$$

The proof of Theorem 4 has been established.

3.5 Moment of the Singh-Maddala Distribution

The k^{th} moment of the Sing-Maddala distribution can be computed using the characteristic function outlined in Equation (26), as shown in Equation (28).

The first moment:

$$E(X) = \delta\beta B\left(1+\frac{1}{\alpha}, \delta - \frac{1}{\alpha}\right) = \beta \frac{\Gamma\left(1+\frac{1}{\alpha}\right)\Gamma\left(\delta-\frac{1}{\alpha}\right)}{\Gamma(\delta)}$$

The second moment:

$$E(X^2) = \delta\beta^2 B\left(1+\frac{2}{\alpha}, \delta - \frac{2}{\alpha}\right) = \beta^2 \frac{\Gamma\left(1+\frac{2}{\alpha}\right)\Gamma\left(\delta-\frac{2}{\alpha}\right)}{\Gamma(\delta)}$$

The k^{th} moment:

$$E(X^k) = \delta\beta^k B\left(1+\frac{k}{\alpha}, \delta - \frac{k}{\alpha}\right) = \beta^k \frac{\Gamma\left(1+\frac{k}{\alpha}\right)\Gamma\left(\delta-\frac{k}{\alpha}\right)}{\Gamma(\delta)} \quad (28)$$

3.6 Approximation of the GB2 Distribution to Dagum Distribution

In relation to the GB2 distribution, the Dagum distribution is a unique instance of the GB2 when the parameter $\delta = 1$ (McDonald, 1984; McDonald and Xu, 1995). For $\delta = 1$, the characteristic function of GB2 in Equation (15) is then represented in Equation (29):

$$\varphi_X(t) = \sum_{n=0}^\infty \frac{(it\beta)^n}{n!} \frac{B\left(\rho+\frac{n}{\alpha}, 1-\frac{n}{\alpha}\right)}{B(\rho, 1)}$$

$$= \sum_{n=0}^\infty \frac{(it\beta)^n}{n!} B\left(\rho+\frac{n}{\alpha}, 1-\frac{n}{\alpha}\right) \frac{\Gamma(\rho+1)}{\Gamma(\rho)\Gamma(1)}$$

$$\varphi_X(t) = \beta^n \sum_{n=0}^\infty \frac{(it)^n}{n!} \frac{\Gamma\left(\rho+\frac{n}{\alpha}\right)\Gamma\left(1-\frac{n}{\alpha}\right)}{\Gamma(\rho)} \quad (29)$$

The characteristic function of the Dagum distribution, which is obtained from the definition given in Theorem 5, is consistent with the explicit formula of the characteristic function of this distribution stated in Equation (29).

Theorem 5. *Let X be a random variable of the Dagum distribution. The characteristic function of X is expressed by Equation* (30):

$$\varphi_X(t) = \beta^n \sum_{n=0}^\infty \frac{(it)^n}{n!} \frac{\Gamma\left(\rho+\frac{n}{\alpha}\right)\Gamma\left(1-\frac{n}{\alpha}\right)}{\Gamma(\rho)} \quad (30)$$

Proof:
The characteristic function of the Dagum distribution is derived from the probability density function presented in Equation (31).

$$\varphi_X(t) = \int_0^\infty e^{itx} f(x;\alpha,\beta,\rho)\,dx$$

$$= \int_0^\infty e^{itx} \frac{\alpha\rho x^{\alpha\rho-1}}{\beta^{\alpha\rho}\left(1+\left(\frac{x}{\beta}\right)^\alpha\right)^{\rho+1}}\,dx \quad (31)$$

