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# A Generalization of Basis and Free Modules Relatives to a Family $\mathcal{F}$ of *R*-Modules

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# A Generalization of Basis and Free Modules Relatives to a Family $\mathcal{U}$ of $R$-Modules

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**Abstract**. Let $\mathcal{U}$ be a family of $R$-modules and $V$ be a submodule of a direct sum of some elements in $\mathcal{U}$. The aim of this paper is to generalize basis and free modules. We use the concept of $\mathcal{U}_V$-generated module and $X$-sublinearly independent to provide the definition of $\mathcal{U}$-basis and $\mathcal{U}$-free module. We construct a $\underline{\mathcal{U}}$-basis of an $R$-module $M$ as a pair $(X, V)$, which a family $\mathcal{U}$ is $X$-sub-linearly independent to $M$ and $M$ is a $\mathcal{U}_V$-generated module. Furthermore, we define $\mathcal{U}$-basis of $M$ as a $\underline{\mathcal{U}}$-basis which has the maximal element on the first component and the minimal element on the second component of a pair $(X, V)$. The results show that the first component of $(X, V)$ in $\mathcal{U}$-basis is closed under submodules and intersections. Moreover, we prove that the second component of $(X, V)$ in $\mathcal{U}$-basis is closed under direct sums. We also determine some $\mathcal{U}$-free modules related to a family $\mathcal{U}$ which contains all $\mathbb{Z}$-module $\mathbb{Z}$ modulo $p$ power of $n$, where $p$ prime and $n \geq 2$.

## 1. Introduction

Let $R$ be a ring, $A$, $B$ and $C$ be $R$-modules and let $A \xrightarrow{f} B \xrightarrow{g} C$ be an exact sequence, i.e. image of $f$ is equal to the kernel of $g$ ($g^{-1}(0)$) [1,2]. Davvaz and Parnian-Garamaleky establish a quasi-exact sequence as a generalization of exact sequence. Let $U$ be a submodule of $C$. A sequence $A \xrightarrow{f} B \xrightarrow{g} C$ is $U$-exact in $B$ if $\mathrm{Im}\, f = g^{-1}(U)$ [3]. For a submodule $V$ of $A$, they also define a $V$-coexact sequence as a dual of a $U$-exact sequence.

Then, Anvariyeh dan Davvaz [4] generalize the Schanuel Lemma by using the quasi-exact sequences. Furthermore, Davvaz and Shabani-Solt [5] give a generalization of homological algebra. In [6], Anvariyeh and Davvaz investigate the connections between projective modules and $U$-split sequences. Then, Madanshekaf [7] gives some results about quasi-exact sequences. In [8], Amizadeh et al. provide a quasi-exact sequence of S-acts.

Motivated by definition of $U$-exact sequence, Fitriani et al. [9] introduce a sub exact sequence as a generalization of an exact sequence of modules. As an application of a sub-exact sequence, Fitriani et al. also establish the notion of an $X$-sub-linearly independent module as a generalization of the linearly independent set in $R$-modules [10]. Furthermore, Fitriani et al. [11] introduce a $\mathcal{U}_V$-generated module by using coexact sequence. We can say that this notion is a dual of $X$-sub-linearly independent module. This concept is motivated by the definition of $\mathcal{U}$-generated module from [12-14].

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In this paper, we use the concept of $\mathcal{U}_V$-generated module and $X$-sub-linearly independent module to construct a $\mathcal{U}$-basis and a $\mathcal{U}$-free module which are a basis and a free module relative to a family $\mathcal{U}$ of $R$-modules. Moreover, we determine some $\mathcal{U}$-free modules, where $\mathcal{U}$ is a family of all $\mathbb{Z}$-modules $\mathbb{Z}$ modulo $p^n$, $p$ prime and $n$ is an integer greater than 2.

## 2. Methods

The aim of this paper is to generalize basis and free modules to basis and free modules relative to a family $\mathcal{U}$ of $R$-modules. If a free module $F$ has a basis $X$, then $F \cong \oplus_{x\in X} R_x$ with each $R_x \cong R$. We can choose $\mathcal{U} = \{R\}$ so that $F$ is a free module relative to $\mathcal{U}$. In this case, a family $\mathcal{U}$ only contain $R$ as an $R$-module.

