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# Generalization of $\mathcal{U}$-Generator and $M$-Subgenerator Related to Category $\sigma[M]$

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**Abstract**

Let $\mathcal{U}$ be a non-empty set of $R$-modules. $R$-module $N$ is generated by $\mathcal{U}$ if there is an epimorphism from $\oplus_\Lambda U_\lambda$ to $N$, where $U_\lambda \in \mathcal{U}$, for every $\lambda \in \Lambda$. $R$-module $M$ is a subgenerator for $N$ if $N$ is isomorphic to a submodule of an $M$-generated module. In this paper, we introduce a $\mathcal{U}_V$-generator, where $V$ be a submodule of $\oplus_\Lambda U_\lambda$, as a generalization of $\mathcal{U}$-generator by using the concept of $V$-coexact sequence. We also provide a $\mathcal{U}_V$-subgenerator motivated by the concept of $M$-subgenerator. Furthermore, we give some properties of $\mathcal{U}_V$-generated and $\mathcal{U}_V$-subgenerated modules related to category $\sigma[M]$. We also investigate the existence of pullback and pushout of a pair of morphisms of $\mathcal{U}_V$-subgenerated modules. We prove that the collection of $\mathcal{U}_V$-subgenerated modules is closed under submodules and factor modules.

**Keywords:** $\mathcal{U}$-generator, $\mathcal{U}_V$-generator, V-coexact sequences, $M$-subgenerator, $\mathcal{U}_V$-subgenerator

## 1. Introduction

The concept of exact sequences of $R$-modules and $R$-module homomorphisms is a useful tool in the study of modules. A sequence $A \to B \to C$ is exact if $Imf = Kerg (= g^{-1}(0))$. Davvaz and Parnian-Garamaleky (1999) provide the generalization of exact sequences, i.e. quasi-exact sequences. They substitute the submodule $\{0\}$ to any submodule $U$ of $C$.

Then Anvariyeh dan Davvaz (2005) investigate further results about quasi-exact sequences. They also introduce the generalization of Schanuel's Lemma. Furthermore, Davvaz and ShabaniSolt (2002) give a generalization of some notions in homological algebra. In 2002, Anvariyeh and Davvaz provide $U$-split sequences. They also establish several connections between $U$-split sequences and projective modules.

Motivated by the definition of $U$-exact and $V$-coexact sequence, Fitriani et al. (2016) provide an $X$-sub exact sequence, which is a generalization of exact sequence. In 2017, they introduce $X$-sublinearly independent module by using the concept of $X$-sub exact sequence.

Let $\mathcal{U}$ be a non-empty set of $R$-modules. An $R$-module $N$ is generated by $\mathcal{U}$ if there is an epimorphism from $\oplus_\Lambda U_\lambda$ to $N$, where $U_\lambda \in \mathcal{U}$, for every $\lambda \in \Lambda$. The trace of $\mathcal{U}$ is defined by $Tr(\mathcal{U}, M) = \sum\{Imh | h : U \to M, \text{ for some } U \in \mathcal{U}\}$. If $\mathcal{U} = \{U\}$ is a singleton, then $Tr(U, M) = \sum\{Imh | h \in Hom_R(U, M)\}$. $Tr(\mathcal{U}, M)$ is the unique largest submodule $L$ of $M$ generated by $\mathcal{U}$ (Wisbauer, 1991). Clearly, $Tr(\mathcal{U}, M) = M$ if and only if $\mathcal{U}$ generates $M$ (Anderson & Fuller, 1992). For an indexed set $(M_\alpha)_{\alpha \in A}$ of modules and class of modules $\mathcal{U}$, the direct sum of the traces $Tr(\mathcal{U}, M)$ is contained in $\oplus_A M_\alpha$. The trace of $M$ in an $R$-module $N$ is the sum of all $M$-generated submodules of $N$ (Clark et al., 2006).

**Proposition 1** (Wisbauer, 1991) *If $(M_\alpha)_{\alpha \in A}$ is an indexed set of modules, then for each module $M$*

$$Tr(\mathcal{U}, \oplus_A M_\alpha) = \oplus_A Tr(\mathcal{U}, M_\alpha).$$

Furthermore, an $M$-subgenerated module is defined as follows.

