Journal of Physics: Conference Series

PAPER • **OPEN ACCESS**

On *X*-sub-linearly independent modules

To cite this article: Fitriani *et al* 2017 *J. Phys.: Conf. Ser.* **893** 012008

IOP Conf. Series: Journal of Physics: Conf. Series **893** (2017) 012008 doi:10.1088/1742-6596/893/1/012008

On X-sub-linearly independent modules

Fitriani[1,2], B Surodjo[1], and I E Wijayanti[1]
[1] Department of Mathematics, Universitas Gadjah Mada
[2] Department of Mathematics, Universitas Lampung
E-mail: fitriani27@mail.ugm.ac.id, {surodjo_b, ind_wijayanti}@ugm.ac.id

Abstract. The notion of X-sub-exact sequence of modules is a generalization of exact sequences. Let K, L, M be R-modules and X a submodule of L. The triple (K, L, M) is said to be X-sub-exact at L if $K \to X \to M$ is exact at X. The exact sequence is a special case of X-sub-exact by taking $X = L$. We introduce an X-sub-linearly independent module which is a generalization of linearly independent relative to an R-module M by using the concept of X-sub-exact sequence.

1. Introduction

Let R be a ring and let M be an R-module, A subset $S \subseteq M$ is R-linearly dependent if there exist distinct $x_1, x_2, .., x_n$ in S and elements $a_1, a_2, .., a_n$ of R, not all of which are 0, such that $a_1x_1 + a_2x_2 + ... + a_nx_n = 0$. A set that is not R-linearly dependent is said to be R-linearly independent [1]. Let N be a left R-module, then N is said linearly independent to R (or N is $R-$linearly independent) if there exists a monomorphism $\varphi : R^{(\Lambda)} \to N$ [5].

Suprapto [6] introduced a generalization of linearly independency relative to an R-module M as follows: Let M be an R-module. The family of R-modules $\mathcal{N} = \{N_\lambda\}_\Lambda$ is said to be linearly independent to M if there exist a monomorphism $f : \coprod_\Lambda N_\lambda \to M$. If $\{N_\lambda = N\}_\Lambda$, then $f : N^{(\Lambda)} \to M$. We can say that $\mathcal{N} = \{N_\lambda\}_\Lambda$ is linearly independent to M if the sequence $0 \to \coprod_\Lambda N_\lambda \xrightarrow{f} M$ is exact at $\coprod_\Lambda N_\lambda$.

Let R be a ring and let $A \xrightarrow{f} B \xrightarrow{g} C$ be an exact sequence of R-modules, i.e. $Im\ f = Ker\ g (= g^{-1}(\{0\}))$. We can generalize the submodule $\{0\}$ to any submodule $U \subseteq C$ as we refer to [2] in which Davvaz and Parnian-Garamaleky introduced the concept of quasi-exact sequences. A sequence of R-modules and R-homomorphisms $A \xrightarrow{f} B \xrightarrow{g} C$ is quasi-exact in B or U-exact in B if there exists a submodule U in C such that $Im\ f = g^{-1}(U)$.

Then, Anvariyeh dan Davvaz [7] proved further results about quasi-exact sequences and introduced generalization of Schanuel Lemma. Moreover, they obtained some relationships between quasi-exact sequences and superfluous (or essential) submodules.

Furthermore, Davvaz and Shabani-Solt introduced a generalization of some notions in homological algebra [3]. They gave a generalization of the Lambek Lemma, Snake Lemma, connecting homomorphism and exact triangle and they established new basic properties of the U-homological algebra. In [8], Anvariyeh and Davvaz studied U-split sequences and established several connections between U-split sequences and projective modules.

Let K, L, M be R-modules and X a submodule of L. The triple (K, L, M) is said to be an X-sub-exact at L if

$$K \to X \to M$$

is exact, i.e. $Im\ f = Ker\ g$. The exact sequence is a special case of X-sub-exact by taking $X = L$ [4].

In this paper, we introduce an X-sub-linearly independent module which is a generalization of linearly independent relative to an R-module M by using the concept of X-sub-exact sequence.