By defining $u = \left(\frac{x}{\beta}\right)^\alpha$, we can express Equation (31) in an alternative form:

$$\varphi_X(t) = \int_0^\infty e^{it\beta u^{\frac{1}{\alpha}}} \frac{\alpha\rho(u^{\frac{1}{\alpha}}\beta)^{\alpha\rho-1}}{\beta^{\alpha\rho}(1+u)^{\rho+1}} \frac{1}{\alpha} u^{\frac{1}{\alpha}-1}\beta\,du$$

$$= \frac{\rho(\beta)^{\alpha\rho-1}\beta}{\beta^{\alpha\rho}} \int_0^\infty e^{it\beta u^{\frac{1}{\alpha}}} \frac{(u^{\frac{1}{\alpha}})^{\alpha\rho-1}}{(1+u)^{\rho+1}} u^{\frac{1}{\alpha}-1}\,du =$$

$$\rho \int_0^\infty e^{it\beta u^{\frac{1}{\alpha}}} \frac{(u^{\frac{1}{\alpha}})^{\alpha\rho-1}}{(1+u)^{\rho+1}} u^{\frac{1}{\alpha}} \frac{1}{u}\,du$$

$$= \rho \int_0^\infty \sum_{n=0}^\infty \frac{(it\beta)^n}{n!} u^{\frac{n}{\alpha}} \frac{u^{\rho-1}}{(1+u)^{\rho+1}}\,du$$

$$= \rho \sum_{n=0}^\infty \frac{(it\beta)^n}{n!} \int_0^\infty \frac{u^{\rho+\frac{n}{\alpha}-1}}{(1+u)^{\rho+1}}\,du$$

$$= \rho \sum_{n=0}^\infty \frac{(it\beta)^n}{n!} \int_0^\infty \frac{u^{\rho+\frac{n}{\alpha}-1}}{(1+u)^{\rho+\frac{n}{\alpha}-1+1-\frac{n}{\alpha}+1}}\,du \quad (32)$$

The characteristic function of the Dagum distribution can be expressed as follows:

$$\varphi_X(t) = \rho\beta^n \sum_{n=0}^{\infty} \frac{(it)^n}{n!} B\left(\rho + \frac{n}{\alpha}, 1 - \frac{n}{\alpha}\right)$$

$$= \rho\beta^n \sum_{n=0}^{\infty} \frac{(it)^n}{n!} B\left(\rho + \frac{n}{\alpha}, 1 - \frac{n}{\alpha}\right)$$

$$= \rho\beta^n \sum_{n=0}^{\infty} \frac{(it)^n}{n!} \frac{\Gamma\left(\rho + \frac{n}{\alpha}\right)\Gamma\left(1 - \frac{n}{\alpha}\right)}{\Gamma\left(\rho + \frac{n}{\alpha} + 1 - \frac{n}{\alpha}\right)}$$

$$= \rho\beta^n \sum_{n=0}^{\infty} \frac{(it)^n}{n!} \frac{\Gamma\left(\rho + \frac{n}{\alpha}\right)\Gamma\left(1 - \frac{n}{\alpha}\right)}{\Gamma(\rho + 1)}$$

$$= \rho\beta^n \sum_{n=0}^{\infty} \frac{(it)^n}{n!} \frac{\Gamma\left(\rho + \frac{n}{\alpha}\right)\Gamma\left(1 - \frac{n}{\alpha}\right)}{\rho\Gamma(\rho)}$$

$$= \beta^n \sum_{n=0}^{\infty} \frac{(it)^n}{n!} \frac{\Gamma\left(\rho + \frac{n}{\alpha}\right)\Gamma\left(1 - \frac{n}{\alpha}\right)}{\Gamma(\rho)}$$

Hence, Theorem 5 has been proven.

Equation (30) represents the characteristic function of the Dagum distribution, as indicated in Equation (29). Thus, this distribution is a special case of the GB2 distribution when $\delta = 1$.