**Definition 2** (Wisbauer, 1991) Let $M$ be an $R$-module. We say that an $R$-module $N$ is subgenerated by $M$, or that $M$ is a subgenerator for $N$, if $N$ is isomorphic to a submodule of an $M$-generated module.

A subcategory $C$ of $R$-$MOD$ is said to be subgenerated by $M$, or $M$ is a subgenerator for $C$, if every object in $C$ is subgenerated by $M$. Category $\sigma[M]$ is the full subcategory of $R - MOD$ whose objects are all $R$-modules subgenerated by $M$. This category is a category closely connected to $M$ and hence reflecting properties of $M$.

The properties of $\sigma[M]$ given by the following proposition:

**Proposition 3** (Wisbauer, 1991) *For an R-module M we have:*

1. *For N in $\sigma[M]$, all factor modules and submodules of N belong to $\sigma[M]$, i.e. $\sigma[M]$ has kernels and cokernels.*
2. *The direct sum of a family of modules in $\sigma[M]$ belong to $\sigma[M]$ and is equal to the coproduct of these modules in $\sigma[M]$.*
3. *Pullback and pushout of morphisms in $\sigma[M]$ belong to $\sigma[M]$.*

As a generalization of exact sequence of $R$-modules, Anvanriyeh and Davvaz (1999) defined $U$-exact sequences as follows: A sequence of $R$-modules $A \xrightarrow{f} B \xrightarrow{g} C$ if there exists a submodule $U$ of $C$ such that $Im\ f = g^{-1}(U)$. In this case, the sequence is said to be $U$-exact (at $B$). If $f(V) = Ker\ g$, where $V$ is a submodule of $A$, then the sequence is said to be $V$-coexact.

Let $\mathcal{U}$ be a family of $R$-modules and $V$ be a submodule of $\oplus_\Lambda U_\lambda$, where $U_\lambda \in \mathcal{U}$, for every $\lambda \in \Lambda$. The aim of this paper is to generalize the concept of $\mathcal{U}$-generator to a $\mathcal{U}_V$-generator, where $V$ is a submodule of $\oplus_\Lambda U_\lambda$. Furthermore, we provide a $\mathcal{U}_V$-subgenerator as a generalization of $M$-subgenerator. We also investigate the properties of $\mathcal{U}_V$-generated modules and $\mathcal{U}_V$-subgenerated modules related to the properties of the category $\sigma[M]$.

## 2. Results

### *2.1 $\mathcal{U}_V$-Generated Modules*

Let $\mathcal{U}$ be a family of $R$-modules. It is possible that an $R$-module $M$ is not a $\mathcal{U}$-generated module, i.e. there no epimorphism from $\oplus_\Lambda U_\lambda$ to $M$, but we can define an epimorphism from a submodule $V$ $\oplus_\Lambda U_\lambda$ to $M$. Therefore we can generalize the concept of a $\mathcal{U}$-generated module to a $\mathcal{U}_V$-generated module by using the definition of $V$-coexact sequence.

**Definition 4** Let $\mathcal{U}$ be a non-empty set of $R$-modules, $V$ be a submodule of $\oplus_\Lambda U_\lambda$, where $U_\lambda \in \mathcal{U}$, for every $\lambda \in \Lambda$. We say that an $R$-module $N$ is generated by $\mathcal{U}_V$ if there exists an epimorphism $V \to N \to 0$.

A set $\{U_\lambda\}_\Lambda$ is called $\mathcal{U}_V$-generator for $N$. Furthermore, the set $\{U_\lambda\}_\Lambda$ is called minimal $\mathcal{U}_V$-generator for $N$ if

$$\Lambda = min\{\Lambda_V | N \text{ is } \mathcal{U}_V - \text{generated}, V \subseteq \oplus_{\Lambda_V} U_\lambda\}.$$

If we take $V = \oplus_\Lambda U_\lambda$, then a $\mathcal{U}_V$-generated module is a $\mathcal{U}$-generated module. Clearly, every $\mathcal{U}$-generated module is $\mathcal{U}_V$-generated. But, a $\mathcal{U}_V$-generated module need not be a $\mathcal{U}$-generated. For example, if we take $\mathcal{U} = \{\mathbb{Q}\}$, then $\mathbb{Z}$-module $\mathbb{Z}$ is a $\mathcal{U}_\mathbb{Z}$-generated module. But, we can not define an epimorphism from $\mathbb{Q}$ to $\mathbb{Z}$ and hence $\mathbb{Z}$-module $\mathbb{Z}$ is not a $\mathcal{U}$-generated module.

