HISTORY OF MANUSCRIPT PUBLICATION

(Indonesian Journal of Combinatorics)

On the Locating Chromatic Number of Barbell Shadow Paths

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I.SUBMISSION PAPER
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II SUBMISSION ACKNOWLEDGEMENT

12 Februari 2021

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[IJC] Submission Acknowledgement External Inbox ×

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to me

Fri, Feb 12, 2021, 12:30 PM

The following message is being delivered on behalf of Indonesian Journal of Combinatorics.

A. Asmiati:

Thank you for submitting the manuscript, "On the Locating Chromatic Number of Barbell Shadow Path Graph" to Indonesian Journal of Combinatorics. With the online journal management system that we are using, you will be able to track its progress through the editorial process by logging in to the journal web site:

Manuscript URL: http://ijc.or.id/index.php/ijc/author/submission/149
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If you have any questions, please contact me. Thank you for considering this journal as a venue for your work.

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III. EDITOR DECISION: REVISION REQUIRED

7 Juli 2021

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[IJC] Editor Decision External Inbox ×

Slamin <slamin@unej.ac.id> — Wed, Jul 7, 2021, 7:24 PM

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The following message is being delivered on behalf of Indonesian Journal of Combinatorics.

Dear Dr. A. Asmiati,

We have reached a decision regarding your submission to Indonesian Journal of Combinatorics, with title "On the Locating Chromatic Number of Barbell Shadow Path Graph".

Our decision is: revision required

Please send your revision version using IJC Template in tex format and please also send the figures (if any).

Slamin and Kiki Sugeng
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Slamins@unej.ac.id

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Indonesian Journal of Combinatorics

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PAPER: *On the locating chromatic number of barbell shadow path graph*
by Asmiati, Maharani Damayanti, and Lyra Yulianti

(1) Research results
[ ] Excellent work, (significant, good) contribution
[X] Solid work, good contribution
[ ] Publishable, (satisfactory, poor) contribution
[ ] Shallow
(2) Concepts
[ ] New concepts and methods
[X] Difficult but standard methods
[ ] Elementary but novel
[ ] Straightforward, relatively simple
(3) Interest
[ ] Extremely /very/ quite interesting
[X] Sufficiently interesting
[ ] Uninteresting
(4) Clarity
[ ] Well prepared and clearly written
[ ] Satisfactory style
[X] Improvement is necessary
[ ] Poorly written
(5) **Recommendation**
[ ] Very good quality, publication recommended
[X] Good quality, publication (not) recommended
[ ] Poor quality, rejection
[ ] Revision required to determine final quality
(6) Revision
[ ] Ready as is
[X] Minor revisions, (no) further review necessary

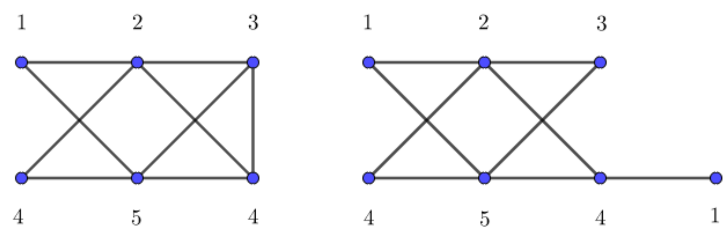
COMMENTS

In this paper, the authors provide an exact value of locating chromatic number of a shadow path graph $D_2(P_n)$ where $n \geq 6$. They also determine the locating chromatic number of a barbell graph containing shadow path. They give some interesting outcomes. I think that the results in the paper are new and correct. Therefore, I RECOMMEND THIS PAPER FOR PUBLICATION in IJC. However, the paper still need some major improvements for publication.

- Abstract and in many places: The authors use two notions "locating chromatic number" and "locating-chromatic number". Please use only one notion throughout the paper.
- Page 1 and in many places: "et al." should be "*et al.*".
- Section 1 first sentence: "... with derived two graphs concept, coloring vertices ..." should be "... by combining two concepts in graph theory, which are vertex coloring ...".
- Page 1 definition of k-coloring: "$1, 2, \ldots, k$" should be "$\{1, 2, \ldots, k\}$".
- Page 2 before Th 1.1: "The following theorem are basics to determine ..." should be "The following two theorems are usefull to determine ...".
- Page 2 before Th 1.1: What is the definition of neighbour of a vertex in a graph?
- Page 2 after Th 1.2: "... a graph is newly interesting ..." should be "... a graph is an interesting ...".
- Page 2 after Th 1.2: "... there is no general theorem for ..." should be "... there is no general algorithm for ...".
- Page 2 after Th 1.2: "... any graph and ..." should be "... any graphs and ..."
- Page 2 after Th 1.2: "... a few results related to the determination of ..." should be "... a few results related to determining ..."
- Page 2 after Th 1.2: "... have succeeded in constructing tree on ..." should be "... provided a tree construction of ...".
- Page 2 definition of shadow path: "Let P_n be a path with $V(P_n) = \{x_i \mid 1 \leq i \leq n\}$ and $E(P_n) = \{x_i x_{i+1} \mid 1 \leq i \leq n-1\}$. The shadow paths graph $D_2(P_n)$ is a graph with the vertex set $\{u_i, v_i \mid 1 \leq i \leq n\}$ where
  - $u_i u_j, v_i v_j \in E(D_2(P_n))$ if and only if $x_i x_j \in E(P_n)$
  - $u_i v_j \in E(D_2(P_n))$ if and only if $x_i x_j \in E(P_n)$"

  Throughout the paper, the label of vertices can be referred to the definition above. The authors do not need to rewrite $V(D_2(P_n))$ on every proof of theorem.

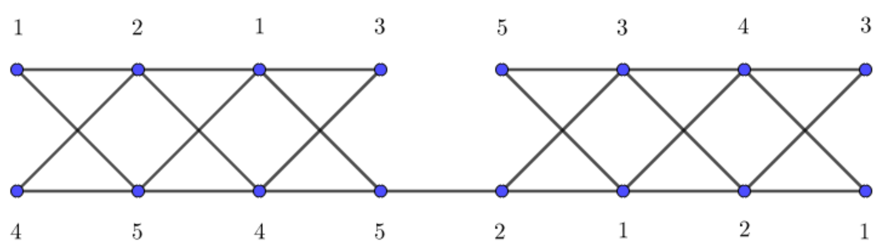
- Page 3 Corollary 2.1: This corollary is not completely true. The counterexamples are two connected graphs below. Both graphs contain $D_2(P_3)$ but their locating chromatic number are at most 5.



- Page 3 Th 2.1: Since Corollary 2.1 is not completely true, therefore, the authors should use another method to prove the lower bound of Theorem 2.1.

- Page 4 function $c_\pi(u_i)$ and in many places: Since a shadow path graph $D_2(P_n)$ is a connected graph, it is clear that $D_2(P_n)$ has only one component. What are 2^{nd}, 3^{rd},4^{th},5^{th}, and 6^{th} components?

- Page 4 Th 2.2: This theorem is also not completely true. A barbell graph in the figure below contains shadow paths $D_2(P_4)$ but its locating chromatic number is at most 5.



- Page 15 Concluding remarks: The authors should revise the conclusion since some of results are not completely true.

- Page 15 References: On [1] the authors write an abbreviation of the name of journal (*ITB J. Sci.*). However, on [2] they write a complete title of the journal (*Far East Journal of Mathematical Sciences*). For all references, please choose either write an abbreviation or write a complete title.

IV. SUBMIT REVISED PAPER

21 Juli 2021

ASMIATI 1976 <asmiati.1976@fmipa.unila.ac.id> Jul 21, 2021, 3:00 PM

to Suhadi

Thank you very much for your information. Please find our revision paper in attachment.

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...

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INDONESIAN JOURNAL OF COMBINATORICS

On the Locating Chromatic Number of Barbell Shadow Path Graphs

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Abstract

The locating chromatic number was introduced by Chartrand in 2002. The locating chromatic number of a graph is a combined concept between the coloring and partition dimension of a graph. The locating chromatic number of a graph is defined as the cardinality of the minimum color classes of the graph. In this paper, we discuss about the locating chromatic number of shadow path graphs and barbell graph containing shadow graph.

Keywords: the locating-chromatic number, shadow path graph, barbell graph
Mathematics Subject Classification : xxxxx

1. Introduction

The locating chromatic number of a graph was introduced by Chatrand et al.[6] by combining two concepts in graph theory, which are vertex coloring and partition dimension of a graph. Let $G = (V, E)$ be a connected graph. A k-coloring of G is a function $c : V(G) \longrightarrow \{1, 2, \cdots, k\}$, where $c(u) \neq c(v)$ for any two adjacent vertices u and v in G. Thus, the coloring c induces a partition Π of $V(G)$ into k color classes (independent sets) $C_1, C_2, \cdots, C_k$, where C_i is the set of all vertices colored by the color i for $1 \leq i \leq k$. The color code $c_\Pi(v)$ of a vertex v in G is defined as the $k-$ordinate $(d(v, C_1), d(v, C_2), \cdots, d(v, C_k))$, where $d(v, C_i) = \min\{d(v, x); x \in C_i\}$ for

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$1 \leq i \leq k$. The $k-$coloring c of G such that all vertices have different color codes is called a *locating coloring* of G. The *locating chromatic number* of G, denoted by $\chi_L(G)$, is the minimum k such that G has a locating coloring.

The following two theorems are useful to determine the lower bound of the locating chromatic of a graph. The set of neighbors of a vertex q in G, denoted by $N(q)$.

Theorem 1.1. *(see [6]). Let c be a locating coloring in a connected graph G. If x and y are distinct vertices of G such that $d(p,w) = d(q,w)$ for all $w \in V(G) - \{p,q\}$, then $c(p) \neq c(q)$. In particular, if p and q are non-adjacent vertices such that $N(p) \neq N(q)$, then $c(p) \neq c(q)$.*

Theorem 1.2. *(see [6]). The locating chromatic number of a cycle graph $C_n (n \geq 3)$ is 3 for odd n and 4 for even n.*

The locating chromatic number of a graph is an interesting topic to study because there is no general algorithm for determining the locating chromatic number of any graphs and there are only a few results related to determining of the locating chromatic number of some graphs. Chartrand *et al.* [6] determined all graphs of order n with locating number n, namely a complete multipartite graph of n vertices. Moreover, Chartrand *et al.*[7] provided a tree construction of n vertices, $n \geq 5$, with locating chromatic number varying from 3 to n, except for $(n-1)$. Next, Asmiati *et al.* [1] obtained the locating chromatic number of amalgamation of stars and non-homogeneous caterpillars and firecracker graphs [2]. In [5] Welyyanti *et al.* determined the locating chromatic number of complete n-ary trees. Next, Sofyan *et al.* [4] determined the locating chromatic number of homogeneous lobster. Recently, Ghanem *et al.* [8] found the locating chromatic number of powers of the path and cycles.

Let P_n be a path with $V(P_n) = \{x_i \mid 1 \leq i \leq n\}$ and $E(P_n) = \{x_i x_{i+1} \mid 1 \leq i \leq n-1\}$. The shadow path graph $D_2(P_n)$ is a graph with the vertex set $\{u_i, v_i \mid 1 \leq i \leq n\}$ where $u_i u_j \in E(D_2(P_n))$ if and only if $x_i x_j \in E(P_n)$ and $u_i v_j \in E(D_2(P_n))$ if and only if $x_i x_j \in E(P_n)$. *A barbell graph containing shadow path graph*, denoted by $B_{D_2(P_n)}$ is obtained by copying a shadow path graph (namely, $D_2'(P_n)$) and connecting the two graphs with a bridge. We assume that $\{u_i', v_i' \mid 1 \leq i \leq n\}$ is a vertex set of $D_2'(P_n)$ and a bridge in $B_{D_2(P_n)}$ connecting $\{u'_{\frac{n+1}{2}} v_{\frac{n+1}{2}}\}$ for odd n and $\{u'_{\frac{n}{2}} v_{\frac{n}{2}}\}$ for even n.

Motivated by the result of Asmiati *et al.* [3] about the determination of the locating chromatic number of certain barbell graphs, in this paper we determine the locating chromatic number of shadow path graphs and barbell graph containing shadow path for $n \geq 3$.

2. Main Results

The following theorem gives the locating chromatic number for shadow path graph $D_2(P_n)$ for $n \geq 3$.

Lemma 2.1. *Let c be a locating-chromatic number for shadow path graph $D_2(P_n)$, with $u_i \in P_i^1$ and $v_i \in P_i^2$. Then $c(u_i) \neq c(v_i)$.*

PROOF. On the shadow path graph $D_2(P_n)$, we can see that $d(u_i, x) = d(v_i, x)$, $i \in [1, n-1]$ for every $x \in ((D_2(P_n)) \setminus \{u_i, v_i\})$. By Theorem 1.1, we have $c(u_i) \neq c(v_i)$. □

Theorem 2.1. *The locating chromatic number of a shadow path graph for* $n \geq 3$, $D_2(P_n)$ *is* 6.

PROOF.

First, we determine the lower bound for the locating-chromatic number of shadow path graph $D_2(P_n)$ for $n \geq 3$. The Shadow path graph $D_2(P_n)$ for $n \geq 3$ consists of minimal two cycles C_4. Pick the first cycle C_4, then by Theorem 1.2, we could assign 4 colors, $\{1, 2, 3, 4\}$ to the first cycle's vertices. Next, in the second C_4, we have two vertices, which intersect with two vertices in the first C_4. By Lemma 2.1, we must assign two different colors to the remaining vertices of the second C_4. Therefore, we have $\chi_L(G) \geq 6$.