Let M be an R-module. The family of R-modules $\mathcal{N} = \{N_\lambda\}_\Lambda$ is said to be X-sub-linearly independent to M if the triple $(0, \coprod_\Lambda N_\lambda, M)$ is X-sub-exact (where X is a submodule of $\coprod_\Lambda N_\lambda$). Then, we collect all submodules X of $\coprod_\Lambda N_\lambda$ such that $\mathcal{N}$ is X-sub-linearly independent to M, we denote it by $\sigma(0, \coprod_\Lambda N_\lambda, M)$. In this paper, we give some basic properties of X-sub-linearly independent modules and $\sigma(0, \coprod_\Lambda N_\lambda, M)$. We will show that $\sigma(0, \coprod_\Lambda N_\lambda, M)$ is closed under submodules and intersections. Furthermore, $\sigma(0, \coprod_\Lambda N_\lambda, M)$ always has a maximal element, for every family of R-modules $\mathcal{N}$ and R-module M. In other words, for every family of R-modules $\mathcal{N} = \{N_\lambda\}_\Lambda$ and R-module M, there exist a submodule X maximal such that $\mathcal{N}$ is an X-sublinearly independent.

2. Main Results

As a generalization of linearly independent relative to an R-module M, we define X-sub-linearly independent by using the concept of X-sub-exact sequence as follows:

Definition 2.1 *Let M be an R-module. The family of R-modules $\mathcal{N} = \{N_\lambda\}_\Lambda$ is said to be X-sublinearly independent to M if the triple $(0, \coprod_\Lambda N_\lambda, M)$ is X-sub-exact (where X is a submodule of $\coprod_\Lambda N_\lambda$), i.e. the sequence*

$$0 \to X \to M$$

is exact.

Example 2.1 Let $\mathcal{N} = \{\mathbb{Z}_2, \mathbb{Z}_5\}$ the family of $\mathbb{Z}$-modules and let $\mathbb{Z}_6$ be $\mathbb{Z}$-module. We define $f : \mathbb{Z}_2 \to \mathbb{Z}_6$, where $f(0) = 0$ and $f(1) = 3$. So, f is a monomorphism. Hence, the sequence

$$0 \to \mathbb{Z}_2 \xrightarrow{f} \mathbb{Z}_6$$

is exact. Therefore, the triple $(0, \mathbb{Z}_2 \oplus \mathbb{Z}_5, \mathbb{Z}_6)$ is $\mathbb{Z}_2$-sub-exact. So, $\mathcal{N}$ is $\mathbb{Z}_2$-sub-linearly independent to $\mathbb{Z}_6$.

Assume f is a monomorphism from $\mathbb{Z}_2 \oplus \mathbb{Z}_5$ to $\mathbb{Z}_6$. Then,

$$0 = f(0, 0) = f(5(0, 1)) = 5f(0, 1).$$

We get $f(0, 1) = f(0, 0) = 0$, a contradiction. So, we can not define a monomorphism from $\mathbb{Z}_2 \oplus \mathbb{Z}_5$ to $\mathbb{Z}_6$. Hence $\mathcal{N}$ is not linearly independent to $\mathbb{Z}_6$.

Example 2.1 shows that if the family of R-modules $\mathcal{N}$ is an X-sub-linearly independent to an R-module M, for some submodule X of $\coprod_\Lambda N_\lambda$, $N_\lambda \in \mathcal{N}$, for all $\lambda \in \Lambda$, then $\mathcal{N}$ is not necessary linearly independent to M.

We already know that any set that containing 0 is linearly dependent since $1.0 = 0$. In the following Proposition, we want to show that the family of R-modules $\mathcal{N}$ is 0-sub-linearly independent to M, for any R-module M.

IOP Conf. Series: Journal of Physics: Conf. Series **893** (2017) 012008 doi:10.1088/1742-6596/893/1/012008

Proposition 2.1 *Let $\mathcal{N} = \{N_\lambda\}_\Lambda$ be a family of R-modules. Then $\mathcal{N}$ is 0-sub-linearly independent to M, for any R-module M.*

Proof. Since the sequence $0 \to 0 \to M$ is exact, the triple $(0, \coprod_\Lambda N_\lambda, M)$ is 0-sub-exact at $\coprod_\Lambda N_\lambda$. Hence, $\mathcal{N}$ is 0-sub-linearly independent to M. □

In fact, we can define a monomorphism from R-module M to itself. So, Any R-module M is M-sub-linearly independent relative to M. We already know that any subset of a linearly independent set is linearly independent. In the following proposition, we will prove that M is X-sub-linearly independent to M, for every submodule X of M.