Now, we give some examples of $\mathcal{U}_V$-generated modules. *Example 1*

1. Let $\mathcal{U}$ be the set of all free $R$-modules and $P$ be projective $R$-module. Since $P$ is projective, $P$ is a direct summand of a free module $F$. Hence $P$ is $\mathcal{U}_F$-generated module.
2. Let $\mathcal{U} = \{\mathbb{Z}_p | p\text{prime}\}$, a family of $\mathbb{Z}$-modules. $\mathbb{Z}$-module $\mathbb{Z}_6$ is a $\mathcal{U}_V$-generated, where $V = \mathbb{Z}_2 \oplus \mathbb{Z}_3$. In general, $\mathbb{Z}$-module $\mathbb{Z}_{pq}$ is a $\mathcal{U}_V$-generated, where $V = \mathbb{Z}_p \oplus \mathbb{Z}_q$, $p$ and $q$ are relative prime.
3. Let $\mathcal{U} = \{\mathbb{Q}\}$. $\mathbb{Z}$-module $\mathbb{Z}_n$, $n \geq 2$, is $\mathcal{U}_V$-generated, where $V = \mathbb{Z}$.
4. Let $R$ be a commutative ring with unit and $\mathcal{U} = \{U_\lambda\}_\Lambda$ be a family of $R$-modules, where $U_\lambda = Hom_R(R, M_\lambda)$, for every $\lambda \in \Lambda$.
   Based on Adkins & Weintraub (1992), we can define
   $$\phi : Hom_R(R, M) \to M,$$
   where $\phi(f) := f(1)$. Then $M_\lambda$ is $\mathcal{U}_{U_\lambda}$-generated.
5. Let $\mathcal{U} = \{\mathbb{Z}_n | n \in \mathbb{Z}\}$ be a family of $\mathbb{Z}$-modules. Let $M = \mathbb{Z}_4^{(\mathbb{N})}$ and $N = \mathbb{Z}_2 \oplus M$ be $\mathbb{Z}$-modules. Then $M$ is $\mathcal{U}_N$-generated and $N$ is $\mathcal{U}_M$-generated.

If there exists a finite index set $E \subseteq \Lambda$ such that $M$ is $\mathcal{U}_V$-generated and $V$ is a submodule of $\oplus_E U_e$, then we define a finitely $\mathcal{U}_V$-generated module as follows:

**Definition 5** Let $\mathcal{U}$ be a non-empty set of $R$-modules and $N$ be an $R$-module. If there exists a finite index set $E \subseteq \Lambda$ such that $V \subseteq \oplus_E U_e$ and $M$ is $\mathcal{U}_V$-generated, then $R$-module $N$ is said to be finitely $\mathcal{U}_V$-generated.

*Example 2* Let $\mathcal{U} = \{\mathbb{Z}_p | p \text{ prime}\}$ be a family of $\mathbb{Z}$-modules. $\mathbb{Z}$-module $\mathbb{Z}_{pq}$ is a finitely $\mathcal{U}_V$-generated, where $V = \mathbb{Z}_p \oplus \mathbb{Z}_q$, $p$ and $q$ are relative prime.

Then, we will give some basic properties of $\mathcal{U}_V$-generated modules. Let $\mathcal{U}$ be a non-empty set of $R$-modules and $N$ be an $R$-module. We define:

$$\mathcal{U}(N) = \{V \subseteq \oplus_\Lambda U_\lambda, U_\lambda \in \mathcal{U} | N \text{ is } \mathcal{U}_V\text{-generated}\}.$$

In this set, we collect all submodules $V$ of $\oplus_\Lambda U_\lambda$ such that $N$ is a $\mathcal{U}_V$-generated module. In the following proposition, we prove that if $V_\lambda \in \mathcal{U}(N_\lambda)$ for every $\lambda \in \Lambda$, then $\oplus_\Lambda V_\lambda \in \mathcal{U}(\oplus_\Lambda N_\lambda)$.