We construct a basis and a free module relative to a family $\mathcal{U} = \{U_\lambda\}_\Lambda$, where $U_\lambda$ is an $R$-module, for every $\lambda \in \Lambda$. We use the concept of $\mathcal{U}_V$-generated module and $X$-sublinearly independent module to provide this concept. We construct a $\underline{\mathcal{U}}$-basis of an $R$-module $M$ as a pair $(X, V)$, which a family $\mathcal{U}$ is $X$-sub-linearly independent to $M$ and $M$ is a $\mathcal{U}_V$-generated module. Next, we define $\mathcal{U}$-basis of $M$ as a $\underline{\mathcal{U}}$-basis which has the maximal element on the first component and the minimal element on the second component of a pair $(X, V)$. Furthermore, we determine some $\mathcal{U}$-free module, where $\mathcal{U}$ is a family of all $\mathbb{Z}$ modulo $p^n$, $p$ prime, $n \in \mathbb{N}$, $n \geq 2$ as a $\mathbb{Z}$-module by using the properties of $\mathbb{Z}_n$ as an Abelian group.

## 3. Results and Discussions

We recall the definition of a $\mathcal{U}_V$-generated module as follows: Given a family $\mathcal{U} = \{U_\lambda\}_\Lambda$ of $R$-modules, $V$ be a submodule of $\oplus_\Lambda U_\lambda$. An $R$-module $N$ is $\mathcal{U}_V$-generated if there exists a surjective homomorphism from $V$ to $N$ [11]. If we take $V = \oplus_\Lambda U_\lambda$, then a $\mathcal{U}_V$-generated module is a $\mathcal{U}$-generated module. From this fact, we can say that every $\mathcal{U}$-generated module is a $\mathcal{U}_V$-generated module. But, the converse need not be true.

Now, we define the following sets:

$$\sigma(0, \oplus_\Lambda U_\lambda, M) = \{X \subseteq \oplus_\Lambda U_\lambda \mid \mathcal{U} \text{ is } X\text{-sub-linearly independent to } M \quad (1)$$

and

$$\mathcal{U}(M) = \{ V \subseteq \oplus_\Lambda U_\lambda \mid M \text{ is } U_V\text{-generated}\} \quad (2)$$

The set $\sigma(0, \oplus_\Lambda U_\lambda, M)$ contains all submodules of $\oplus_\Lambda U_\lambda$ which is $X$-sub-linearly independent to $M$. Hence, if there is an injective homomorphism from $Y$ to $M$, where $Y$ is a submodule of $\oplus_\Lambda U_\lambda$, then $Y$ is in the set $\sigma(0, \oplus_\Lambda U_\lambda, M)$.

Suppose that $X$ is in $\sigma(0, \oplus_\Lambda U_\lambda, M)$. Consequently, a family $\mathcal{U}$ is $X$-sub-linearly independent to $M$. Therefore, there exists a monomorphism $f$ from $X$ to $M$. We already know that for every submodule $X'$ of $X$, we always have a monomorphism $i$ from $X'$ to $X$. Then $\mathcal{U}$ is also $X'$-sub-linearly independent to $M$, for every submodule $X'$ of $X$ [10]. Similarly, if a family $\mathcal{U}$ of $R$-modules is $X_i$–sub-linearly independent to an R-module $M$ for every $i \in I$, then a family $\mathcal{U}$ is also $\cap_{i\in I} X_i$–sub-linearly independent to $M$.

Consider the set $\mathcal{U}(M) = \{ V \subseteq \oplus_\Lambda U_\lambda \mid M$ is $\mathcal{U}_V$-generated$\}$. In this set, we collect all submodules $V$ of $\oplus_\Lambda U_\lambda$ which $M$ is $\mathcal{U}_V$-generated. If $V$ is in $\mathcal{U}(M)$, we have a surective homomorphism $g$ from $V$ to $M$. If $R$-module $X_1$ is $\mathcal{U}_{V_1}$-generated and $R$-module $X_1$ is $\mathcal{U}_{V_2}$-generated, then $X_1 \oplus X_2$ is $\mathcal{U}_{V_1 \oplus V_2}$-generated, where $V_1$ and $V_2$ be submodules of $\oplus_\Lambda U_\lambda$, $U_\lambda \in \mathcal{U}$, for every $\lambda \in \Lambda$. Based on [11], we have the set $\mathcal{U}(M)$ is closed under direct sums and homomorphic images. We will use the properties of the set $\sigma(0, \oplus_\Lambda U_\lambda, M)$ and $\mathcal{U}(M)$ to investigate some characteristics of $\mathcal{U}$-basis and $\mathcal{U}$-free module.