3.7 Moment of the Dagum Distribution

Referring to the characteristic function presented in Equation (29), the k^{th} moment of the Dagum distribution can be derived as shown in Equation (33). The first moment of the Dagum distribution:

$$E(X) = \rho\beta B\left(\rho + \frac{1}{\alpha}, 1 - \frac{1}{\alpha}\right) = \beta\frac{\Gamma\left(\rho + \frac{1}{\alpha}\right)\Gamma\left(1 - \frac{1}{\alpha}\right)}{\Gamma(\rho)}$$

The second moment of the Dagum distribution:

$$E(X^2) = \rho\beta^2 B\left(\rho + \frac{2}{\alpha}, 1 - \frac{2}{\alpha}\right) = \beta^2\frac{\Gamma\left(\rho + \frac{2}{\alpha}\right)\Gamma\left(1 - \frac{2}{\alpha}\right)}{\Gamma(\rho)}$$

The k^{th} moment of the Dagum distribution:

$$E(X^k) = \rho\beta^k B\left(\rho + \frac{k}{\alpha}, 1 - \frac{k}{\alpha}\right) = \beta^k\frac{\Gamma\left(\rho + \frac{k}{\alpha}\right)\Gamma\left(1 - \frac{k}{\alpha}\right)}{\Gamma(\rho)} \quad (33)$$

3.8 Approximation of the GB2 Distribution to Fisk Distribution

For $\rho = 1$ and $\delta = 1$, the characteristic function of the GB2 distribution is expressed as the characteristic function of the Fisk distribution, as shown in Equation (34):

$$\varphi_X(t) = \sum_{n=0}^{\infty} \frac{(it\beta)^n}{n!} \frac{B\left(1 + \frac{n}{\alpha}, 1 - \frac{n}{\alpha}\right)}{B(1,1)}$$

$$= \sum_{n=0}^{\infty} \frac{(it\beta)^n}{n!} B\left(1 + \frac{n}{\alpha}, 1 - \frac{n}{\alpha}\right)$$

$$\varphi_X(t) = \beta^n \sum_{n=0}^{\infty} \frac{(it)^n}{n!} \Gamma\left(1 + \frac{n}{\alpha}\right)\Gamma\left(1 - \frac{n}{\alpha}\right) \quad (34)$$

The expression given in Equation (34) represents the characteristic function of the Fisk distribution, as defined in Theorem 6.

Theorem 6. *Let X be a random variable of Fisk distribution. The characteristic function of X is stated by Equation* (35):

$$\varphi_X(t) = \beta^n \sum_{n=0}^{\infty} \frac{(it)^n}{n!} \Gamma\left(1 + \frac{n}{\alpha}\right)\Gamma\left(1 - \frac{n}{\alpha}\right) \quad (35)$$

Proof:
The characteristic function of the Fisk distribution is expressed in Equation (36), as derived from the probability density function presented in Equation (4).

$$\varphi_X(t) = \int_0^{\infty} e^{itx} f(x; \alpha, \beta)dx = \int_0^{\infty} e^{itx} \frac{\alpha x^{\alpha-1}}{\beta^\alpha(1 + (x/\beta)^\alpha)^2}dx \quad (36)$$

By defining $u = (x/\beta)^\alpha$, Equation (36) of $\varphi_X(t)$ can be reformulated as follows.

$$\varphi_X(t) = \int_0^{\infty} e^{it\beta u^{1/\alpha}} \frac{\alpha(u^{1/\alpha}\beta)^{\alpha-1}}{\beta^\alpha(1+u)^2} \frac{1}{\alpha} u^{\frac{1}{\alpha}-1}\beta\, du$$

$$= \frac{\alpha\beta^{\alpha-1}\beta}{\beta^\alpha} \int_0^{\infty} e^{it\beta u^{1/\alpha}} \frac{(u^{1/\alpha})^{\alpha-1}}{(1+u)^2} \frac{1}{\alpha} u^{\frac{1}{\alpha}-1} du$$

$$= \int_0^{\infty} \frac{e^{it\beta u^{1/\alpha}}}{(1+u)^2} du = \int_0^{\infty} \sum_{n=0}^{\infty} \frac{(it\beta)^n}{n!} u^{n/\alpha} \frac{1}{(1+u)^2} du$$

$$= \sum_{n=0}^{\infty} \frac{(it\beta)^n}{n!} \int_0^{\infty} \frac{u^{n/\alpha}}{(1+u)^2} du$$

$$= \sum_{n=0}^{\infty} \frac{(it\beta)^n}{n!} \int_0^{\infty} \frac{u^{n/\alpha+1-1}}{(1+u)^{n/\alpha+1-1+1-n/\alpha+1}} du$$