Next, we determine the upper bound of the locating chromatic number of the shadow path graph for $n \geq 3$. Let c be a coloring using 6 colors as follow : $c(u_i) = \begin{cases} 1, & \text{for } i = 1 \\ 2, & \text{for } i = 2n\ , n \geq 1 \\ 3, & \text{for } i = 2n + 1\ , n \geq 1 \end{cases}$

$$c(v_i) = \begin{cases} 4, & \text{for } i = 1 \\ 5, & \text{for } i = 2n\ , n \geq 1 \\ 6, & \text{for } i = 2n + 1\ , n \geq 1 \end{cases}$$

The color codes of $D_2(P_n)$ are :

$$c_\pi(u_i) = \begin{cases} i - 1, & \text{for } 1^{st} \text{ ordinate}, i \geq 1; \\ & \text{for } 4^{th} \text{ordinate}, i \geq 2; \\ 0, & \text{for } 2^{nd} \text{ ordinate, even i}\ , 2 \leq i \leq n; \\ & \text{for } 3^{rd} \text{ ordinate, odd i}\ , 3 \leq i \leq n; \\ 2, & \text{for } 5^{th} \text{ ordinate, even i}\ , 2 \leq i \leq n; \\ & \text{for } 6^{th} \text{ ordinate, odd i}\ , 1 \leq i \leq n; \\ & \text{for } 3^{rd} \text{ ordinate}, i = 1 \\ & \text{for } 4^{th} \text{ ordinate}, i = 1 \\ 1, & \text{otherwise.} \end{cases}$$

$$c_\pi(v_i) = \begin{cases} i - 1, & \text{for } 1^{st} \text{ ordinate}, i \geq 2; \\ & \text{for } 4^{th} \text{ordinate}, i \geq 1; \\ 0, & \text{for } 5^{nd} \text{ ordinate, even i}\ , 2 \leq i \leq n; \\ & \text{for } 6^{th} \text{ ordinate, odd i}\ , 3 \leq i \leq n; \\ 2, & \text{for } 2^{nd} \text{ ordinate, even i}\ , 2 \leq i \leq n; \\ & \text{for } 3^{rd} \text{ ordinate, odd i}\ , 3 \leq i \leq n; \\ & \text{for } 1^{st} \text{ ordinate}, i = 1 \\ & \text{for } 6^{th} \text{ ordinate}, i = 1 \\ 1, & \text{otherwise.} \end{cases}$$

Since all vertices in $D_2(P_n)$ for $n \geq 3$ have distinct color codes, then c is a locating coloring using 6 colors. As a result $\chi_L D_2(P_n) \leq 6$. Thus $\chi_L D_2(P_n) = 6$. $\square$

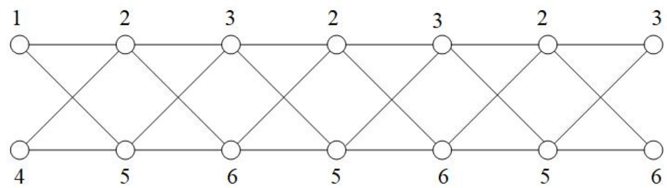

Figure 1. A locating coloring of $D_2(P_7)$.

Theorem 2.2. *The locating chromatic number of a barbell graph containing shadow path for* $n \geq 3$ *is* 6.

PROOF.

First, we determine the lower bound of $\chi_L B_{D_2(P_n)}$ for $n \geq 3$. Since the barbell graph $B_{D_2(P_n)}$ containing $D_2(P_n)$, then by Theorem 2.3 we have $\chi_L(B_{D_2(P_n)}) \geq 6$. To prove the upper bound, consider the following three cases.

CASE 1 ($n = 3$). Let c be a locating coloring using 6 colors as follows :

$$c(u_i) = \begin{cases} 1, & \text{for } i = 2; \\ 2, & \text{for } i = 1 \\ 3, & \text{for } i = 3 \end{cases}$$

$$c(u_i') = \begin{cases} 4, & \text{for } i = 2; \\ 5, & \text{for } i = 1 \\ 6, & \text{for } i = 3 \end{cases}$$

$$c(v_i) = \begin{cases} 1, & \text{for } i = 1; \\ 5, & \text{for } i = 3 \\ 6, & \text{for } i = 2 \end{cases}$$

$$c(v_i') = \begin{cases} 2, & \text{for } i = 1; \\ 3, & \text{for } i = 3 \\ 4, & \text{for } i = 2 \end{cases}$$

The color codes of $B_{D_2(P_3)}$ are

$$c_\pi(u_i) = \begin{cases} 0, & \text{for } 1^{st} \text{ ordinate}, \ i = 2; \\ & \text{for } 2^{nd} \text{ ordinate}, \ i = 1; \\ & \text{for } 3^{rd} \text{ ordinate}, \ i = 3; \\ 2, & \text{for } 3^{th} \text{ ordinate}, \ i = 1; \\ & \text{for } 4^{th} \text{ ordinate}, \ i = 2; \\ & \text{for } 2^{nd} \text{ ordinate}, \ i = 3; \\ & \text{for } 5^{th} \ and \ 6^{th} \text{ ordinate}, \ i = 1 \ and \ 3; \\ 1, & \text{otherwise.} \end{cases}$$

$$c_\pi(u'_i) = \begin{cases} 0, & \text{for } 5^{th} \text{ ordinate}, \ i = 1; \\ & \text{for } 4^{th} \text{ ordinate}, \ i = 2; \\ & \text{for } 6^{th} \text{ ordinate}, \ i = 3; \\ 2, & \text{for } 6^{th} \text{ ordinate}, \ i = 1; \\ & \text{for } 1^{nd} \text{ ordinate}, \ i = 2; \\ & \text{for } 5^{th} \text{ ordinate}, \ i = 3; \\ & \text{for } 2^{nd} \ and \ 3^{rd} \text{ ordinate}, \ i = 1 \ and \ 3; \\ 1, & \text{otherwise.} \end{cases}$$

$$c_\pi(v_i) = \begin{cases} 0, & \text{for } 1^{st} \text{ ordinate}, \ i = 1; \\ & \text{for } 6^{th} \text{ ordinate}, \ i = 2; \\ & \text{for } 5^{th} \text{ ordinate}, \ i = 3; \\ 2, & \text{for } 5^{th} \text{ ordinate}, \ i = 1; \\ & \text{for } 1^{st} \text{ ordinate}, \ i = 3; \\ & \text{for } 2^{nd} \ and \ 3^{rd} \text{ ordinate}, \ i = 1 \ and \ 3; \\ 1, & \text{otherwise.} \end{cases}$$

$$c_\pi(v'_i) = \begin{cases} 0, & \text{for } 2^{nd} \text{ ordinate}, \ i = 1; \\ & \text{for } 4^{th} \text{ ordinate}, \ i = 2; \\ & \text{for } 3^{rd} \text{ ordinate}, \ i = 3; \\ 2, & \text{for } 1^{st} \text{ ordinate}, \ i = 1 \ and \ 3; \\ & \text{for } 6^{th} \text{ ordinate}, \ i = 2; \\ & \text{for } 2^{nd} \text{ ordinate}, \ i = 3; \\ & \text{for } 3^{rd} \ and \ 5^{th} \text{ ordinate}, \ i = 3; \\ 1, & \text{otherwise.} \end{cases}$$

Since all vertices in $B_{D_2(P_3)}$ have distinct color codes, then c is a locating coloring using 6 colors. As a result $\chi_L B_{D_2(P_3)} \leq 6$. Thus $\chi_L B_{D_2(P_3)} = 6$.

CASE 2 (n odd). Let c be a locating coloring using 6 colors as follows :

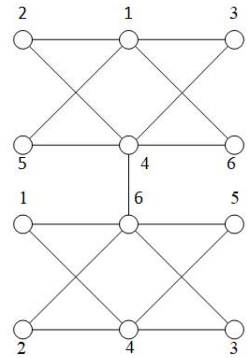

Figure 2. A locating coloring of $B_{D_2(P_3)}$.

$$c(u_i)=\begin{cases}1, & \text{for } i=\dfrac{n+1}{2};\\ 2, & \text{for odd i; } i<\dfrac{n+1}{2},\ n=4j+3, j\geq 1 \text{ and } i>\dfrac{n+1}{2},\ n=4j+1, j\geq 1\\ & \text{for even 1; } i<\dfrac{n+1}{2},\ n=4j+1, j\geq 1 \text{ and } i>\dfrac{n+1}{2},\ n=4j+3, j\geq 1\\ 3, & \text{for odd i; } i<\dfrac{n+1}{2},\ n=4j+1, j\geq 1 \text{ and } i>\dfrac{n+1}{2},\ n=4j+3, j\geq 1\\ & \text{for even i; } i<\dfrac{n+1}{2},\ n=4j+3, j\geq 1 \text{ and } i>\dfrac{n+1}{2},\ n=4j+1, j\geq 1\end{cases}$$

$$c(u'_i)=\begin{cases}4, & \text{for } i=\dfrac{n+1}{2};\\ 5, & \text{for odd i; } i<\dfrac{n+1}{2},\ n=4j+3, j\geq 1 \text{ and } i>\dfrac{n+1}{2},\ n=4j+1, j\geq 1\\ & \text{for even 1; } i<\dfrac{n+1}{2},\ n=4j+1, j\geq 1 \text{ and } i>\dfrac{n+1}{2},\ n=4j+3, j\geq 1\\ 6, & \text{for odd i; } i<\dfrac{n+1}{2},\ n=4j+1, j\geq 1 \text{ and } i>\dfrac{n+1}{2},\ n=4j+3, j\geq 1\\ & \text{for even i; } i<\dfrac{n+1}{2},\ n=4j+3, j\geq 1 \text{ and } i>\dfrac{n+1}{2},\ n=4j+1, j\geq 1\end{cases}$$

$$c(v_i)=\begin{cases}1, & \text{for odd i; } i<\dfrac{n+1}{2},\ n=4j+3, j\geq 1 \text{ and } i>\dfrac{n+1}{2},\ n=4j+1, j\geq 1\\ & \text{for even i; } i<\dfrac{n+1}{2},\ n=4j+1, j\geq 1 \text{ and } i>\dfrac{n+1}{2},\ n=4j+3, j\geq 1\\ 5, & \text{for odd i; } i<\dfrac{n+1}{2},\ n=4j+1, j\geq 1 \text{ and } i>\dfrac{n+1}{2},\ n=4j+3, j\geq 1\\ & \text{for even i; } i<\dfrac{n+1}{2},\ n=4j+3, j\geq 1 \text{ and } i>\dfrac{n+1}{2},\ n=4j+1, j\geq 1\\ 6, & \text{for } i=\dfrac{n+1}{2}.\end{cases}$$

$$c(v'_i) = \begin{cases} 2, & \text{for odd i}; i < \frac{n+1}{2},\ n = 4j+3, j \geq 1 \text{ and } i > \frac{n+1}{2},\ n = 4j+1, j \geq 1 \\ & \text{for even i}; i < \frac{n+1}{2},\ n = 4j+1, j \geq 1 \text{ and } i > \frac{n+1}{2},\ n = 4j+3, j \geq 1 \\ 4, & \text{for } i = \frac{n+1}{2}; \\ 3, & \text{for odd i}; i < \frac{n+1}{2},\ n = 4j+1, j \geq 1 \text{ and } i > \frac{n+1}{2},\ n = 4j+3, j \geq 1 \\ & \text{for even i}; i < \frac{n+1}{2},\ n = 4j+3, j \geq 1 \text{ and } i > \frac{n+1}{2},\ n = 4j+1, j \geq 1 \end{cases}$$

The color codes of $B_{D_2(P_n)}$ are

$$c_\pi(u_i) = \begin{cases} (\frac{n+1}{2}) - i, & \text{for } 1^{st} \text{ ordinate}, i \leq \frac{n+1}{2}; \\ & \text{for } 4^{th} \text{ ordinate}, i < \frac{n+1}{2}; \\ i - (\frac{n+1}{2}), & \text{for } 1^{st} \text{ ordinate}, i > \frac{n+1}{2}; \\ & \text{for } 4^{th} \text{ ordinate}, i > \frac{n+1}{2}; \\ 0, & \text{for } 2^{nd} \text{ ordinate, odd i}, i < \frac{n+1}{2}; \\ & \text{for } 3^{rd} \text{ ordinate, even i}, i < \frac{n+1}{2}; \\ & \text{for } 2^{th} \text{ ordinate, even i}, i > \frac{n+1}{2}; \\ & \text{for } 3^{rd} \text{ ordinate, odd i}, i > \frac{n+1}{2}; \\ 2, & \text{for } 5^{th} \text{ ordinate, odd i}, i < \frac{n+1}{2}; \\ & \text{for } 5^{th} \text{ ordinate, even i}, i > \frac{n+1}{2}; \\ & \text{for } 6^{th} \text{ ordinate, odd i}, i < \frac{n+1}{2}; \\ & \text{for } 6^{th} \text{ ordinate, even i}, i > \frac{n+1}{2}; \\ & \text{for } 4^{th} \text{ ordinate}, i = \frac{n+1}{2}; \\ 1, & \text{otherwise.} \end{cases}$$

$$c_\pi(u_i') = \begin{cases} (\frac{n+1}{2}) - i, & \text{for } 1^{st} \text{ ordinate}, i < \frac{n+1}{2}; \\ & \text{for } 4^{th} \text{ ordinate}, i \leq \frac{n+1}{2}; \\ i - (\frac{n+1}{2}), & \text{for } 1^{st} \text{ ordinate}, i > \frac{n+1}{2}; \\ & \text{for } 4^{th} \text{ ordinate}, i > \frac{n+1}{2}; \\ 0, & \text{for } 5^{th} \text{ ordinate, odd i}, i < \frac{n+1}{2}; \\ & \text{for } 5^{th} \text{ ordinate, even i}, i > \frac{n+1}{2}; \\ & \text{for } 6^{th} \text{ ordinate, odd i}, i > \frac{n+1}{2}; \\ & \text{for } 6^{th} \text{ ordinate, even i}, i < \frac{n+1}{2}; \\ 2, & \text{for } 2^{nd} \text{ ordinate, odd i}, i < \frac{n+1}{2}; \\ & \text{for } 2^{nd} \text{ ordinate, even i}, i > \frac{n+1}{2}; \\ & \text{for } 3^{rd} \text{ ordinate, odd i}, i > \frac{n+1}{2}; \\ & \text{for } 3^{rd} \text{ ordinate, even i}, i < \frac{n+1}{2}; \\ & \text{for } 1^{st} \text{ ordinate}, i = \frac{n+1}{2}; \\ 1, & \text{otherwise.} \end{cases}$$