Proposition 2.2 *For any R-module M, M is X-sub-linearly independent to M, for every submodule X of M.*

Proof. Let M be an R-module and let X be a submodule of M. We have the inclusion $i : X \to M$ such that the sequence $0 \to X \xrightarrow{i} M$ is exact. Hence, the triple $(0, M, M)$ is X-sub-exact. Therefore M is X-sub-linearly independent to M. □

Then, we will give some properties of X-sub-linearly independent relative to an R-module M.

Clearly, we can define a monomorphism from N_λ to $\coprod_\Lambda N_\lambda$. So, we have the following proposition:

Proposition 2.3 *Let $\mathcal{N} = \{N_\lambda\}_\Lambda$ be a family of R-modules. Then $\mathcal{N}$ is N_λ-sub-linearly independent to $\coprod_\Lambda N_\lambda$, for every $\lambda \in \Lambda$.*

Proof. For every $\lambda \in \Lambda$, we have the inclusion $i : N_\lambda \to \coprod_\Lambda N_\lambda$ such that the sequence $0 \to N_\lambda \xrightarrow{i} \coprod_\Lambda N_\lambda$ is exact. Therefore, the triple $(0, \coprod_\Lambda N_\lambda, \coprod_\Lambda N_\lambda)$ is N_λ-sub-exact at $\coprod_\Lambda N_\lambda$. So, $\mathcal{N}$ is N_λ-sub-linearly independent to $\coprod_\Lambda N_\lambda$, for every $\lambda \in \Lambda$. □

Since for any submodule X of N_λ, we can define a monomorphism from X to $\coprod_\Lambda N_\lambda$, we have the following proposition.

Proposition 2.4 *Let $\mathcal{N} = \{N_\lambda\}_\Lambda$ be a family of R-modules. Then, for every $\lambda \in \Lambda$, $\mathcal{N}$ is X-sub-linearly independent to N_λ for any submodule X of N_λ.*

Proof. Let X be a submodule of $N_\lambda \subset \coprod_\Lambda N_\lambda$. We have the inclusion $i : X \to N_\lambda$ such that the sequence $0 \to X \to N_\lambda$ is exact. This implies the triple $(0, \coprod_\Lambda N_\lambda, N_\lambda)$ is X-sub-exact sequence at $\coprod_\Lambda N_\lambda$. Hence $\mathcal{N}$ is X-sublinearly independent to N_λ. □

Let K, L, M be R-modules. We define

$$\sigma(K, L, M) = \{X \leq L | (K, L, M) \ X\text{-sub-exact at } L\}.$$

Then $\sigma(K, L, M) \neq \emptyset$ since $0 \in \sigma(K, L, M)$.

Let $\mathcal{N}$ be a family of R-modules. If we take $K = 0$, $L = \coprod_\Lambda N_\lambda$ and $K = M$, then

$$\begin{aligned}\sigma(0, L, M) &= \{X \leq L | (0, L, M) \text{ is } X\text{-sub-exact at } L\} \\ &= \{X \leq L | \mathcal{N} \text{ is } X\text{-sublinearly independent to } M\}.\end{aligned}$$

We recall the properties of $\sigma(0, L, M)$ as follows:

Proposition 2.5 *[4] Let L, M be two R-modules and X_λ be a submodule of L, for every $\lambda \in \Lambda$. If $X_\lambda \in \sigma(0, L, M)$, for every $\lambda \in \lambda$, then $\cap_\Lambda X_\lambda \in \sigma(0, L, M)$.*

In the following Proposition, we will prove that $\sigma(0, \coprod_\Lambda N_\lambda, M)$ is closed under intersections, i.e. if $X_i \in \sigma(0, \coprod_\Lambda N_\lambda, M)$, for every $i \in I$, then $\cap_{i\in I} X_i \in \sigma(0, \coprod_\Lambda N_\lambda, M)$.