**Proposition 6** *Let $\mathcal{U}$ be a non-empty set of $R$-modules, $V_\lambda$ be a submodule of $\oplus_\Lambda U_\lambda$, where $U_\lambda \in \Lambda$ for every $\lambda \in \Lambda$. If $N_\lambda$ is $\mathcal{U}_{V_\lambda}$-generated, for every $\lambda \in \Lambda$, then $\oplus_\Lambda N_\lambda$ is $\mathcal{U}_{\oplus_\Lambda V_\lambda}$-generated.*

*Proof.* Since $N_\lambda$ is $\mathcal{U}_{V_\lambda}$-generated, for every $\lambda \in \Lambda$, the sequences $V_\lambda \to N_\lambda \to 0$ is exact for every $\lambda \in \Lambda$. Therefore, the sequence

$$\oplus_\Lambda V_\lambda \to \oplus_\Lambda N_\lambda \to 0$$

is exact. Hence, $\oplus_\Lambda N_\lambda$ is $\mathcal{U}_{\oplus_\Lambda V_\lambda}$-generated. So, we can say that if $V_\lambda \in \mathcal{U}(N_\lambda)$ for every $\lambda \in \Lambda$, then $\oplus_\Lambda V_\lambda \in \mathcal{U}(\oplus_\Lambda N_\lambda)$.

As a corollary of Proposition 6, we obtain:

**Corollary 7** *Let $\mathcal{U}$ be a non-empty set of $R$-modules. If $R$-module $N_i$ is $\mathcal{U}_{V_i}$-generated for every $i = 1, 2, ..., n$, then $\oplus_{i=1}^n X_i$ is $\mathcal{U}_{\oplus_{i=1}^n V_i}$-generated, where $V_i$ be submodule of $\oplus_\Lambda U_\lambda$, $U_\lambda \in \Lambda$, for every $i = 1, 2, ..., n$ and $\lambda \in \Lambda$.*

In the following proposition, we will show that if $V \in \mathcal{U}(N)$, for an $R$-module $N$, then $V$ is in $\mathcal{U}(N')$, for every homomorphic image $N'$ of $N$.

**Proposition 8** *Let $\mathcal{U}$ be a non-empty set of $R$-modules. If $R$-module $N$ is $\mathcal{U}_V$-generated, then $N'$ is $\mathcal{U}_V$-generated, for every homomorphic image $N'$ of $N$.*

*Proof.* If $R$-module $N$ is $\mathcal{U}_V$-generated, then the sequence

$$\oplus_\Lambda U_\lambda \xrightarrow{f} N \to 0$$

is $V$-coexact. Let $N'$ be homomorphic image of $N$, then there is an epimorphism $p : N \to N'$. Hence, $g = p \circ f$ is a homomorphism from $V$ to $N'$. Since $f$ and $p$ are epimorphisms, then $g$ is an epimorphism. So, $N'$ is $\mathcal{U}_V$-generated.

In the next proposition, we will prove that $\mathcal{U}_V(N)$ is closed under direct sum, i.e. if $V_\lambda$ is in $\mathcal{U}(N)$ for every $\lambda \in \Lambda$, then $\oplus_{\lambda \in \Lambda} V_\lambda$ is in $\mathcal{U}(N)$.

**Proposition 9** *Let $\mathcal{U}$ be a non-empty set of $R$-modules and $V_\alpha$ be submodules of $\oplus_\Lambda U_\lambda$, $U_\lambda \in \mathcal{U}$ for every $\lambda \in \Lambda$. If $R$-module $M$ is $\mathcal{U}_{V_\alpha}$-generated, for every $\alpha \in A$, then $M$ is $\mathcal{U}_{\oplus_{\alpha \in A} V_\alpha}$-generated.*

*Proof.* Since $R$-module $M$ is $\mathcal{U}_{V_\alpha}$-generated for every $\alpha \in A$, there is an epimorphism $f_\alpha$ such that the sequence: $V_\alpha \xrightarrow{f_\alpha} M \to 0$ is exact for every $\alpha \in A$. We can define $f : \oplus_{\alpha \in A} V_\alpha \to M$, where $f((v_\alpha)_A) = f_{\alpha_i}(v_{\alpha_i})$, $\alpha_i \in A$. From this, we have $f$ is an epimorphism from $\oplus_{\alpha \in A}$ to $M$. Hence, $M$ is $\mathcal{U}_{\oplus_{\alpha \in A} V_\alpha}$-generated.