$$c_\pi(v_i) = \begin{cases} (\frac{n+1}{2}) - i, & \text{for } 4^{th} \text{ ordinate}, i < \frac{n+1}{2}; \\ & \text{for } 6^{th} \text{ ordinate}, i \leq \frac{n+1}{2}; \\ i - (\frac{n+1}{2}), & \text{for } 4^{th} \text{ ordinate}, i > \frac{n+1}{2}; \\ & \text{for } 6^{th} \text{ ordinate}, i > \frac{n+1}{2}; \\ 0, & \text{for } 1^{st} \text{ ordinate}, \text{ odd i }, i < \frac{n+1}{2}; \\ & \text{for } 1^{st} \text{ ordinate}, \text{ even i }, i > \frac{n+1}{2}; \\ & \text{for } 5^{th} \text{ ordinate}, \text{ odd i }, i > \frac{n+1}{2}; \\ & \text{for } 5^{th} \text{ ordinate}, \text{ even i }, i < \frac{n+1}{2}; \\ 2, & \text{for } 2^{nd} \text{ ordinate}, \text{ odd i }, i < \frac{n+1}{2}; \\ & \text{for } 2^{nd} \text{ ordinate}, \text{ even i }, i > \frac{n+1}{2}; \\ & \text{for } 3^{rd} \text{ ordinate}, \text{ odd i }, i > \frac{n+1}{2}; \\ & \text{for } 3^{rd} \text{ ordinate}, \text{ even i }, i < \frac{n+1}{2}; \\ & \text{for } 1^{st} \text{ ordinate}, i = \frac{n+1}{2}; \\ 1, & \text{otherwise.} \end{cases}$$

$$c_\pi(v'_i) = \begin{cases} (\frac{n+1}{2}) - i, & \text{for } 4^{th} \text{ ordinate}, i \leq \frac{n+1}{2}; \\ & \text{for } 6^{th} \text{ ordinate}, i > \frac{n+1}{2}; \\ i - (\frac{n+1}{2}), & \text{for } 4^{th} \text{ ordinate}, i > \frac{n+1}{2}; \\ & \text{for } 6^{th} \text{ ordinate}, i > \frac{n+1}{2}; \\ 0, & \text{for } 2^{nd} \text{ ordinate, odd i}, i < \frac{n+1}{2}; \\ & \text{for } 2^{nd} \text{ ordinate, even i}, i > \frac{n+1}{2}; \\ & \text{for } 3^{rd} \text{ ordinate, odd i}, i > \frac{n+1}{2}; \\ & \text{for } 3^{rd} \text{ ordinate, even i}, i < \frac{n+1}{2}; \\ 2, & \text{for } 1^{st} \text{ ordinate, odd i}, i < \frac{n+1}{2}; \\ & \text{for } 1^{st} \text{ ordinate, even i}, i > \frac{n+1}{2}; \\ & \text{for } 5^{th} \text{ ordinate, odd i}, i > \frac{n+1}{2}; \\ & \text{for } 5^{th} \text{ ordinate, even i}, i < \frac{n+1}{2}; \\ & \text{for } 6^{th} \text{ ordinate}, i = \frac{n+1}{2}; \\ 1, & \text{otherwise.} \end{cases}$$

Since all vertices in $B_{D_2(P_n)}$, $n > 3$ for odd n have distinct color codes, then c is a locating coloring using 6 colors. As a result $\chi_L B_{D_2(P_n)} \leq 6$. Thus $\chi_L B_{D_2(P_n)} = 6$.

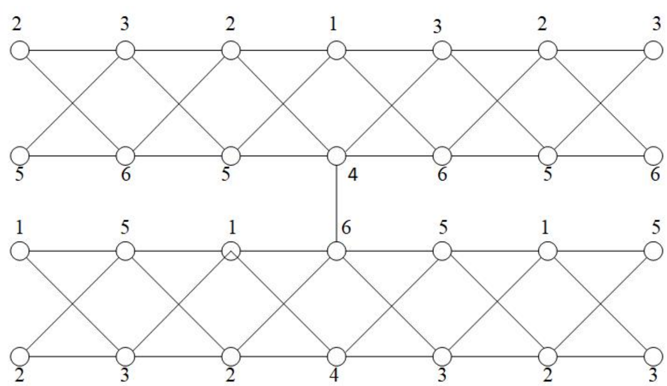

Figure 3. A locating coloring of $B_{D_2(P_7)}$

CASE 3 (n even). Let c be a locating coloring using 6 colors as follows :

$$c(u_i)=\begin{cases}1, & \text{for } i=\frac{n}{2};\\ 2, & \text{for odd i}; i<\frac{n}{2},\ n=4j, j\geq 1 \text{ and } i>\frac{n}{2},\ n=4j+2, j\geq 1\\ & \text{for even i}; i<\frac{n}{2},\ n=4j+2, j\geq 1 \text{ and } i>\frac{n}{2},\ n=4j, j\geq 1\\ 3, & \text{for odd i}; i<\frac{n}{2},\ n=4j+2, j\geq 1 \text{ and } i>\frac{n}{2},\ n=4j, j\geq 1\\ & \text{for even i}; i<\frac{n}{2},\ n=4j, j\geq 1 \text{ and } i>\frac{n}{2},\ n=4j+2, j\geq 1\end{cases}$$

$$c(u'_i)=\begin{cases}4, & \text{for } i=\frac{n}{2};\\ 5, & \text{for odd i}; i<\frac{n}{2},\ n=4j, j\geq 1 \text{ and } i>\frac{n}{2},\ n=4j+2, j\geq 1\\ & \text{for even i}; i<\frac{n}{2},\ n=4j+2, j\geq 1 \text{ and } i>\frac{n}{2},\ n=4j, j\geq 1\\ 6, & \text{for odd i}; i<\frac{n}{2},\ n=4j+2, j\geq 1 \text{ and } i>\frac{n}{2},\ n=4j, j\geq 1\\ & \text{for even i}; i<\frac{n}{2},\ n=4j, j\geq 1 \text{ and } i>\frac{n}{2},\ n=4j+2, j\geq 1\end{cases}$$

$$c(v_i)=\begin{cases}1, & \text{for odd i}; i<\frac{n}{2},\ n=4j, j\geq 1 \text{ and } i>\frac{n}{2},\ n=4j+2, j\geq 1\\ & \text{for even i}; i<\frac{n}{2},\ n=4j+2, j\geq 1 \text{ and } i>\frac{n}{2},\ n=4j, j\geq 1\\ 5, & \text{for odd i}; i<\frac{n}{2},\ n=4j+2, j\geq 1 \text{ and } i>\frac{n}{2},\ n=4j, j\geq 1\\ & \text{for even i}; i<\frac{n}{2},\ n=4j, j\geq 1 \text{ and } i>\frac{n}{2},\ n=4j+2, j\geq 1\\ 6, & \text{for } i=\frac{n}{2};\end{cases}$$

$$c(v'_i)=\begin{cases}2, & \text{for odd i}; i<\frac{n}{2},\ n=4j, j\geq 1 \text{ and } i>\frac{n}{2},\ n=4j+2, j\geq 1\\ & \text{for even i}; i<\frac{n}{2},\ n=4j+2, j\geq 1 \text{ and } i>\frac{n}{2},\ n=4j, j\geq 1\\ 3, & \text{for odd i}; i<\frac{n}{2},\ n=4j+2, j\geq 1 \text{ and } i>\frac{n}{2},\ n=4j, j\geq 1\\ & \text{for even i}; i<\frac{n}{2},\ n=4j, j\geq 1 \text{ and } i>\frac{n}{2},\ n=4j+2, j\geq 1\\ 4, & \text{for } i=\frac{n}{2};\end{cases}$$

The color codes of $B_{D_2(P_n)}$ are

$$c_\pi(u_i) = \begin{cases} (\frac{n}{2}) - i, & \text{for } 1^{st} \text{ ordinate}, i \leq \frac{n}{2}; \\ & \text{for } 4^{th} \text{ ordinate}, i < \frac{n}{2}; \\ i - (\frac{n}{2}), & \text{for } 1^{st} \text{ ordinate}, i > \frac{n}{2}; \\ & \text{for } 4^{th} \text{ ordinate}, i > \frac{n}{2}; \\ 0, & \text{for } 2^{nd} \text{ ordinate}, \text{ odd i}, i < \frac{n}{2}; \\ & \text{for } 3^{rd} \text{ ordinate}, \text{ even i}, i < \frac{n}{2}; \\ & \text{for } 2^{th} \text{ ordinate}, \text{ even i}, i > \frac{n}{2}; \\ & \text{for } 3^{rd} \text{ ordinate}, \text{ odd i}, i > \frac{n}{2}; \\ 2, & \text{for } 5^{th} \text{ ordinate}, \text{ odd i}, i < \frac{n}{2}; \\ & \text{for } 5^{th} \text{ ordinate}, \text{ even i}, i > \frac{n}{2}; \\ & \text{for } 6^{th} \text{ ordinate}, \text{ odd i}, i < \frac{n}{2}; \\ & \text{for } 6^{th} \text{ ordinate}, \text{ even i}, i > \frac{n}{2}; \\ & \text{for } 4^{th} \text{ ordinate}, i = \frac{n}{2}; \\ 1, & \text{otherwise.} \end{cases}$$

$$c_\pi(u'_i) = \begin{cases} (\frac{n}{2}) - i, & \text{for } 1^{st} \text{ ordinate}, i < \frac{n}{2}; \\ & \text{for } 4^{th} \text{ ordinate}, i \leq \frac{n}{2}; \\ i - (\frac{n}{2}), & \text{for } 1^{st} \text{ ordinate}, i > \frac{n}{2}; \\ & \text{for } 4^{th} \text{ ordinate}, i > \frac{n}{2}; \\ 0, & \text{for } 5^{th} \text{ ordinate}, \text{ odd i}, i < \frac{n}{2}; \\ & \text{for } 5^{th} \text{ ordinate}, \text{ even i}, i > \frac{n}{2}; \\ & \text{for } 6^{th} \text{ ordinate}, \text{ odd i}, i > \frac{n}{2}; \\ & \text{for } 6^{th} \text{ ordinate}, \text{ even i}, i < \frac{n}{2}; \\ 2, & \text{for } 2^{nd} \text{ ordinate}, \text{ odd i}, i < \frac{n}{2}; \\ & \text{for } 2^{nd} \text{ ordinate}, \text{ even i}, i > \frac{n}{2}; \\ & \text{for } 3^{rd} \text{ ordinate}, \text{ odd i}, i > \frac{n}{2}; \\ & \text{for } 3^{rd} \text{ ordinate}, \text{ even i}, i < \frac{n}{2}; \\ & \text{for } 1^{st} \text{ ordinate}, i = \frac{n}{2}; \\ 1, & \text{otherwise.} \end{cases}$$

$$c_\pi(v_i) = \begin{cases} (\frac{n}{2}) - i, & \text{for } 4^{th} \text{ ordinate}, i < \frac{n}{2}; \\ & \text{for } 6^{th} \text{ ordinate}, i \leq \frac{n}{2}; \\ i - (\frac{n}{2}), & \text{for } 4^{th} \text{ ordinate}, i > \frac{n}{2}; \\ & \text{for } 6^{th} \text{ ordinate}, i > \frac{n}{2}; \\ 0, & \text{for } 1^{st} \text{ ordinate, odd i}, i < \frac{n}{2}; \\ & \text{for } 1^{st} \text{ ordinate, even i}, i > \frac{n}{2}; \\ & \text{for } 5^{th} \text{ ordinate, odd i}, i > \frac{n}{2}; \\ & \text{for } 5^{th} \text{ ordinate, even i}, i < \frac{n}{2}; \\ 2, & \text{for } 2^{nd} \text{ ordinate, odd i}, i < \frac{n}{2}; \\ & \text{for } 2^{nd} \text{ ordinate, even i}, i > \frac{n}{2}; \\ & \text{for } 3^{rd} \text{ ordinate, odd i}, i > \frac{n}{2}; \\ & \text{for } 3^{rd} \text{ ordinate, even i}, i < \frac{n}{2}; \\ & \text{for } 1^{st} \text{ ordinate}, i = \frac{n}{2}; \\ 1, & \text{otherwise.} \end{cases}$$

$$c_\pi(v_i') = \begin{cases} (\frac{n}{2}) - i, & \text{for } 4^{th} \text{ ordinate}, i \leq \frac{n}{2}; \\ & \text{for } 6^{th} \text{ ordinate}, i > \frac{n}{2}; \\ i - (\frac{n}{2}), & \text{for } 4^{th} \text{ ordinate}, i > \frac{n}{2}; \\ & \text{for } 6^{th} \text{ ordinate}, i > \frac{n}{2}; \\ 0, & \text{for } 2^{nd} \text{ ordinate, odd i}, i < \frac{n}{2}; \\ & \text{for } 2^{nd} \text{ ordinate, even i}, i > \frac{n}{2}; \\ & \text{for } 3^{rd} \text{ ordinate, odd i}, i > \frac{n}{2}; \\ & \text{for } 3^{rd} \text{ ordinate, even i}, i < \frac{n}{2}; \\ 2, & \text{for } 1^{st} \text{ ordinate, odd i}, i < \frac{n}{2}; \\ & \text{for } 1^{st} \text{ ordinate, even i}, i > \frac{n}{2}; \\ & \text{for } 5^{th} \text{ ordinate, odd i}, i > \frac{n}{2}; \\ & \text{for } 5^{th} \text{ ordinate, even i}, i < \frac{n}{2}; \\ & \text{for } 6^{th} \text{ ordinate}, i = \frac{n}{2}; \\ 1, & \text{otherwise.} \end{cases}$$

Since all vertices in $B_{D_2(P_n)}$, $n \geq 3$ for even n have distinct color codes, then c is a locating coloring using 6 colors. As a result $\chi_L B_{D_2(P_n)} \leq 6$. Thus $\chi_L B_{D_2(P_n)} = 6$. $\square$

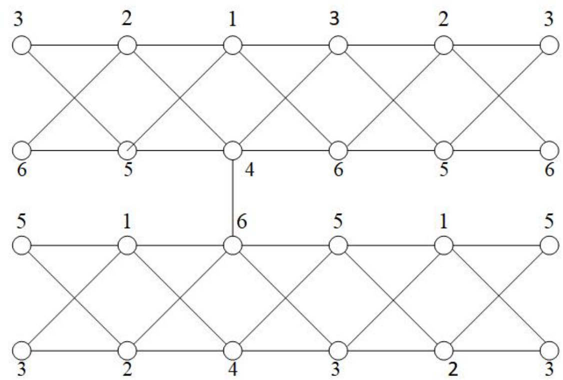

Figure 4. A locating coloring of $B_{D_2(P_6)}$

3. Concluding Remarks

The locating chromatic number of a shadow path graphs and the barbell graph containing a shadow path graph is similar, which is 6.