Proposition 2.6 *Let $\mathcal{N} = \{N_\lambda\}_\Lambda$ be a family of R-modules and let M be an R-module. If $\mathcal{N}$ is X_i-sub-linearly independent to M, for every $i \in I$, then $\mathcal{N}$ is $\cap_{i\in I} X_i$-sub-linearly independent to M. In other words, $\sigma(0, \coprod_\Lambda N_\lambda, M)$ is closed under intersections.*

Proof. Since $\mathcal{N}$ is X_i-sub-linearly independent to M, for every $i \in I$, then the triple $(0, \coprod_\Lambda N_\lambda, M)$ is X_i-sub-exact. Therefore, $X_i \in \sigma(0, \coprod_\Lambda N_\lambda, M)$, for every $i \in I$. By Proposition 2.5, we get $\cap_{i\in I} X_i \in \sigma(0, \coprod_\Lambda N_\lambda, M)$. So, $\mathcal{N}$ is $\cap_{i\in I} X_i$-sub-linearly independent to M. □

Furthermore, in the following proposition, we want to show that $\sigma(0, \coprod_\Lambda N_\lambda, M)$ is closed under submodules.

Proposition 2.7 *Let $\mathcal{N} = \{N_\lambda\}_\Lambda$ be a family of R-modules and M be an R-module. If $\mathcal{N}$ is X-sub-linearly independent to M, then $\mathcal{N}$ is X'-sub-linearly independent to M, for every submodule X' of X. In other words, $\sigma(0, \coprod_\Lambda N_\lambda, M)$ is closed under submodules.*

Proof. Since $\mathcal{N}$ is X-sub-linearly independent to M, then there is a monomorphism $f : X \to M$. Let X' be a submodule of X. Then, we can define the inclusion $i : X' \to X$. So, $f \circ i : X' \to M$ is a monomorphism. Hence, $\mathcal{N}$ is X'-sub-linearly independent to M. Therefore, if $X \in \sigma(0, \coprod_\Lambda N_\lambda, M)$, then $X' \in \sigma(0, \coprod_\Lambda N_\lambda, M)$, for every submodule X' of X. □

We already know that a basis for a free R-module F is a maximal linearly independent set in R-module F. So, we will investigate the maximal element of $\sigma(0, \coprod_\Lambda N_\lambda, M)$, i.e the maximal subset X of $\coprod_\Lambda N_\lambda$ such that $\mathcal{N}$ is X-sub-linearly independent to an R-module M. If there is a monomorphism $f : \coprod_\Lambda N_\lambda \to M$, then $\coprod_\Lambda N_\lambda$ is the maximal element in $\sigma(0, \coprod_\Lambda N_\lambda, M)$, i.e. for every $X \in \sigma(0, \coprod_\Lambda N_\lambda, M)$, if $\coprod_\Lambda N_\lambda \subseteq X$, then $\coprod_\Lambda N_\lambda = X$. But, $\coprod_\Lambda N_\lambda$ is not necessary belong to $\sigma(0, \coprod_\Lambda N_\lambda, M)$.

The most important criterion for the existence of maximal elements in a partially ordered set is Zorn's lemma. We recall Zorn's lemma as follows:

Proposition 2.8 ***(Zorn's Lemma)****[1] Let X be a partially ordered set and assume that every chain in X has an upper bound. Then X has a maximal element.*

By using Zorn's lemma, we want to show that there exist a submodule X in $\coprod_\Lambda N_\lambda$ maximal such that $\mathcal{N}$ is an X-sublinearly independent to M, for every family of R-modules $\mathcal{N}$ and R-module M.

Theorem 2.1 *Let $\mathcal{N} = \{N_\lambda\}_\Lambda$ be a family of R-modules and M be an R-module. Then there exist a submodule X in $\coprod_\Lambda N_\lambda$ maximal such that $\mathcal{N}$ is an X-sublinearly independent to M. In other words, $\sigma(0, \coprod_\Lambda N_\lambda, M)$ always has a maximal element.*

Proof. Let $\mathcal{X} = \{X | X \subseteq \coprod_\Lambda N_\lambda \text{ and } X \subseteq M\}$. The set $\mathcal{X}$ is not empty since $0 \in \mathcal{X}$. Let $\{X_i\}_{i\in I}$ be a chain (totally ordered set) in $\mathcal{X}$. Let $Y = \bigcup_{i\in I} X_i$, where $X_i \in \mathcal{X}$, for all $i \in I$. As a set, Y certainly contains all the X_i's. Since a union of submodules is not usually a submodule, we will show that Y is a submodule of $\coprod_\Lambda N_\lambda$.