The characteristic function of the Fisk distribution can be derived as follows.

$$\varphi_X(t) = \beta^n \sum_{n=0}^{\infty} \frac{(it)^n}{n!} B\left(1 + \frac{n}{\alpha}, 1 - \frac{n}{\alpha}\right)$$

$$= \beta^n \sum_{n=0}^{\infty} \frac{(it)^n}{n!} \frac{\Gamma\left(1+\frac{n}{\alpha}\right)\Gamma\left(1-\frac{n}{\alpha}\right)}{\Gamma\left(1+\frac{n}{\alpha}+1-\frac{n}{\alpha}\right)}$$

$$= \beta^n \sum_{n=0}^{\infty} \frac{(it)^n}{n!} \frac{\Gamma\left(1+\frac{n}{\alpha}\right)\Gamma\left(1-\frac{n}{\alpha}\right)}{\Gamma(2)}$$

$$= \beta^n \sum_{n=0}^{\infty} \frac{(it)^n}{n!} \Gamma\left(1+\frac{n}{\alpha}\right)\Gamma\left(1-\frac{n}{\alpha}\right)$$

Therefore, the Theorem 6 has been verified. The Fisk distribution is identified as a specific instance of the GB2 distribution when the parameters ρ and δ are both set to 1 in relation to the characteristic function.

3.9 Moment of the Fisk Distribution

The k-moment of the Dagum distribution can be derived directly from the characteristic function presented in Equation (35), as shown in Equation (37). The first moment of the Fisk distribution:

$$E(X) = \beta B\left(1+\frac{1}{\alpha}, 1-\frac{1}{\alpha}\right) = \beta\Gamma\left(1+\frac{1}{\alpha}\right)\Gamma\left(1-\frac{1}{\alpha}\right)$$

The second moment of the Fisk distribution:

$$E(X^2) = \beta^2 B\left(1+\frac{2}{\alpha}, 1-\frac{2}{\alpha}\right) = \beta^2\Gamma\left(1+\frac{2}{\alpha}\right)\Gamma\left(1-\frac{2}{\alpha}\right)$$

The k^{th} moment of the Fisk distribution:

$$E(X^k) = \beta^k B\left(1+\frac{k}{\alpha}, 1-\frac{k}{\alpha}\right) = \beta^k\Gamma\left(1+\frac{k}{\alpha}\right)\Gamma\left(1-\frac{k}{\alpha}\right) \quad (37)$$

4. CONCLUSIONS

This study presents a mathematical expression for the characteristic function of the four-parameter GB2 distribution. A thorough analysis is provided to show how this characteristic function unveils key properties of the GB2 distribution, especially its ability to compute moments like the mean and variance. The results demonstrate that the closed-form expression for the GB2 characteristic function can be obtained using the McLaurin series, the beta function, and the gamma function. Furthermore, the moments, including the mean and variance, are directly derived from the characteristic function of the GB2 distribution. By reparameterizing the closed-form characteristic function of the GB2 distribution, the Singh-Maddala, Dagum, and Fisk distributions can be efficiently approximated. Specifically, the characteristic functions for the Singh-Maddala and Dagum distributions are obtained by setting ρ = 1 and δ = 1 in the GB2 characteristic function, respectively. Similarly, the characteristic function for the Fisk distribution is derived by substituting both $\rho = 1$ and $\delta = 1$ into the GB2 distribution's characteristic function. As a result, the Singh-Maddala, Dagum, and Fisk distributions emerge as special cases of the GB2 distribution when considering its characteristic function.

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