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On the locating chromatic number of barbell shadow path graphs

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Abstract

The locating chromatic number was introduced by Chartrand in 2002. The locating chromatic number of a graph is a combined concept between the coloring and partition dimension of a graph. The locating chromatic number of a graph is defined as the cardinality of the minimum color classes of the graph. In this paper, we discuss about the locating chromatic number of shadow path graphs and barbell graph containing shadow graph.

Keywords: the locating-chromatic number, shadow path graph, barbell graph
Mathematics Subject Classification: 05C12, 05C15
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1. Introduction

The locating chromatic number of a graph was introduced by Chatrand et al.[6] by combining two concepts in graph theory, which are vertex coloring and partition dimension of a graph. Let $G = (V, E)$ be a connected graph. A k-coloring of G is a function $c : V(G) \longrightarrow \{1, 2, \cdots, k\}$, where $c(u) \neq c(v)$ for any two adjacent vertices u and v in G. Thus, the coloring c induces a partition Π of $V(G)$ into k color classes (independent sets) $C_1, C_2, \cdots, C_k$, where C_i is the set of all vertices colored by the color i for $1 \leq i \leq k$. The color code $c_\Pi(v)$ of a vertex v in G is defined

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as the $k-$ordinate $(d(v, C_1), d(v, C_2), \cdots, d(v, C_k))$, where $d(v, C_i) = \min\{d(v, x); x \in C_i\}$ for $1 \leq i \leq k$. The $k-$coloring c of G such that all vertices have different color codes is called a *locating coloring* of G. The *locating chromatic number* of G, denoted by $\chi_L(G)$, is the minimum k such that G has a locating coloring.

The following two theorems are useful to determine the lower bound of the locating chromatic of a graph. The set of neighbors of a vertex q in G, denoted by $N(q)$.

Theorem 1.1. *(see [6]). Let c be a locating coloring in a connected graph G. If x and y are distinct vertices of G such that $d(p, w) = d(q, w)$ for all $w \in V(G) - \{p, q\}$, then $c(p) \neq c(q)$. In particular, if p and q are non-adjacent vertices such that $N(p) \neq N(q)$, then $c(p) \neq c(q)$.*

Theorem 1.2. *(see [6]). The locating chromatic number of a cycle graph $C_n(n \geq 3)$ is 3 for odd n and 4 for even n.*

The locating chromatic number of a graph is an interesting topic to study because there is no general algorithm for determining the locating chromatic number of any graphs and there are only a few results related to determining of the locating chromatic number of some graphs. Chartrand *et al.* [6] determined all graphs of order n with locating number n, namely a complete multipartite graph of n vertices. Moreover, Chartrand *et al.*[7] provided a tree construction of n vertices, $n \geq 5$, with locating chromatic number varying from 3 to n, except for $(n - 1)$. Next, Asmiati *et al.* [1] obtained the locating chromatic number of amalgamation of stars and non-homogeneous caterpillars and firecracker graphs [2]. In [5] Welyyanti *et al.* determined the locating chromatic number of complete n-ary trees. Next, Sofyan *et al.* [4] determined the locating chromatic number of homogeneous lobster. Recently, Ghanem *et al.* [8] found the locating chromatic number of powers of the path and cycles.

Let P_n be a path with $V(P_n) = \{x_i \mid 1 \leq i \leq n\}$ and $E(P_n) = \{x_i x_{i+1} \mid 1 \leq i \leq n-1\}$. The shadow path graph $D_2(P_n)$ is a graph with the vertex set $\{u_i, v_i \mid 1 \leq i \leq n\}$ where $u_i u_j \in E(D_2(P_n))$ if and only if $x_i x_j \in E(P_n)$ and $u_i v_j \in E(D_2(P_n))$ if and only if $x_i x_j \in E(P_n)$. *A barbell graph containing shadow path graph*, denoted by $B_{D_2(P_n)}$ is obtained by copying a shadow path graph (namely, $D_2'(P_n)$) and connecting the two graphs with a bridge. We assume that $\{u_i', v_i' \mid 1 \leq i \leq n\}$ is a vertex set of $D_2'(P_n)$ and a bridge in $B_{D_2(P_n)}$ connecting $\{u'_{\frac{n+1}{2}} v_{\frac{n+1}{2}}\}$ for odd n and $\{u'_{\frac{n}{2}} v_{\frac{n}{2}}\}$ for even n.

Motivated by the result of Asmiati *et al.* [3] about the determination of the locating chromatic number of certain barbell graphs, in this paper we determine the locating chromatic number of shadow path graphs and barbell graph containing shadow path for $n \geq 3$.

2. Main Results

The following theorem gives the locating chromatic number for shadow path graph $D_2(P_n)$ for $n \geq 3$.

Lemma 2.1. *Let c be a locating-chromatic number for shadow path graph $D_2(P_n)$, with $u_i \in P_i^1$ and $v_i \in P_i^2$. Then $c(u_i) \neq c(v_i)$.*

Proof. On the shadow path graph $D_2(P_n)$, we can see that $d(u_i, x) = d(v_i, x)$, $i \in [1, n-1]$ for every $x \in ((D_2(P_n)) \setminus \{u_i, v_i\})$. By Theorem 1.1, we have $c(u_i) \neq c(v_i)$. □

Theorem 2.1. *The locating chromatic number of a shadow path graph for $n \geq 3, D_2(P_n)$ is 6.*

Proof. First, we determine the lower bound for the locating-chromatic number of shadow path graph $D_2(P_n)$ for $n \geq 3$. The Shadow path graph $D_2(P_n)$ for $n \geq 3$ consists of minimal two cycles C_4. Pick the first cycle C_4, then by Theorem 1.2, we could assign 4 colors, $\{1, 2, 3, 4\}$ to the first cycle's vertices. Next, in the second C_4, we have two vertices, which intersect with two vertices in the first C_4. By Lemma 2.1, we must assign two different colors to the remaining vertices of the second C_4. Therefore, we have $\chi_L(G) \geq 6$.

Next, we determine the upper bound of the locating chromatic number of the shadow path graph for $n \geq 3$. Let c be a coloring using 6 colors as follow:

$$c(u_i) = \begin{cases} 1, & \text{for } i = 1, \\ 2, & \text{for } i = 2n \text{ }, n \geq 1, \\ 3, & \text{for } i = 2n+1 \text{ }, n \geq 1, \end{cases}$$

$$c(v_i) = \begin{cases} 4, & \text{for } i = 1, \\ 5, & \text{for } i = 2n \text{ }, n \geq 1, \\ 6, & \text{for } i = 2n+1 \text{ }, n \geq 1. \end{cases}$$

The color codes of $D_2(P_n)$ are :

$$c_\pi(u_i) = \begin{cases} i-1, & \text{for } 1^{st} \text{ ordinate}, i \geq 1, \\ & \text{for } 4^{th} \text{ordinate}, i \geq 2, \\ 0, & \text{for } 2^{nd} \text{ ordinate, even i }, 2 \leq i \leq n, \\ & \text{for } 3^{rd} \text{ ordinate, odd i }, 3 \leq i \leq n, \\ 2, & \text{for } 5^{th} \text{ ordinate, even i }, 2 \leq i \leq n, \\ & \text{for } 6^{th} \text{ ordinate, odd i }, 1 \leq i \leq n, \\ & \text{for } 3^{rd} \text{ ordinate}, i = 1, \\ & \text{for } 4^{th} \text{ ordinate}, i = 1, \\ 1, & \text{otherwise}, \end{cases}$$

$$c_\pi(v_i) = \begin{cases} i-1, & \text{for } 1^{st} \text{ ordinate}, i \geq 2, \\ & \text{for } 4^{th} \text{ordinate}, i \geq 1, \\ 0, & \text{for } 5^{nd} \text{ ordinate, even i }, 2 \leq i \leq n, \\ & \text{for } 6^{th} \text{ ordinate, odd i }, 3 \leq i \leq n; \\ 2, & \text{for } 2^{nd} \text{ ordinate, even i }, 2 \leq i \leq n, \\ & \text{for } 3^{rd} \text{ ordinate, odd i }, 3 \leq i \leq n, \\ & \text{for } 1^{st} \text{ ordinate}, i = 1, \\ & \text{for } 6^{th} \text{ ordinate}, i = 1, \\ 1, & \text{otherwise.} \end{cases}$$

Since all vertices in $D_2(P_n)$ for $n \geq 3$ have distinct color codes, then c is a locating coloring using 6 colors. As a result $\chi_L D_2(P_n) \leq 6$. Thus $\chi_L D_2(P_n) = 6$. $\square$

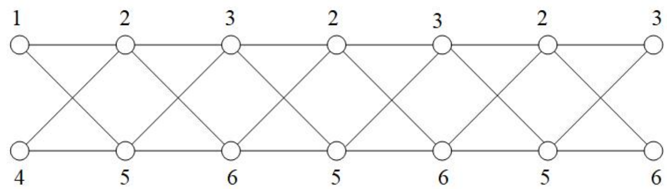

Figure 1. A minimum locating coloring of $D_2(P_7)$.

Theorem 2.2. *The locating chromatic number of a barbell graph containing shadow path for* $n \geq 3$ *is* 6.

Proof. First, we determine the lower bound of $\chi_L B_{D_2(P_n)}$ for $n \geq 3$. Since the barbell graph $B_{D_2(P_n)}$ containing $D_2(P_n)$, then by Theorem 2.3 we have $\chi_L(B_{D_2(P_n)}) \geq 6$. To prove the upper bound, consider the following three cases.

CASE 1 ($n = 3$). Let c be a locating coloring using 6 colors as follows :

$$c(u_i) = \begin{cases} 1, & \text{for } i = 2, \\ 2, & \text{for } i = 1, \\ 3, & \text{for } i = 3, \end{cases}$$

$$c(u_i') = \begin{cases} 4, & \text{for } i = 2, \\ 5, & \text{for } i = 1, \\ 6, & \text{for } i = 3, \end{cases}$$

$$c(v_i) = \begin{cases} 1, & \text{for } i = 1, \\ 5, & \text{for } i = 3, \\ 6, & \text{for } i = 2, \end{cases}$$

$$c(v_i') = \begin{cases} 2, & \text{for } i = 1, \\ 3, & \text{for } i = 3, \\ 4, & \text{for } i = 2. \end{cases}$$

The color codes of $B_{D_2(P_3)}$ are

$$c_\pi(u_i) = \begin{cases} 0, & \text{for } 1^{st} \text{ ordinate}, \ i = 2, \\ & \text{for } 2^{nd} \text{ ordinate}, \ i = 1, \\ & \text{for } 3^{rd} \text{ ordinate}, \ i = 3, \\ 2, & \text{for } 3^{th} \text{ ordinate}, \ i = 1, \\ & \text{for } 4^{th} \text{ ordinate}, \ i = 2, \\ & \text{for } 2^{nd} \text{ ordinate}, \ i = 3, \\ & \text{for } 5^{th} \ and \ 6^{th} \text{ ordinate}, \ i = 1 \text{ and } 3, \\ 1, & \text{otherwise}, \end{cases}$$

$$c_\pi(u_i') = \begin{cases} 0, & \text{for } 5^{th} \text{ ordinate}, \ i = 1, \\ & \text{for } 4^{th} \text{ ordinate}, \ i = 2, \\ & \text{for } 6^{th} \text{ ordinate}, \ i = 3, \\ 2, & \text{for } 6^{th} \text{ ordinate}, \ i = 1, \\ & \text{for } 1^{nd} \text{ ordinate}, \ i = 2, \\ & \text{for } 5^{th} \text{ ordinate}, \ i = 3, \\ & \text{for } 2^{nd} \ and \ 3^{rd} \text{ ordinate}, \ i = 1 \text{ and } 3, \\ 1, & \text{otherwise}, \end{cases}$$

$$c_\pi(v_i) = \begin{cases} 0, & \text{for } 1^{st} \text{ ordinate}, \ i = 1, \\ & \text{for } 6^{th} \text{ ordinate}, \ i = 2, \\ & \text{for } 5^{th} \text{ ordinate}, \ i = 3, \\ 2, & \text{for } 5^{th} \text{ ordinate}, \ i = 1, \\ & \text{for } 1^{st} \text{ ordinate}, \ i = 3, \\ & \text{for } 2^{nd} \ and \ 3^{rd} \text{ ordinate}, \ i = 1 \text{ and } 3, \\ 1, & \text{otherwise}, \end{cases}$$

$$c_\pi(v_i') = \begin{cases} 0, & \text{for } 2^{nd} \text{ ordinate}, \ i = 1, \\ & \text{for } 4^{th} \text{ ordinate}, \ i = 2, \\ & \text{for } 3^{rd} \text{ ordinate}, \ i = 3, \\ 2, & \text{for } 1^{st} \text{ ordinate}, \ i = 1 \text{ and } 3, \\ & \text{for } 6^{th} \text{ ordinate}, \ i = 2, \\ & \text{for } 2^{nd} \text{ ordinate}, \ i = 3, \\ & \text{for } 3^{rd} \ and \ 5^{th} \text{ ordinate}, \ i = 3, \\ 1, & \text{otherwise}. \end{cases}$$

Since all vertices in $B_{D_2(P_3)}$ have distinct color codes, then c is a locating coloring using 6 colors. As a result $\chi_L B_{D_2(P_3)} \leq 6$. Thus $\chi_L B_{D_2(P_3)} = 6$.

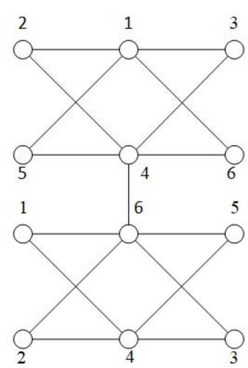

Figure 2. A minimum locating coloring of $B_{D_2(P_3)}$.