If x and y are in Y, then $x \in X_i$ and $y \in X_j$, for two of the submodules X_i and X_j of $\coprod_\Lambda N_\lambda$. Since the set of submodules $\{X_i\}_i \in I$ is totally ordered,

$$X_i \subset X_j \text{ or } X_j \subset X_i.$$

Without loss of generality, $X_i \subset X_j$. Therefore x and y are in X_j, so $x + y \in X_j \subset Y$ and $rx \in X_j \subset Y$, for every $r \in R$. We can conclude that $Y = \cup_{i\in I} X_i$ is a submodule of $\coprod_\Lambda N_\lambda$. Similarly, we obtain Y is a submodule of M.

Since Y contains every X_i, for all $i \in I$, Y is an upper bound on the totally ordered set $\{X_i\}_{i\in I}$. By Zorn's lemma, $\mathcal{X}$ contains a maximal element. This maximal element is a submodule of $\coprod_\Lambda N_\lambda$ and M that is maximal for inclusion among all submodule of $\coprod_\Lambda N_\lambda$ and M. We can conclude that there exist a submodule X in $\coprod_\Lambda N_\lambda$ maximal such that $\mathcal{N}$ is an X-sublinearly independent to M or $\sigma(0, \coprod_\Lambda N_\lambda, M)$ always has a maximal element. □

3. Conclusion

The family of R-modules $\mathcal{N} = \{N_\lambda\}_\Lambda$ is an X-sub-linearly independent to M if the triple $(0, \coprod_\Lambda N_\lambda, M)$ is X-sub-exact (where X is a submodule of $\coprod_\Lambda N_\lambda$). If we take $X = \coprod_\Lambda N_\lambda$, then $\mathcal{N} = \{N_\lambda\}_\Lambda$ is linearly independent. Hence, sub-linearly independent module is a generalization of linearly independent module.

Then, we collect all submodules X of $\coprod_\Lambda N_\lambda$ such that $\mathcal{N}$ is X-sub-linearly independent to M, we denote it by $\sigma(0, \coprod_\Lambda N_\lambda, M)$. We have proved that $\sigma(0, \coprod_\Lambda N_\lambda, M)$ is closed under submodules and intersections. Furthermore, for every family of R-modules $\mathcal{N} = \{N_\lambda\}_\Lambda$ and R-module M, there exist X maximal such that $\mathcal{N}$ is an X-sublinearly independent. In other words, $\sigma(0, \coprod_\Lambda N_\lambda, M)$ always has a maximal element, for every family of R-modules $\mathcal{N}$ and R-module M.

Acknowledgment

We would like to express deep gratitude to Professor Alexander Zimmermann for his contribution to this research work. We would also like to thank the Indonesian Mathematical Society (IndoMS) which support the first author to join the Asian Mathematical Conference (AMC) 2016.

References

[1] Adkins W A and Weintraub S H 1992 *Algebra : An Approach via Module Theory* (New York : Springer-Verlag)
[2] Davvaz B and Parnian-Garamaleky Y A 1999 A note on exact sequences *Bull.Malays. Math. Sci. Soc.* **22**(1) p 53
[3] Davvaz B and Shabani-Solt H 2001 A generalization of homological algebra*J. Korean Math. Soc.* **39**(6) p 881
[4] Fitriani, Surodjo B and Wijayanti I E 2016 on X-sub-exact sequences, *Far East J. Math. Sci* **100**(7) p 1055
[5] Maclane S and Birkhoff G 1979 *Algebra* (New York : Mac Millan Publishing Co)

[6] Suprapto 2015 *Structure of Family of Linearly Independent Modules and Cohereditary Modules on a Category* $\tau[M]$ Dissertation Universitas Gadjah Mada
[7] Anvariyeh S M and Davvaz B 2005 on quasi-exact sequences *Bull. Korean Math. Soc.* **42**(1) p 149
[8] Anvariyeh S M and Davvaz B 2002 U-split-exact sequences *Far East J. Math. Sci.* **4**(2) p 209
[9] Wisbauer R 1991 *Foundation of Module and Ring Theory* (USA : Gordon and Breach)