As a corollary of Proposition 9, we obtain:

**Proposition 10** *Let $\mathcal{U}$ be a non-empty set of $R$-modules. If $R$-module $M$ is $\mathcal{U}_{V_i}$-generated for every $i = 1, 2, ..., n$, then $M$ is $\mathcal{U}_{\oplus_{i=1}^n V_i}$-generated, where $V_i$ be submodule of $\oplus_\Lambda U_\lambda$ for every $i = 1, 2, ..., n$.*

If $V_2 \in \mathcal{U}(N)$ and $V_1 \in \mathcal{U}(V_2)$ i.e. $N$ is $\mathcal{U}_{V_1}$-generated and $V_2$ is $\mathcal{U}_{V_1}$-generated, with modules $V_1$ and $V_2$ are submodules of $\oplus_\Lambda U_\lambda$, $U_\lambda \in \mathcal{U}$, then we will show that $V_1 \in \mathcal{U}(N)$, i.e. $N$ is $\mathcal{U}_{V_1}$-generated module.

**Proposition 11** *Let $\mathcal{U}$ be a non-empty set of $R$-modules. If $R$-module $N$ is $\mathcal{U}_{V_2}$-generated and $V_2$ is $\mathcal{U}_{V_1}$-generated, then $N$ is $\mathcal{U}_{V_1}$-generated, where $V_1, V_2$ be submodules of $\oplus_\Lambda U_\lambda$, $U_\lambda \in \Lambda$, for every $\lambda \in \Lambda$.*

*Proof.* Since $N$ is $\mathcal{U}_{V_2}$-generated and $V_2$ is $\mathcal{U}_{V_1}$-generated, there exists epimorphisms $\alpha : V_2 \to N$ and $\beta : V_1 \to V_2$. So, we can define $g = \alpha \circ \beta : V_1 \to N$. Since $\alpha$ and $\beta$ are epimorphisms, $g$ is an epimorphism. Finally, $N$ is $\mathcal{U}_{V_1}$-generated.

As a corollary we obtain:

**Corollary 12** *Let $\mathcal{U}$ be a non-empty set of $R$-modules. If $R$-module $N$ is $\mathcal{U}_V$-generated and $V$ is $\mathcal{U}$-generated, then $N$ is $\mathcal{U}$-generated, where $V$ be submodule of $\oplus_\Lambda U_\lambda$, $U_\lambda \in \Lambda$, for every $\lambda \in \Lambda$.*

*Proof.* Since $R$-module $N$ is $\mathcal{U}_V$-generated and $V$ is $\mathcal{U}$-generated, by Proposition 11, we have $N$ is $\mathcal{U}_{\oplus_\Lambda U_\lambda}$-generated. In other words, $N$ is $\mathcal{U}$-generated.

**Corollary 12** *Let $\mathcal{U}$ be a non-empty set of R-modules and $V \subset \oplus_\Lambda U_\lambda$, with modules $U_\lambda \in \mathcal{U}$. If R-module $M$ is $\mathcal{U}_V$-subgenerated and V is a $\mathcal{U}$-generated module, then the sequence*

$$\oplus_\Lambda U_\lambda \to M \to 0$$

*is $V$-coexact.*

*Proof.* Since $R$-module $M$ is $\mathcal{U}_V$-subgenerated, there is an epimorphism $\alpha : V \to M$. By asumption, $V$ is a $\mathcal{U}$-generated module. So, there is an epimorphism $\pi : \oplus_\Lambda U_\lambda \to V$. Hence, $g = \alpha \circ \pi$ is an epimorphism from $\oplus_\Lambda U_\lambda$ to $M$ such that $g|_V = \alpha$. We have the sequence

$$\oplus_\Lambda U_\lambda \xrightarrow{g} M \to 0$$

is $V$-coexact.