CASE 2 (n odd). Let c be a locating coloring using 6 colors as follows :

$$c(u_i)=\begin{cases}1, & \text{for } i=\frac{n+1}{2},\\ 2, & \text{for odd i; } i<\frac{n+1}{2},\ n=4j+3, j\geq 1 \text{ and } i>\frac{n+1}{2},\ n=4j+1, j\geq 1,\\ & \text{for even 1; } i<\frac{n+1}{2},\ n=4j+1, j\geq 1 \text{ and } i>\frac{n+1}{2},\ n=4j+3, j\geq 1,\\ 3, & \text{for odd i; } i<\frac{n+1}{2},\ n=4j+1, j\geq 1 \text{ and } i>\frac{n+1}{2},\ n=4j+3, j\geq 1,\\ & \text{for even i; } i<\frac{n+1}{2},\ n=4j+3, j\geq 1 \text{ and } i>\frac{n+1}{2},\ n=4j+1, j\geq 1,\end{cases}$$

$$c(u_i')=\begin{cases}4, & \text{for } i=\frac{n+1}{2},\\ 5, & \text{for odd i; } i<\frac{n+1}{2},\ n=4j+3, j\geq 1 \text{ and } i>\frac{n+1}{2},\ n=4j+1, j\geq 1,\\ & \text{for even 1; } i<\frac{n+1}{2},\ n=4j+1, j\geq 1 \text{ and } i>\frac{n+1}{2},\ n=4j+3, j\geq 1,\\ 6, & \text{for odd i; } i<\frac{n+1}{2},\ n=4j+1, j\geq 1 \text{ and } i>\frac{n+1}{2},\ n=4j+3, j\geq 1,\\ & \text{for even i; } i<\frac{n+1}{2},\ n=4j+3, j\geq 1 \text{ and } i>\frac{n+1}{2},\ n=4j+1, j\geq 1,\end{cases}$$

$$c(v_i)=\begin{cases}1, & \text{for odd i; } i<\frac{n+1}{2},\ n=4j+3, j\geq 1 \text{ and } i>\frac{n+1}{2},\ n=4j+1, j\geq 1,\\ & \text{for even i; } i<\frac{n+1}{2},\ n=4j+1, j\geq 1 \text{ and } i>\frac{n+1}{2},\ n=4j+3, j\geq 1,\\ 5, & \text{for odd i; } i<\frac{n+1}{2},\ n=4j+1, j\geq 1 \text{ and } i>\frac{n+1}{2},\ n=4j+3, j\geq 1,\\ & \text{for even i; } i<\frac{n+1}{2},\ n=4j+3, j\geq 1 \text{ and } i>\frac{n+1}{2},\ n=4j+1, j\geq 1,\\ 6, & \text{for } i=\frac{n+1}{2},\end{cases}$$

$$c(v_i')=\begin{cases}2, & \text{for odd i; } i<\frac{n+1}{2},\ n=4j+3, j\geq 1 \text{ and } i>\frac{n+1}{2},\ n=4j+1, j\geq 1,\\ & \text{for even i; } i<\frac{n+1}{2},\ n=4j+1, j\geq 1 \text{ and } i>\frac{n+1}{2},\ n=4j+3, j\geq 1,\\ 4, & \text{for } i=\frac{n+1}{2},\\ 3, & \text{for odd i; } i<\frac{n+1}{2},\ n=4j+1, j\geq 1 \text{ and } i>\frac{n+1}{2},\ n=4j+3, j\geq 1,\\ & \text{for even i; } i<\frac{n+1}{2},\ n=4j+3, j\geq 1 \text{ and } i>\frac{n+1}{2},\ n=4j+1, j\geq 1.\end{cases}$$

The color codes of $B_{D_2(P_n)}$ are

$$c_\pi(u_i)=\begin{cases}(\frac{n+1}{2})-i, & \text{for } 1^{st} \text{ ordinate}, i\leq \frac{n+1}{2},\\ & \text{for } 4^{th} \text{ ordinate}, i< \frac{n+1}{2},\\ i-(\frac{n+1}{2}), & \text{for } 1^{st} \text{ ordinate}, i> \frac{n+1}{2},\\ & \text{for } 4^{th} \text{ ordinate}, i> \frac{n+1}{2},\\ 0, & \text{for } 2^{nd} \text{ ordinate, odd i }, i< \frac{n+1}{2},\\ & \text{for } 3^{rd} \text{ ordinate, even i }, i< \frac{n+1}{2},\\ & \text{for } 2^{th} \text{ ordinate, even i }, i> \frac{n+1}{2},\\ & \text{for } 3^{rd} \text{ ordinate, odd i }, i> \frac{n+1}{2},\\ 2, & \text{for } 5^{th} \text{ ordinate, odd i }, i< \frac{n+1}{2},\\ & \text{for } 5^{th} \text{ ordinate, even i }, i> \frac{n+1}{2},\\ & \text{for } 6^{th} \text{ ordinate, odd i }, i< \frac{n+1}{2},\\ & \text{for } 6^{th} \text{ ordinate, even i }, i> \frac{n+1}{2},\\ & \text{for } 4^{th} \text{ ordinate}, i= \frac{n+1}{2},\\ 1, & \text{otherwise},\end{cases}$$

$$c_\pi(u_i')=\begin{cases}(\frac{n+1}{2})-i, & \text{for } 1^{st} \text{ ordinate}, i< \frac{n+1}{2},\\ & \text{for } 4^{th} \text{ ordinate}, i\leq \frac{n+1}{2},\\ i-(\frac{n+1}{2}), & \text{for } 1^{st} \text{ ordinate}, i> \frac{n+1}{2},\\ & \text{for } 4^{th} \text{ ordinate}, i> \frac{n+1}{2},\\ 0, & \text{for } 5^{th} \text{ ordinate, odd i }, i< \frac{n+1}{2},\\ & \text{for } 5^{th} \text{ ordinate, even i }, i> \frac{n+1}{2},\\ & \text{for } 6^{th} \text{ ordinate, odd i }, i> \frac{n+1}{2},\\ & \text{for } 6^{th} \text{ ordinate, even i }, i< \frac{n+1}{2},\\ 2, & \text{for } 2^{nd} \text{ ordinate, odd i }, i< \frac{n+1}{2},\\ & \text{for } 2^{nd} \text{ ordinate, even i }, i> \frac{n+1}{2},\\ & \text{for } 3^{rd} \text{ ordinate, odd i }, i> \frac{n+1}{2},\\ & \text{for } 3^{rd} \text{ ordinate, even i }, i< \frac{n+1}{2},\\ & \text{for } 1^{st} \text{ ordinate}, i= \frac{n+1}{2},\\ 1, & \text{otherwise},\end{cases}$$

$$c_\pi(v_i)=\begin{cases}(\frac{n+1}{2})-i, & \text{for } 4^{th} \text{ ordinate}, i<\frac{n+1}{2},\\ & \text{for } 6^{th} \text{ ordinate}, i\le\frac{n+1}{2},\\ i-(\frac{n+1}{2}), & \text{for } 4^{th} \text{ ordinate}, i>\frac{n+1}{2},\\ & \text{for } 6^{th} \text{ ordinate}, i>\frac{n+1}{2},\\ 0, & \text{for } 1^{st} \text{ ordinate, odd i}, i<\frac{n+1}{2},\\ & \text{for } 1^{st} \text{ ordinate, even i}, i>\frac{n+1}{2},\\ & \text{for } 5^{th} \text{ ordinate, odd i}, i>\frac{n+1}{2},\\ & \text{for } 5^{th} \text{ ordinate, even i}, i<\frac{n+1}{2},\\ 2, & \text{for } 2^{nd} \text{ ordinate, odd i}, i<\frac{n+1}{2},\\ & \text{for } 2^{nd} \text{ ordinate, even i}, i>\frac{n+1}{2},\\ & \text{for } 3^{rd} \text{ ordinate, odd i}, i>\frac{n+1}{2},\\ & \text{for } 3^{rd} \text{ ordinate, even i}, i<\frac{n+1}{2},\\ & \text{for } 1^{st} \text{ ordinate}, i=\frac{n+1}{2},\\ 1, & \text{otherwise},\end{cases}$$

$$c_\pi(v'_i)=\begin{cases}(\frac{n+1}{2})-i, & \text{for } 4^{th} \text{ ordinate}, i\le\frac{n+1}{2},\\ & \text{for } 6^{th} \text{ ordinate}, i>\frac{n+1}{2},\\ i-(\frac{n+1}{2}), & \text{for } 4^{th} \text{ ordinate}, i>\frac{n+1}{2},\\ & \text{for } 6^{th} \text{ ordinate}, i>\frac{n+1}{2},\\ 0, & \text{for } 2^{nd} \text{ ordinate, odd i}, i<\frac{n+1}{2},\\ & \text{for } 2^{nd} \text{ ordinate, even i}, i>\frac{n+1}{2},\\ & \text{for } 3^{rd} \text{ ordinate, odd i}, i>\frac{n+1}{2},\\ & \text{for } 3^{rd} \text{ ordinate, even i}, i<\frac{n+1}{2},\\ 2, & \text{for } 1^{st} \text{ ordinate, odd i}, i<\frac{n+1}{2},\\ & \text{for } 1^{st} \text{ ordinate, even i}, i>\frac{n+1}{2},\\ & \text{for } 5^{th} \text{ ordinate, odd i}, i>\frac{n+1}{2},\\ & \text{for } 5^{th} \text{ ordinate, even i}, i<\frac{n+1}{2},\\ & \text{for } 6^{th} \text{ ordinate}, i=\frac{n+1}{2},\\ 1, & \text{otherwise}.\end{cases}$$

Since all vertices in $B_{D_2(P_n)}, n>3$ for odd n have distinct color codes, then c is a locating coloring using 6 colors. As a result $\chi_L B_{D_2(P_n)} \le 6$. Thus $\chi_L B_{D_2(P_n)} = 6$.

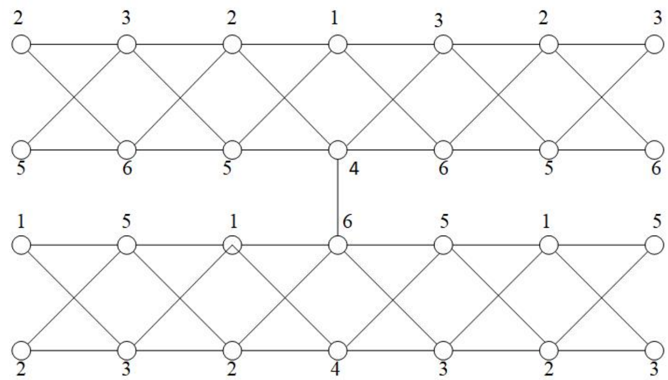

Figure 3. A minimum locating coloring of $B_{D_2(P_7)}$

CASE 3 (n even). Let c be a locating coloring using 6 colors as follows:

$$c(u_i) = \begin{cases} 1, & \text{for } i = \frac{n}{2}, \\ 2, & \text{for odd i; } i < \frac{n}{2},\ n = 4j, j \geq 1 \text{ and } i > \frac{n}{2},\ n = 4j+2, j \geq 1, \\ & \text{for even i; } i < \frac{n}{2},\ n = 4j+2, j \geq 1 \text{ and } i > \frac{n}{2},\ n = 4j, j \geq 1, \\ 3, & \text{for odd i; } i < \frac{n}{2},\ n = 4j+2, j \geq 1 \text{ and } i > \frac{n}{2},\ n = 4j, j \geq 1, \\ & \text{for even i; } i < \frac{n}{2},\ n = 4j, j \geq 1 \text{ and } i > \frac{n}{2},\ n = 4j+2, j \geq 1, \end{cases}$$

$$c(u_i') = \begin{cases} 4, & \text{for } i = \frac{n}{2}, \\ 5, & \text{for odd i; } i < \frac{n}{2},\ n = 4j, j \geq 1 \text{ and } i > \frac{n}{2},\ n = 4j+2, j \geq 1, \\ & \text{for even i; } i < \frac{n}{2},\ n = 4j+2, j \geq 1 \text{ and } i > \frac{n}{2},\ n = 4j, j \geq 1, \\ 6, & \text{for odd i; } i < \frac{n}{2},\ n = 4j+2, j \geq 1 \text{ and } i > \frac{n}{2},\ n = 4j, j \geq 1, \\ & \text{for even i; } i < \frac{n}{2},\ n = 4j, j \geq 1 \text{ and } i > \frac{n}{2},\ n = 4j+2, j \geq 1, \end{cases}$$

$$c(v_i) = \begin{cases} 1, & \text{for odd i; } i < \frac{n}{2},\ n = 4j, j \geq 1 \text{ and } i > \frac{n}{2},\ n = 4j+2, j \geq 1, \\ & \text{for even i; } i < \frac{n}{2},\ n = 4j+2, j \geq 1 \text{ and } i > \frac{n}{2},\ n = 4j, j \geq 1, \\ 5, & \text{for odd i; } i < \frac{n}{2},\ n = 4j+2, j \geq 1 \text{ and } i > \frac{n}{2},\ n = 4j, j \geq 1, \\ & \text{for even i; } i < \frac{n}{2},\ n = 4j, j \geq 1 \text{ and } i > \frac{n}{2},\ n = 4j+2, j \geq 1, \\ 6, & \text{for } i = \frac{n}{2}, \end{cases}$$

$$c(v_i') = \begin{cases} 2, & \text{for odd i; } i < \frac{n}{2},\ n = 4j, j \geq 1 \text{ and } i > \frac{n}{2},\ n = 4j+2, j \geq 1, \\ & \text{for even i; } i < \frac{n}{2},\ n = 4j+2, j \geq 1 \text{ and } i > \frac{n}{2},\ n = 4j, j \geq 1, \\ 3, & \text{for odd i; } i < \frac{n}{2},\ n = 4j+2, j \geq 1 \text{ and } i > \frac{n}{2},\ n = 4j, j \geq 1, \\ & \text{for even i; } i < \frac{n}{2},\ n = 4j, j \geq 1 \text{ and } i > \frac{n}{2},\ n = 4j+2, j \geq 1, \\ 4, & \text{for } i = \frac{n}{2}. \end{cases}$$