**Corollary 13** *Let $\mathcal{U}$ be a non-empty set of semisimple R-modules. If R-module $M$ is $\mathcal{U}_V$-generated, then $M$ is $\mathcal{U}$-generated, where V is a submodule of $\oplus_\Lambda U_\lambda$.*

*Proof.* We assume that $R$-module $M$ is a $\mathcal{U}_V$-generated. Since every submodule of semisimple module $\oplus_\Lambda U_\lambda$ is a direct summand, $M$ is $\mathcal{U}$-generated by using Proposition 11.

*2.2 $\mathcal{U}_V$-Subgenerated Modules*

We already know that an $M$-subgenerated module is a generalization of a $\mathcal{U}$-generated module. In the similar way, we can obtain a $\mathcal{U}_V$-subgenerated module as a generalization of $\mathcal{U}_V$-generated module.

**Definition 14** *Let $\mathcal{U}$ be a non-empty set of R-modules, V be a submodule of $\oplus_\Lambda U_\lambda$. We say that an R-module N is subgenerated by $\mathcal{U}_V$ if N isomorphic to a submodule of a $\mathcal{U}_V$-generated module.*

$M$-subgenerated module is a special case of $\mathcal{U}_V$-subgenerated modules by taking $\mathcal{U} = \{M\}$ and $V = M^{(\Lambda)}$. By Definition 14, every $\mathcal{U}_V$-generated module is a $\mathcal{U}_V$-subgenerated module. But the converse need not be true. For example, let $\mathcal{U}$ the set of all $\mathbb{Z}$-modules. $\mathbb{Z}$-module $\mathbb{Z}$ is $\mathcal{U}_{\mathbb{Q}}$-subgenerated. But, $\mathbb{Z}$-module $\mathbb{Z}$ is not $\mathcal{U}_{\mathbb{Q}}$-generated.

**Proposition 15** *Let $\mathcal{U}$ be a non-empty set of R-modules and V be a submodule of $\oplus_\Lambda U_\lambda$. If R-module N is $\mathcal{U}_V$-subgenerated and N is a direct summand of a $\mathcal{U}_V$-generated module, then N is $\mathcal{U}_V$-generated module.*

Let $\mathcal{U}$ be a non-empty set of $R$-modules and $N$ be an $R$-module. In $\sigma[M]$, Wisbauer (1991) collect all $R$-modules subgenerated by $M$. In the similar way, we will collect all $R$-modules subgenerated by $\mathcal{U}_V$, we denote it by $\sigma_V(\mathcal{U})$:

$$\sigma_V(\mathcal{U}) = \{N | N \text{ is } \mathcal{U}_V\text{-subgenerated}\}.$$

The full subcategory $\sigma[M]$ of $R-MOD$ is a special case of $\sigma_V(\mathcal{U})$ by taking $\mathcal{U} = \{M\}$ and $V = M^{(\Lambda)}$. Next, we will show that $\sigma_V(\mathcal{U})$ is closed under submodules and factor modules.

**Proposition 16** *Let $\mathcal{U}$ be a non-empty set of R-modules and V be a submodule of $\oplus_\Lambda U_\lambda$. If R-module N is $\mathcal{U}_V$-subgenerated, then $N'$ is a $\mathcal{U}_V$-subgenerated module, for every submodule $N'$ of N.*

*Proof.* Since $N$ is a $\mathcal{U}_V$-subgenerated, then $N$ isomorphic to a submodule of a $\mathcal{U}_V$-generated module. So, there is an epimorphism:

$$V \xrightarrow{f} K \to 0$$

and $N$ is isomorphic to a submodule of $K$. Let $N'$ be a submodule of $N$. We have $N'$ is somorphic to a submodule of $K$ and $N'$ is a $\mathcal{U}_V$-subgenerated module.

**Proposition 17** *Let $\mathcal{U}$ be a non-empty set of R-modules and V be a submodule of $\oplus_\Lambda U_\lambda$. If R-module N is $\mathcal{U}_V$-subgenerated, then $N/L$ is $\mathcal{U}_V$-subgenerated module, for every factor module $N/L$ of N.*

*Proof.* Since $N$ is a $\mathcal{U}_V$-subgenerated, there is a $\mathcal{U}_V$-generated module $K$ and an epimorphism:

$$V \xrightarrow{f} K \to 0$$

and $N$ is isomorphic to a submodule of $K$. Let $L$ be a submodule of $N$. We have $L$ is isomorphic to a submodule of $K$ and hence $N/L$ is is isomorphic to a submodule of $K/L'$, where $L \cong L'$. Since $K/L'$ is a $\mathcal{U}_V$-generated module, we get $N/L$ is a $\mathcal{U}_V$-subgenerated module.

As a corolarry of Proposition 16 and 17, we obtain:

**Corollary 18** *Let $\mathcal{U}$ be a non-empty set of R-modules, V be a submodule of $\oplus_\Lambda U_\lambda$ and*

$$0 \to K \to L \to M \to 0$$

*be an exact sequence of R-modules. If L is a $\mathcal{U}_V$-subgenerated module, then K and M are $\mathcal{U}_V$-subgenerated modules.*

If $R$-module $N_1$ and $N_2$ are $\mathcal{U}_V$-subgenerated, then we have two exact sequences: $V \to M_1 \to 0$ and $V \to M_1 \to 0$. Furthermore, $N_1$ and $N_2$ are isomorphic to submodules of $M_1$ and $M_2$, respectively. Hence $Tr(V, M_1) = M_1$ and $Tr(V, M_2) = M_2$. By Proposition 1, we have $Tr(V, M_1 \oplus M_2) = Tr(V, M_1) \oplus Tr(V, M_2) = M_1 \oplus M_2$. But, $N_1 \oplus N_2$ need not be a $\mathcal{U}_V$-subgenerated module. By Proposition 6, we have $N_1 \oplus N_2$ is a $\mathcal{U}_{V_1 \oplus V_2}$-subgenerated module.

In the following proposition, we will show the existence of pullback and pushout of a pair of morphisms of $\mathcal{U}_V$-subgenerated modules.

**Proposition 19** *Let $\mathcal{U}$ be a non-empty set of R-modules. If $N_1$ is $\mathcal{U}_{V_1}$-subgenerated and $N_2$ is $\mathcal{U}_{V_2}$-subgenerated, then pullback of $f_1 : N_1 \to N$ and $f_2 : N_2 \to N$ is $\mathcal{U}_{V_1 \oplus V_2}$-subgenerated module, where $V_1, V_2$ are submodules of $\oplus_\Lambda U_\lambda$.*

*Proof.* Since $N_1$ is $\mathcal{U}_{V_1}$-subgenerated and $N_2$ is $\mathcal{U}_{V_2}$-subgenerated, $N_1$ and $N_2$ are $\mathcal{U}_{V_1 \oplus V_2}$-subgenerated. Let $f_1 : N_1 \to M$, $f_2 : N_2 \to M$ be a pair of morphisms of $\mathcal{U}_{V_1 \oplus V_2}$-subgenerated modules. We have $N_1 \oplus N_2$ is $\mathcal{U}_{V_1 \oplus V_2}$-subgenerated module. Based on Wisbauer (1991), pullback of $(f_1, f_2)$ is a submodule of $N_1 \oplus N_2$. Since every submodule of $\mathcal{U}_{V_1 \oplus V_2}$-subgenerated module is a $\mathcal{U}_{V_1 \oplus V_2}$-subgenerated, the pullback of $(f_1, f_2)$ is a $\mathcal{U}_{V_1 \oplus V_2}$-subgenerated module.

**Proposition 20** *Let $\mathcal{U}$ be a non-empty set of R-modules. If $N_1$ is $\mathcal{U}_{V_1}$-subgenerated and $N_2$ is $\mathcal{U}_{V_2}$-subgenerated, then pushout of $g_1 : X \to N_1$ and $g_2 : X \to N_2$ is $\mathcal{U}_{V_1 \oplus V_2}$-subgenerated module, where $V_1, V_2$ are submodules of $\oplus_\Lambda U_\lambda$.*