The color codes of $B_{D_2(P_n)}$ are

$$c_\pi(u_i) = \begin{cases} (\frac{n}{2}) - i, & \text{for } 1^{st} \text{ ordinate}, i \leq \frac{n}{2}, \\ & \text{for } 4^{th} \text{ ordinate}, i < \dfrac{n}{2}; \\ i - (\frac{n}{2}), & \text{for } 1^{st} \text{ ordinate}, i > \frac{n}{2}, \\ & \text{for } 4^{th} \text{ ordinate}, i > \dfrac{n}{2}; \\ 0, & \text{for } 2^{nd} \text{ ordinate, odd i}, i < \frac{n}{2}, \\ & \text{for } 3^{rd} \text{ ordinate, even i}, i < \frac{n}{2}, \\ & \text{for } 2^{th} \text{ ordinate, even i}, i > \frac{n}{2}, \\ & \text{for } 3^{rd} \text{ ordinate, odd i}, i > \frac{n}{2}, \\ 2, & \text{for } 5^{th} \text{ ordinate, odd i}, i < \frac{n}{2}, \\ & \text{for } 5^{th} \text{ ordinate, even i}, i > \frac{n}{2}, \\ & \text{for } 6^{th} \text{ ordinate, odd i}, i < \frac{n}{2}, \\ & \text{for } 6^{th} \text{ ordinate, even i}, i > \frac{n}{2}, \\ & \text{for } 4^{th} \text{ ordinate}, i = \frac{n}{2}, \\ 1, & \text{otherwise}, \end{cases}$$

$$c_\pi(u'_i) = \begin{cases} (\frac{n}{2}) - i, & \text{for } 1^{st} \text{ ordinate}, i < \frac{n}{2}, \\ & \text{for } 4^{th} \text{ ordinate}, i \leq \frac{n}{2}, \\ i - (\frac{n}{2}), & \text{for } 1^{st} \text{ ordinate}, i > \frac{n}{2}, \\ & \text{for } 4^{th} \text{ ordinate}, i > \frac{n}{2}, \\ 0, & \text{for } 5^{th} \text{ ordinate, odd i}, i < \frac{n}{2}, \\ & \text{for } 5^{th} \text{ ordinate, even i}, i > \frac{n}{2}, \\ & \text{for } 6^{th} \text{ ordinate, odd i}, i > \frac{n}{2}, \\ & \text{for } 6^{th} \text{ ordinate, even i}, i < \frac{n}{2}, \\ 2, & \text{for } 2^{nd} \text{ ordinate, odd i}, i < \frac{n}{2}, \\ & \text{for } 2^{nd} \text{ ordinate, even i}, i > \frac{n}{2}, \\ & \text{for } 3^{rd} \text{ ordinate, odd i}, i > \frac{n}{2}, \\ & \text{for } 3^{rd} \text{ ordinate, even i}, i < \frac{n}{2}, \\ & \text{for } 1^{st} \text{ ordinate}, i = \frac{n}{2}, \\ 1, & \text{otherwise}, \end{cases}$$

$$c_\pi(v_i) = \begin{cases} (\dfrac{n}{2}) - i, & \text{for } 4^{th} \text{ ordinate}, i < \frac{n}{2}, \\ & \text{for } 6^{th} \text{ ordinate}, i \leq \frac{n}{2}, \\ i - (\dfrac{n}{2}), & \text{for } 4^{th} \text{ ordinate}, i > \frac{n}{2}, \\ & \text{for } 6^{th} \text{ ordinate}, i > \frac{n}{2}, \\ 0, & \text{for } 1^{st} \text{ ordinate, odd i}, i < \frac{n}{2}, \\ & \text{for } 1^{st} \text{ ordinate, even i}, i > \frac{n}{2}, \\ & \text{for } 5^{th} \text{ ordinate, odd i}, i > \frac{n}{2}, \\ & \text{for } 5^{th} \text{ ordinate, even i}, i < \frac{n}{2}, \\ 2, & \text{for } 2^{nd} \text{ ordinate, odd i}, i < \frac{n}{2}, \\ & \text{for } 2^{nd} \text{ ordinate, even i}, i > \frac{n}{2}, \\ & \text{for } 3^{rd} \text{ ordinate, odd i}, i > \frac{n}{2}, \\ & \text{for } 3^{rd} \text{ ordinate, even i}, i < \frac{n}{2}, \\ & \text{for } 1^{st} \text{ ordinate}, i = \frac{n}{2}, \\ 1, & \text{otherwise}, \end{cases}$$

$$c_\pi(v_i') = \begin{cases} (\dfrac{n}{2}) - i, & \text{for } 4^{th} \text{ ordinate}, i \leq \frac{n}{2}, \\ & \text{for } 6^{th} \text{ ordinate}, i > \frac{n}{2}, \\ i - (\dfrac{n}{2}), & \text{for } 4^{th} \text{ ordinate}, i > \frac{n}{2}, \\ & \text{for } 6^{th} \text{ ordinate}, i > \frac{n}{2}, \\ 0, & \text{for } 2^{nd} \text{ ordinate, odd i}, i < \frac{n}{2}, \\ & \text{for } 2^{nd} \text{ ordinate, even i}, i > \frac{n}{2}, \\ & \text{for } 3^{rd} \text{ ordinate, odd i}, i > \frac{n}{2}, \\ & \text{for } 3^{rd} \text{ ordinate, even i}, i < \frac{n}{2}, \\ 2, & \text{for } 1^{st} \text{ ordinate, odd i}, i < \frac{n}{2}, \\ & \text{for } 1^{st} \text{ ordinate, even i}, i > \frac{n}{2}, \\ & \text{for } 5^{th} \text{ ordinate, odd i}, i > \frac{n}{2}, \\ & \text{for } 5^{th} \text{ ordinate, even i}, i < \frac{n}{2}, \\ & \text{for } 6^{th} \text{ ordinate}, i = \frac{n}{2}, \\ 1, & \text{otherwise}. \end{cases}$$

Since all vertices in $B_{D_2(P_n)}$, $n \geq 3$ for even n have distinct color codes, then c is a locating coloring using 6 colors. As a result $\chi_L B_{D_2(P_n)} \leq 6$. Thus $\chi_L B_{D_2(P_n)} = 6$. □

3. Concluding Remarks

The locating chromatic number of a shadow path graphs and the barbell graph containing a shadow path graph is similar, which is 6.

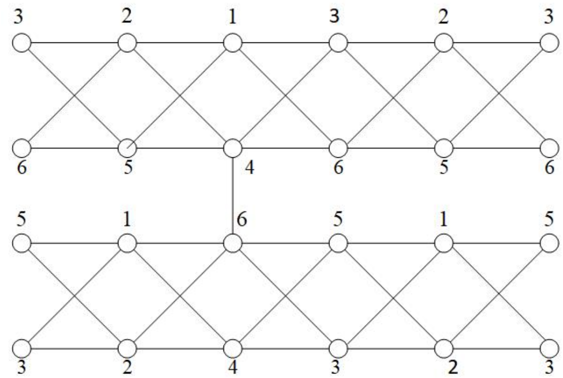

Figure 4. A minimum locating coloring of $B_{D_2(P_6)}$

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On the Locating Chromatic Number of Barbell Shadow Path Graph

A. Asmiati, Maharani Damayanti, Lyra Yulianti

Abstract

The locating-chromatic number was introduced by Chartrand in 2002. The locating chromatic number of a graph is a combined concept between the coloring and partition dimension of a graph. The locating chromatic number of a graph is defined as the cardinality of the minimum color classes of the graph. In this paper, we discuss about the locating-chromatic number of shadow path graph and barbell graph containing shadow graph.

Keywords

the locating-chromatic number, shadow path graph, barbell graph

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On the locating chromatic number of barbell shadow path graphs

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Abstract

The locating chromatic number was introduced by Chartrand in 2002. The locating chromatic number of a graph is a combined concept between the coloring and partition dimension of a graph. The locating chromatic number of a graph is defined as the cardinality of the minimum color classes of the graph. In this paper, we discuss about the locating chromatic number of shadow path graphs and barbell graph containing shadow graph.

Keywords: the locating-chromatic number, shadow path graph, barbell graph
Mathematics Subject Classification: 05C12, 05C15
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1. Introduction

The locating chromatic number of a graph was introduced by Chatrand et al.[6] by combining two concepts in graph theory, which are vertex coloring and partition dimension of a graph. Let $G = (V, E)$ be a connected graph. A k-coloring of G is a function $c : V(G) \longrightarrow \{1, 2, \cdots, k\}$, where $c(u) \neq c(v)$ for any two adjacent vertices u and v in G. Thus, the coloring c induces a partition Π of $V(G)$ into k color classes (independent sets) $C_1, C_2, \cdots, C_k$, where C_i is the set of all vertices colored by the color i for $1 \leq i \leq k$. The color code $c_\Pi(v)$ of a vertex v in G is defined

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as the $k-$ordinate $(d(v, C_1), d(v, C_2), \cdots, d(v, C_k))$, where $d(v, C_i) = \min\{d(v, x); x \in C_i\}$ for $1 \leq i \leq k$. The $k-$coloring c of G such that all vertices have different color codes is called a *locating coloring* of G. The *locating chromatic number* of G, denoted by $\chi_L(G)$, is the minimum k such that G has a locating coloring.

The following two theorems are useful to determine the lower bound of the locating chromatic of a graph. The set of neighbors of a vertex q in G, denoted by $N(q)$.

Theorem 1.1. *(see [6]). Let c be a locating coloring in a connected graph G. If x and y are distinct vertices of G such that $d(p, w) = d(q, w)$ for all $w \in V(G) - \{p, q\}$, then $c(p) \neq c(q)$. In particular, if p and q are non-adjacent vertices such that $N(p) \neq N(q)$, then $c(p) \neq c(q)$.*

Theorem 1.2. *(see [6]). The locating chromatic number of a cycle graph $C_n(n \geq 3)$ is 3 for odd n and 4 for even n.*

The locating chromatic number of a graph is an interesting topic to study because there is no general algorithm for determining the locating chromatic number of any graphs and there are only a few results related to determining of the locating chromatic number of some graphs. Chartrand *et al*. [6] determined all graphs of order n with locating number n, namely a complete multipartite graph of n vertices. Moreover, Chartrand *et al*.[7] provided a tree construction of n vertices, $n \geq 5$, with locating chromatic number varying from 3 to n, except for $(n-1)$. Next, Asmiati *et al*. [1] obtained the locating chromatic number of amalgamation of stars and non-homogeneous caterpillars and firecracker graphs [2]. In [5] Welyyanti *et al*. determined the locating chromatic number of complete n-ary trees. Next, Sofyan *et al*. [4] determined the locating chromatic number of homogeneous lobster. Recently, Ghanem *et al*. [8] found the locating chromatic number of powers of the path and cycles.

Let P_n be a path with $V(P_n) = \{x_i \mid 1 \leq i \leq n\}$ and $E(P_n) = \{x_i x_{i+1} \mid 1 \leq i \leq n-1\}$. The shadow path graph $D_2(P_n)$ is a graph with the vertex set $\{u_i, v_i \mid 1 \leq i \leq n\}$ where $u_i u_j \in E(D_2(P_n))$ if and only if $x_i x_j \in E(P_n)$ and $u_i v_j \in E(D_2(P_n))$ if and only if $x_i x_j \in E(P_n)$. *A barbell graph containing shadow path graph*, denoted by $B_{D_2(P_n)}$ is obtained by copying a shadow path graph (namely, $D_2'(P_n)$) and connecting the two graphs with a bridge. We assume that $\{u_i', v_i' \mid 1 \leq i \leq n\}$ is a vertex set of $D_2'(P_n)$ and a bridge in $B_{D_2(P_n)}$ connecting $\{u'_{\frac{n+1}{2}} v_{\frac{n+1}{2}}\}$ for odd n and $\{u'_{\frac{n}{2}} v_{\frac{n}{2}}\}$ for even n.

Motivated by the result of Asmiati *et al*. [3] about the determination of the locating chromatic number of certain barbell graphs, in this paper we determine the locating chromatic number of shadow path graphs and barbell graph containing shadow path for $n \geq 3$.

2. Main Results

The following theorem gives the locating chromatic number for shadow path graph $D_2(P_n)$ for $n \geq 3$.

Lemma 2.1. *Let c be a locating-chromatic number for shadow path graph $D_2(P_n)$, with $u_i \in P_i^1$ and $v_i \in P_i^2$. Then $c(u_i) \neq c(v_i)$.*

Proof. On the shadow path graph $D_2(P_n)$, we can see that $d(u_i, x) = d(v_i, x)$, $i \in [1, n-1]$ for every $x \in ((D_2(P_n)) \setminus \{u_i, v_i\})$. By Theorem 1.1, we have $c(u_i) \neq c(v_i)$. $\square$

Theorem 2.1. *The locating chromatic number of a shadow path graph for $n \geq 3, D_2(P_n)$ is 6.*

Proof. First, we determine the lower bound for the locating-chromatic number of shadow path graph $D_2(P_n)$ for $n \geq 3$. The Shadow path graph $D_2(P_n)$ for $n \geq 3$ consists of minimal two cycles C_4. Pick the first cycle C_4, then by Theorem 1.2, we could assign 4 colors, $\{1, 2, 3, 4\}$ to the first cycle's vertices. Next, in the second C_4, we have two vertices, which intersect with two vertices in the first C_4. By Lemma 2.1, we must assign two different colors to the remaining vertices of the second C_4. Therefore, we have $\chi_L(G) \geq 6$.

Next, we determine the upper bound of the locating chromatic number of the shadow path graph for $n \geq 3$. Let c be a coloring using 6 colors as follow:

$$c(u_i) = \begin{cases} 1, & \text{for } i = 1, \\ 2, & \text{for } i = 2n\ , n \geq 1, \\ 3, & \text{for } i = 2n+1\ , n \geq 1, \end{cases}$$

$$c(v_i) = \begin{cases} 4, & \text{for } i = 1, \\ 5, & \text{for } i = 2n\ , n \geq 1, \\ 6, & \text{for } i = 2n+1\ , n \geq 1. \end{cases}$$

The color codes of $D_2(P_n)$ are :

$$c_\pi(u_i) = \begin{cases} i-1, & \text{for } 1^{st} \text{ ordinate}, i \geq 1, \\ & \text{for } 4^{th}\text{ordinate}, i \geq 2, \\ 0, & \text{for } 2^{nd} \text{ ordinate, even i}\ , 2 \leq i \leq n, \\ & \text{for } 3^{rd} \text{ ordinate, odd i}\ , 3 \leq i \leq n, \\ 2, & \text{for } 5^{th} \text{ ordinate, even i}\ , 2 \leq i \leq n, \\ & \text{for } 6^{th} \text{ ordinate, odd i}\ , 1 \leq i \leq n, \\ & \text{for } 3^{rd} \text{ ordinate}, i = 1, \\ & \text{for } 4^{th} \text{ ordinate}, i = 1, \\ 1, & \text{otherwise}, \end{cases}$$

$$c_\pi(v_i) = \begin{cases} i-1, & \text{for } 1^{st} \text{ ordinate}, i \geq 2, \\ & \text{for } 4^{th}\text{ordinate}, i \geq 1, \\ 0, & \text{for } 5^{nd} \text{ ordinate, even i}\ , 2 \leq i \leq n, \\ & \text{for } 6^{th} \text{ ordinate, odd i}\ , 3 \leq i \leq n; \\ 2, & \text{for } 2^{nd} \text{ ordinate, even i}\ , 2 \leq i \leq n, \\ & \text{for } 3^{rd} \text{ ordinate, odd i}\ , 3 \leq i \leq n, \\ & \text{for } 1^{st} \text{ ordinate}, i = 1, \\ & \text{for } 6^{th} \text{ ordinate}, i = 1, \\ 1, & \text{otherwise}. \end{cases}$$

Since all vertices in $D_2(P_n)$ for $n \geq 3$ have distinct color codes, then c is a locating coloring using 6 colors. As a result $\chi_L D_2(P_n) \leq 6$. Thus $\chi_L D_2(P_n) = 6$. □

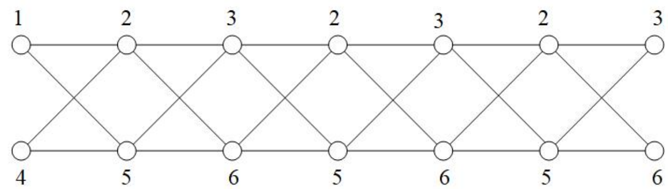

Figure 1. A minimum locating coloring of $D_2(P_7)$.

Theorem 2.2. *The locating chromatic number of a barbell graph containing shadow path for* $n \geq 3$ *is* 6.

Proof. First, we determine the lower bound of $\chi_L B_{D_2(P_n)}$ for $n \geq 3$. Since the barbell graph $B_{D_2(P_n)}$ containing $D_2(P_n)$, then by Theorem 2.3 we have $\chi_L(B_{D_2(P_n)}) \geq 6$. To prove the upper bound, consider the following three cases.

CASE 1 ($n = 3$). Let c be a locating coloring using 6 colors as follows :

$$c(u_i) = \begin{cases} 1, & \text{for } i = 2, \\ 2, & \text{for } i = 1, \\ 3, & \text{for } i = 3, \end{cases}$$

$$c(u_i') = \begin{cases} 4, & \text{for } i = 2, \\ 5, & \text{for } i = 1, \\ 6, & \text{for } i = 3, \end{cases}$$

$$c(v_i) = \begin{cases} 1, & \text{for } i = 1, \\ 5, & \text{for } i = 3, \\ 6, & \text{for } i = 2, \end{cases}$$

$$c(v_i') = \begin{cases} 2, & \text{for } i = 1, \\ 3, & \text{for } i = 3, \\ 4, & \text{for } i = 2. \end{cases}$$

The color codes of $B_{D_2(P_3)}$ are

$$c_\pi(u_i) = \begin{cases} 0, & \text{for } 1^{st} \text{ ordinate}, \ i = 2, \\ & \text{for } 2^{nd} \text{ ordinate}, \ i = 1, \\ & \text{for } 3^{rd} \text{ ordinate}, \ i = 3, \\ 2, & \text{for } 3^{th} \text{ ordinate}, \ i = 1, \\ & \text{for } 4^{th} \text{ ordinate}, \ i = 2, \\ & \text{for } 2^{nd} \text{ ordinate}, \ i = 3, \\ & \text{for } 5^{th} \ and \ 6^{th} \text{ ordinate}, \ i = 1 \text{ and } 3, \\ 1, & \text{otherwise}, \end{cases}$$

$$c_\pi(u'_i) = \begin{cases} 0, & \text{for } 5^{th} \text{ ordinate}, \ i=1, \\ & \text{for } 4^{th} \text{ ordinate}, \ i=2, \\ & \text{for } 6^{th} \text{ ordinate}, \ i=3, \\ 2, & \text{for } 6^{th} \text{ ordinate}, \ i=1, \\ & \text{for } 1^{nd} \text{ ordinate}, \ i=2, \\ & \text{for } 5^{th} \text{ ordinate}, \ i=3, \\ & \text{for } 2^{nd} \; and \; 3^{rd} \text{ ordinate}, \ i=1 \text{ and } 3, \\ 1, & \text{otherwise}, \end{cases}$$

$$c_\pi(v_i) = \begin{cases} 0, & \text{for } 1^{st} \text{ ordinate}, \ i=1, \\ & \text{for } 6^{th} \text{ ordinate}, \ i=2, \\ & \text{for } 5^{th} \text{ ordinate}, \ i=3, \\ 2, & \text{for } 5^{th} \text{ ordinate}, \ i=1, \\ & \text{for } 1^{st} \text{ ordinate}, \ i=3, \\ & \text{for } 2^{nd} \; and \; 3^{rd} \text{ ordinate}, \ i=1 \text{ and } 3, \\ 1, & \text{otherwise}, \end{cases}$$

$$c_\pi(v'_i) = \begin{cases} 0, & \text{for } 2^{nd} \text{ ordinate}, \ i=1, \\ & \text{for } 4^{th} \text{ ordinate}, \ i=2, \\ & \text{for } 3^{rd} \text{ ordinate}, \ i=3, \\ 2, & \text{for } 1^{st} \text{ ordinate}, \ i=1 \text{ and } 3, \\ & \text{for } 6^{th} \text{ ordinate}, \ i=2, \\ & \text{for } 2^{nd} \text{ ordinate}, \ i=3, \\ & \text{for } 3^{rd} \; and \; 5^{th} \text{ ordinate}, \ i=3, \\ 1, & \text{otherwise}. \end{cases}$$

Since all vertices in $B_{D_2(P_3)}$ have distinct color codes, then c is a locating coloring using 6 colors. As a result $\chi_L B_{D_2(P_3)} \leq 6$. Thus $\chi_L B_{D_2(P_3)} = 6$.

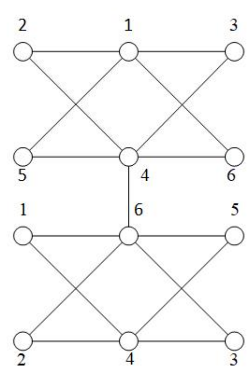

Figure 2. A minimum locating coloring of $B_{D_2(P_3)}$.

CASE 2 (n odd). Let c be a locating coloring using 6 colors as follows :

$$c(u_i)=\begin{cases}1, & \text{for } i=\dfrac{n+1}{2},\\ 2, & \text{for odd i; } i<\dfrac{n+1}{2},\ n=4j+3, j\geq 1 \text{ and } i>\dfrac{n+1}{2},\ n=4j+1, j\geq 1,\\ & \text{for even 1; } i<\dfrac{n+1}{2},\ n=4j+1, j\geq 1 \text{ and } i>\dfrac{n+1}{2},\ n=4j+3, j\geq 1,\\ 3, & \text{for odd i; } i<\dfrac{n+1}{2},\ n=4j+1, j\geq 1 \text{ and } i>\dfrac{n+1}{2},\ n=4j+3, j\geq 1,\\ & \text{for even i; } i<\dfrac{n+1}{2},\ n=4j+3, j\geq 1 \text{ and } i>\dfrac{n+1}{2},\ n=4j+1, j\geq 1,\end{cases}$$

$$c(u'_i)=\begin{cases}4, & \text{for } i=\dfrac{n+1}{2},\\ 5, & \text{for odd i; } i<\dfrac{n+1}{2},\ n=4j+3, j\geq 1 \text{ and } i>\dfrac{n+1}{2},\ n=4j+1, j\geq 1,\\ & \text{for even 1; } i<\dfrac{n+1}{2},\ n=4j+1, j\geq 1 \text{ and } i>\dfrac{n+1}{2},\ n=4j+3, j\geq 1,\\ 6, & \text{for odd i; } i<\dfrac{n+1}{2},\ n=4j+1, j\geq 1 \text{ and } i>\dfrac{n+1}{2},\ n=4j+3, j\geq 1,\\ & \text{for even i; } i<\dfrac{n+1}{2},\ n=4j+3, j\geq 1 \text{ and } i>\dfrac{n+1}{2},\ n=4j+1, j\geq 1,\end{cases}$$

$$c(v_i)=\begin{cases}1, & \text{for odd i; } i<\dfrac{n+1}{2},\ n=4j+3, j\geq 1 \text{ and } i>\dfrac{n+1}{2},\ n=4j+1, j\geq 1,\\ & \text{for even i; } i<\dfrac{n+1}{2},\ n=4j+1, j\geq 1 \text{ and } i>\dfrac{n+1}{2},\ n=4j+3, j\geq 1,\\ 5, & \text{for odd i; } i<\dfrac{n+1}{2},\ n=4j+1, j\geq 1 \text{ and } i>\dfrac{n+1}{2},\ n=4j+3, j\geq 1,\\ & \text{for even i; } i<\dfrac{n+1}{2},\ n=4j+3, j\geq 1 \text{ and } i>\dfrac{n+1}{2},\ n=4j+1, j\geq 1,\\ 6, & \text{for } i=\dfrac{n+1}{2},\end{cases}$$

$$c(v'_i)=\begin{cases}2, & \text{for odd i; } i<\dfrac{n+1}{2},\ n=4j+3, j\geq 1 \text{ and } i>\dfrac{n+1}{2},\ n=4j+1, j\geq 1,\\ & \text{for even i; } i<\dfrac{n+1}{2},\ n=4j+1, j\geq 1 \text{ and } i>\dfrac{n+1}{2},\ n=4j+3, j\geq 1,\\ 4, & \text{for } i=\dfrac{n+1}{2},\\ 3, & \text{for odd i; } i<\dfrac{n+1}{2},\ n=4j+1, j\geq 1 \text{ and } i>\dfrac{n+1}{2},\ n=4j+3, j\geq 1,\\ & \text{for even i; } i<\dfrac{n+1}{2},\ n=4j+3, j\geq 1 \text{ and } i>\dfrac{n+1}{2},\ n=4j+1, j\geq 1.\end{cases}$$

The color codes of $B_{D_2(P_n)}$ are

$$c_\pi(u_i)=\begin{cases}(\frac{n+1}{2})-i, & \text{for } 1^{st} \text{ ordinate}, i\le \frac{n+1}{2},\\ & \text{for } 4^{th} \text{ ordinate}, i< \frac{n+1}{2},\\ i-(\frac{n+1}{2}), & \text{for } 1^{st} \text{ ordinate}, i> \frac{n+1}{2},\\ & \text{for } 4^{th} \text{ ordinate}, i> \frac{n+1}{2},\\ 0, & \text{for } 2^{nd} \text{ ordinate, odd i}, i< \frac{n+1}{2},\\ & \text{for } 3^{rd} \text{ ordinate, even i}, i< \frac{n+1}{2},\\ & \text{for } 2^{th} \text{ ordinate, even i}, i> \frac{n+1}{2},\\ & \text{for } 3^{rd} \text{ ordinate, odd i}, i> \frac{n+1}{2},\\ 2, & \text{for } 5^{th} \text{ ordinate, odd i}, i< \frac{n+1}{2},\\ & \text{for } 5^{th} \text{ ordinate, even i}, i> \frac{n+1}{2},\\ & \text{for } 6^{th} \text{ ordinate, odd i}, i< \frac{n+1}{2},\\ & \text{for } 6^{th} \text{ ordinate, even i}, i> \frac{n+1}{2},\\ & \text{for } 4^{th} \text{ ordinate}, i= \frac{n+1}{2},\\ 1, & \text{otherwise},\end{cases}$$

$$c_\pi(u'_i)=\begin{cases}(\frac{n+1}{2})-i, & \text{for } 1^{st} \text{ ordinate}, i< \frac{n+1}{2},\\ & \text{for } 4^{th} \text{ ordinate}, i\le \frac{n+1}{2},\\ i-(\frac{n+1}{2}), & \text{for } 1^{st} \text{ ordinate}, i> \frac{n+1}{2},\\ & \text{for } 4^{th} \text{ ordinate}, i> \frac{n+1}{2},\\ 0, & \text{for } 5^{th} \text{ ordinate, odd i}, i< \frac{n+1}{2},\\ & \text{for } 5^{th} \text{ ordinate, even i}, i> \frac{n+1}{2},\\ & \text{for } 6^{th} \text{ ordinate, odd i}, i> \frac{n+1}{2},\\ & \text{for } 6^{th} \text{ ordinate, even i}, i< \frac{n+1}{2},\\ 2, & \text{for } 2^{nd} \text{ ordinate, odd i}, i< \frac{n+1}{2},\\ & \text{for } 2^{nd} \text{ ordinate, even i}, i> \frac{n+1}{2},\\ & \text{for } 3^{rd} \text{ ordinate, odd i}, i> \frac{n+1}{2},\\ & \text{for } 3^{rd} \text{ ordinate, even i}, i< \frac{n+1}{2},\\ & \text{for } 1^{st} \text{ ordinate}, i= \frac{n+1}{2},\\ 1, & \text{otherwise},\end{cases}$$