*Proof.* Since $N_1$ is $\mathcal{U}_{V_1}$-subgenerated and $N_2$ is $\mathcal{U}_{V_2}$-subgenerated, $N_1$ and $N_2$ are $\mathcal{U}_{V_1 \oplus V_2}$-subgenerated. Let $g_1 : X \to N_1$, $g_2 : X \to N_2$ be a pair of morphisms of $\mathcal{U}_{V_1 \oplus V_2}$-subgenerated module. We have $N_1 \oplus N_2$ is $\mathcal{U}_{V_1 \oplus V_2}$-subgenerated modules. Based on Wisbauer (1991), pushout of $(g_1, g_2)$ is a factor module of $N_1 \oplus N_2$. Since every factor module of $\mathcal{U}_{V_1 \oplus V_2}$-subgenerated module is a $\mathcal{U}_{V_1 \oplus V_2}$-subgenerated, the pushout of $(g_1, g_2)$ is a $\mathcal{U}_{V_1 \oplus V_2}$-subgenerated module.

A submodule $N$ of $R$-module $M$ is called fully invariant if $f(N)$ is contained in $N$ for every $R$-endomorphism $f$ of $M$. $M$ is called a duo module provided every submodule of $M$ is fully invariant (Özcan et al., 2006).

The following theorem shows that the properties of $R$-modules in $\sigma_V\mathcal{U}$ are reflecting the properties of $V$.

**Theorem 21** *Let $\mathcal{U}$ be a non-empty set of R-modules and V be a submodule of $\oplus_\Lambda U_\lambda$, $U_\lambda \in \mathcal{U}$, for every $\lambda \in \Lambda$.*

1. *If R-module U is V-injective (V-projective), then U is N-injective (N-projective), for every $N \in \sigma_V(\mathcal{U})$.*
2. *If V is semisimple, then every module in $\sigma_V(\mathcal{U})$ is semisimple.*
3. *If V is Noetherian (Artinian), then N is Noetherian (Artinian), for every $N \in \sigma_V(\mathcal{U})$.*
4. *If V is a duo module, quasi-injective and quasi-projective, then N is a duo module, V-projective and V-injective, for every $N \in \sigma_V(\mathcal{U})$.*

*Proof.*

1. Let $N \in \sigma_V\mathcal{U}$. Then $N$ is isomorphic to a submodule of $\mathcal{U}_V$-generated module, say $M$. We have the following exact sequence:

$$0 \to Ker\, f \to V \xrightarrow{f} M \to 0.$$

Based on Wisbauer (1991), if $U$ is $V$-injective, then $U$ is $M$-injective. Therefore by Wisbauer (1991) 16.3, $U$ is $N$-injective.
2 and 3 can be shown in a similar way to 1.

4 Based on Özcan et. al. (2006), if $V$ is a duo module and quasi-injective, then every submodule of $V$ is a duo module. Futhermore, if $V$ is a duo module and quasi-projective, then every homomorphic image of $V$ is a duo module. From 1, we have $N$ is $V$-projective and $V$-injective, for every $N$ in $\sigma_V(\mathcal{U})$.

### 3. Conclusions

A $\mathcal{U}_V$-generator is a generalization of $\mathcal{U}$-generator. If an $R$-module $N$ is $\mathcal{U}_V$-generated, then every homomorphic image of $N$ is also $\mathcal{U}_V$-generated. Furthermore, direct sums of $\mathcal{U}_V$-generated $R$-modules are $\mathcal{U}_{V'}$-generated, for some submodules $V'$ of $\oplus_\Lambda U_\lambda$. In the set $\mathcal{U}(N)$, we collect all submodules $V$ of $\oplus_\Lambda U_\lambda$ such that $N$ is a $\mathcal{U}_V$-generated module and we have $\mathcal{U}(N)$ is closed under direct sums.

In the set $\sigma_V(\mathcal{U})$, we collect all $R$-modules subgenerated by $\mathcal{U}_V$. The full subcategory $\sigma[M]$ of $R-MOD$ is a special case of $\sigma_V(\mathcal{U})$ by taking $\mathcal{U} = \{M\}$ and $V = M^{(\Lambda)}$. The set $\sigma_V(\mathcal{U})$ is closed under submodules and factor modules. Furthermore, the properties of $R$-modules in $\sigma_V(\mathcal{U})$ are reflecting the properties of $V$.

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