$$c_\pi(v_i)=\begin{cases}(\frac{n+1}{2})-i, & \text{for } 4^{th} \text{ ordinate}, i<\frac{n+1}{2},\\ & \text{for } 6^{th} \text{ ordinate}, i\le\frac{n+1}{2},\\ i-(\frac{n+1}{2}), & \text{for } 4^{th} \text{ ordinate}, i>\frac{n+1}{2},\\ & \text{for } 6^{th} \text{ ordinate}, i>\frac{n+1}{2},\\ 0, & \text{for } 1^{st} \text{ ordinate, odd i }, i<\frac{n+1}{2},\\ & \text{for } 1^{st} \text{ ordinate, even i }, i>\frac{n+1}{2},\\ & \text{for } 5^{th} \text{ ordinate, odd i }, i>\frac{n+1}{2},\\ & \text{for } 5^{th} \text{ ordinate, even i }, i<\frac{n+1}{2},\\ 2, & \text{for } 2^{nd} \text{ ordinate, odd i }, i<\frac{n+1}{2},\\ & \text{for } 2^{nd} \text{ ordinate, even i }, i>\frac{n+1}{2},\\ & \text{for } 3^{rd} \text{ ordinate, odd i }, i>\frac{n+1}{2},\\ & \text{for } 3^{rd} \text{ ordinate, even i }, i<\frac{n+1}{2},\\ & \text{for } 1^{st} \text{ ordinate}, i=\frac{n+1}{2},\\ 1, & \text{otherwise},\end{cases}$$

$$c_\pi(v_i')=\begin{cases}(\frac{n+1}{2})-i, & \text{for } 4^{th} \text{ ordinate}, i\le\frac{n+1}{2},\\ & \text{for } 6^{th} \text{ ordinate}, i>\frac{n+1}{2},\\ i-(\frac{n+1}{2}), & \text{for } 4^{th} \text{ ordinate}, i>\frac{n+1}{2},\\ & \text{for } 6^{th} \text{ ordinate}, i>\frac{n+1}{2},\\ 0, & \text{for } 2^{nd} \text{ ordinate, odd i}, i<\frac{n+1}{2},\\ & \text{for } 2^{nd} \text{ ordinate, even i}, i>\frac{n+1}{2},\\ & \text{for } 3^{rd} \text{ ordinate, odd i}, i>\frac{n+1}{2},\\ & \text{for } 3^{rd} \text{ ordinate, even i}, i<\frac{n+1}{2},\\ 2, & \text{for } 1^{st} \text{ ordinate, odd i}, i<\frac{n+1}{2},\\ & \text{for } 1^{st} \text{ ordinate, even i}, i>\frac{n+1}{2},\\ & \text{for } 5^{th} \text{ ordinate, odd i}, i>\frac{n+1}{2},\\ & \text{for } 5^{th} \text{ ordinate, even i}, i<\frac{n+1}{2},\\ & \text{for } 6^{th} \text{ ordinate}, i=\frac{n+1}{2},\\ 1, & \text{otherwise}.\end{cases}$$

Since all vertices in $B_{D_2(P_n)}, n>3$ for odd n have distinct color codes, then c is a locating coloring using 6 colors. As a result $\chi_L B_{D_2(P_n)}\le 6$. Thus $\chi_L B_{D_2(P_n)}=6$.

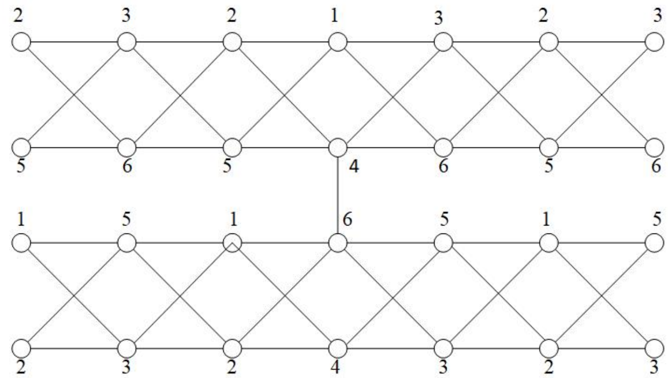

Figure 3. A minimum locating coloring of $B_{D_2(P_7)}$

CASE 3 (n even). Let c be a locating coloring using 6 colors as follows:

$$c(u_i) = \begin{cases} 1, & \text{for } i = \frac{n}{2}, \\ 2, & \text{for odd i}; i < \frac{n}{2},\ n = 4j, j \geq 1 \text{ and } i > \frac{n}{2},\ n = 4j+2, j \geq 1, \\ & \text{for even i}; i < \frac{n}{2},\ n = 4j+2, j \geq 1 \text{ and } i > \frac{n}{2},\ n = 4j, j \geq 1, \\ 3, & \text{for odd i}; i < \frac{n}{2},\ n = 4j+2, j \geq 1 \text{ and } i > \frac{n}{2},\ n = 4j, j \geq 1, \\ & \text{for even i}; i < \frac{n}{2},\ n = 4j, j \geq 1 \text{ and } i > \frac{n}{2},\ n = 4j+2, j \geq 1, \end{cases}$$

$$c(u_i') = \begin{cases} 4, & \text{for } i = \frac{n}{2}, \\ 5, & \text{for odd i}; i < \frac{n}{2},\ n = 4j, j \geq 1 \text{ and } i > \frac{n}{2},\ n = 4j+2, j \geq 1, \\ & \text{for even i}; i < \frac{n}{2},\ n = 4j+2, j \geq 1 \text{ and } i > \frac{n}{2},\ n = 4j, j \geq 1, \\ 6, & \text{for odd i}; i < \frac{n}{2},\ n = 4j+2, j \geq 1 \text{ and } i > \frac{n}{2},\ n = 4j, j \geq 1, \\ & \text{for even i}; i < \frac{n}{2},\ n = 4j, j \geq 1 \text{ and } i > \frac{n}{2},\ n = 4j+2, j \geq 1, \end{cases}$$

$$c(v_i) = \begin{cases} 1, & \text{for odd i}; i < \frac{n}{2},\ n = 4j, j \geq 1 \text{ and } i > \frac{n}{2},\ n = 4j+2, j \geq 1, \\ & \text{for even i}; i < \frac{n}{2},\ n = 4j+2, j \geq 1 \text{ and } i > \frac{n}{2},\ n = 4j, j \geq 1, \\ 5, & \text{for odd i}; i < \frac{n}{2},\ n = 4j+2, j \geq 1 \text{ and } i > \frac{n}{2},\ n = 4j, j \geq 1, \\ & \text{for even i}; i < \frac{n}{2},\ n = 4j, j \geq 1 \text{ and } i > \frac{n}{2},\ n = 4j+2, j \geq 1, \\ 6, & \text{for } i = \frac{n}{2}, \end{cases}$$

$$c(v_i') = \begin{cases} 2, & \text{for odd i}; i < \frac{n}{2},\ n = 4j, j \geq 1 \text{ and } i > \frac{n}{2},\ n = 4j+2, j \geq 1, \\ & \text{for even i}; i < \frac{n}{2},\ n = 4j+2, j \geq 1 \text{ and } i > \frac{n}{2},\ n = 4j, j \geq 1, \\ 3, & \text{for odd i}; i < \frac{n}{2},\ n = 4j+2, j \geq 1 \text{ and } i > \frac{n}{2},\ n = 4j, j \geq 1, \\ & \text{for even i}; i < \frac{n}{2},\ n = 4j, j \geq 1 \text{ and } i > \frac{n}{2},\ n = 4j+2, j \geq 1, \\ 4, & \text{for } i = \frac{n}{2}. \end{cases}$$

The color codes of $B_{D_2(P_n)}$ are

$$c_\pi(u_i) = \begin{cases} (\dfrac{n}{2}) - i, & \text{for } 1^{st} \text{ ordinate}, i \le \frac{n}{2}, \\ & \text{for } 4^{th} \text{ ordinate}, i < \dfrac{n}{2}; \\ i - (\dfrac{n}{2}), & \text{for } 1^{st} \text{ ordinate}, i > \frac{n}{2}, \\ & \text{for } 4^{th} \text{ ordinate}, i > \dfrac{n}{2}; \\ 0, & \text{for } 2^{nd} \text{ ordinate, odd i}, i < \frac{n}{2}, \\ & \text{for } 3^{rd} \text{ ordinate, even i}, i < \frac{n}{2}, \\ & \text{for } 2^{th} \text{ ordinate, even i}, i > \frac{n}{2}, \\ & \text{for } 3^{rd} \text{ ordinate, odd i}, i > \frac{n}{2}, \\ 2, & \text{for } 5^{th} \text{ ordinate, odd i}, i < \frac{n}{2}, \\ & \text{for } 5^{th} \text{ ordinate, even i}, i > \frac{n}{2}, \\ & \text{for } 6^{th} \text{ ordinate, odd i}, i < \frac{n}{2}, \\ & \text{for } 6^{th} \text{ ordinate, even i}, i > \frac{n}{2}, \\ & \text{for } 4^{th} \text{ ordinate}, i = \frac{n}{2}, \\ 1, & \text{otherwise}, \end{cases}$$

$$c_\pi(u'_i) = \begin{cases} (\dfrac{n}{2}) - i, & \text{for } 1^{st} \text{ ordinate}, i < \frac{n}{2}, \\ & \text{for } 4^{th} \text{ ordinate}, i \le \frac{n}{2}, \\ i - (\dfrac{n}{2}), & \text{for } 1^{st} \text{ ordinate}, i > \frac{n}{2}, \\ & \text{for } 4^{th} \text{ ordinate}, i > \frac{n}{2}, \\ 0, & \text{for } 5^{th} \text{ ordinate, odd i}, i < \frac{n}{2}, \\ & \text{for } 5^{th} \text{ ordinate, even i}, i > \frac{n}{2}, \\ & \text{for } 6^{th} \text{ ordinate, odd i}, i > \frac{n}{2}, \\ & \text{for } 6^{th} \text{ ordinate, even i}, i < \frac{n}{2}, \\ 2, & \text{for } 2^{nd} \text{ ordinate, odd i}, i < \frac{n}{2}, \\ & \text{for } 2^{nd} \text{ ordinate, even i}, i > \frac{n}{2}, \\ & \text{for } 3^{rd} \text{ ordinate, odd i}, i > \frac{n}{2}, \\ & \text{for } 3^{rd} \text{ ordinate, even i}, i < \frac{n}{2}, \\ & \text{for } 1^{st} \text{ ordinate}, i = \frac{n}{2}, \\ 1, & \text{otherwise}, \end{cases}$$

$$c_\pi(v_i) = \begin{cases} (\dfrac{n}{2}) - i, & \text{for } 4^{th} \text{ ordinate}, i < \frac{n}{2}, \\ & \text{for } 6^{th} \text{ ordinate}, i \leq \frac{n}{2}, \\ i - (\dfrac{n}{2}), & \text{for } 4^{th} \text{ ordinate}, i > \frac{n}{2}, \\ & \text{for } 6^{th} \text{ ordinate}, i > \frac{n}{2}, \\ 0, & \text{for } 1^{st} \text{ ordinate, odd i}, i < \frac{n}{2}, \\ & \text{for } 1^{st} \text{ ordinate, even i}, i > \frac{n}{2}, \\ & \text{for } 5^{th} \text{ ordinate, odd i}, i > \frac{n}{2}, \\ & \text{for } 5^{th} \text{ ordinate, even i}, i < \frac{n}{2}, \\ 2, & \text{for } 2^{nd} \text{ ordinate, odd i}, i < \frac{n}{2}, \\ & \text{for } 2^{nd} \text{ ordinate, even i}, i > \frac{n}{2}, \\ & \text{for } 3^{rd} \text{ ordinate, odd i}, i > \frac{n}{2}, \\ & \text{for } 3^{rd} \text{ ordinate, even i}, i < \frac{n}{2}, \\ & \text{for } 1^{st} \text{ ordinate}, i = \frac{n}{2}, \\ 1, & \text{otherwise}, \end{cases}$$

$$c_\pi(v'_i) = \begin{cases} (\dfrac{n}{2}) - i, & \text{for } 4^{th} \text{ ordinate}, i \leq \frac{n}{2}, \\ & \text{for } 6^{th} \text{ ordinate}, i > \frac{n}{2}, \\ i - (\dfrac{n}{2}), & \text{for } 4^{th} \text{ ordinate}, i > \frac{n}{2}, \\ & \text{for } 6^{th} \text{ ordinate}, i > \frac{n}{2}, \\ 0, & \text{for } 2^{nd} \text{ ordinate, odd i}, i < \frac{n}{2}, \\ & \text{for } 2^{nd} \text{ ordinate, even i}, i > \frac{n}{2}, \\ & \text{for } 3^{rd} \text{ ordinate, odd i}, i > \frac{n}{2}, \\ & \text{for } 3^{rd} \text{ ordinate, even i}, i < \frac{n}{2}, \\ 2, & \text{for } 1^{st} \text{ ordinate, odd i}, i < \frac{n}{2}, \\ & \text{for } 1^{st} \text{ ordinate, even i}, i > \frac{n}{2}, \\ & \text{for } 5^{th} \text{ ordinate, odd i}, i > \frac{n}{2}, \\ & \text{for } 5^{th} \text{ ordinate, even i}, i < \frac{n}{2}, \\ & \text{for } 6^{th} \text{ ordinate}, i = \frac{n}{2}, \\ 1, & \text{otherwise}. \end{cases}$$

Since all vertices in $B_{D_2(P_n)}$, $n \geq 3$ for even n have distinct color codes, then c is a locating coloring using 6 colors. As a result $\chi_L B_{D_2(P_n)} \leq 6$. Thus $\chi_L B_{D_2(P_n)} = 6$. □

3. Concluding Remarks

The locating chromatic number of a shadow path graphs and the barbell graph containing a shadow path graph is similar, which is 6.

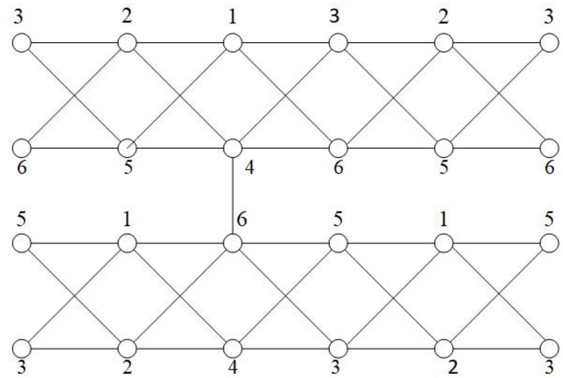

Figure 4. A minimum locating coloring of $B_{D_2(P_6)}$

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