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Now, we will construct the definition of $\mathscr{U}$-basis and $\mathscr{U}$-free module by using the concept of $X$-sub-linearly independent to an $R$-module $M$ and $\mathscr{U}_V$-generated module.

**Definition 2.1.** *Given a family $\mathscr{U}$ of R-modules. A pair of submodules $(X, V)$ of $\oplus_\Lambda U_\lambda$ is said to be a $\mathscr{U}$-basis of R-module M if $\mathscr{U}$ is an X-sublinearly independent to M and M is a $\mathscr{U}_V$-generated module.*

From Definition 2.1, $\mathscr{U}$-basis of $R$-module $M$ is a pair of two submodules $X$ and $V$ of $\oplus_\Lambda U_\lambda$ which $\mathscr{U}$ is an $X$-sublinearly independent to $M$ and $M$ is a $\mathscr{U}_V$-generated module. In other words, if $(X, V)$ is a $\mathscr{U}$-basis of $R$-module $M$, then there are a monomorphism $f$ from $X$ to $M$ and an epimorphism $g$ from $V$ to $M$. Then we will give some examples of $\mathscr{U}$-basis of an $R$-module.

**Example 2.2**. Let $\mathscr{U} = \{\mathbb{Z}_p \mid p \text{ prime}\}$, a family of $\mathbb{Z}$-modules, where $\mathbb{Z}$ is a set of integers. We consider $\mathbb{Z}_6$ as a $\mathbb{Z}$-module. We will find $\mathscr{U}$-basis of $\mathbb{Z}$-module $\mathbb{Z}_6$. We can define monomorphisms from 0, $\mathbb{Z}_2$, $\mathbb{Z}_3$ and $\mathbb{Z}_2 \oplus \mathbb{Z}_3$ to $\mathbb{Z}_6$. Also, we can define an epimorphism from $\mathbb{Z}_2 \oplus \mathbb{Z}_3$ to $\mathbb{Z}_6$. Therefore, we have some $\mathscr{U}$-basis of $\mathbb{Z}$-module $\mathbb{Z}_6$ as follows: $(0, \mathbb{Z}_2\oplus\mathbb{Z}_3)$, $(\mathbb{Z}_2, \mathbb{Z}_2\oplus\mathbb{Z}_3)$, $(\mathbb{Z}_3, \mathbb{Z}_2\oplus\mathbb{Z}_3)$, $(\mathbb{Z}_2\oplus\mathbb{Z}_3, \mathbb{Z}_2\oplus\mathbb{Z}_3)$.

**Example 2.3.** Let $\mathscr{U} = \{\mathbb{Z}\}$, a family of $\mathbb{Z}$-module, where $\mathbb{Z}$ is a set of integers. We will find $\mathscr{U}$-basis of $\mathbb{Z}$-module $\mathbb{Z}_4$. Clearly, there is a monomorphism from 0 to $\mathbb{Z}_4$ and hence a family $\mathscr{U}$ is 0-sub-linearly independent to $\mathbb{Z}_4$. Furthermore, we can define an epimorphism from $\mathbb{Z}$ to $\mathbb{Z}_4$. As a consequence, $(0, \mathbb{Z})$ is a $\mathscr{U}$-basis of $\mathbb{Z}$-module $\mathbb{Z}_4$. In general, we can show that $(0, \mathbb{Z})$ is a $\mathscr{U}$-basis of $\mathbb{Z}$-module $\mathbb{Z}_n$, for every $n \geq 2$.

Now, we will give some properties of $\mathscr{U}$-basis of an $R$-module $M$. We already know that the set $\sigma(0, \oplus_\Lambda U_\lambda, M)$ is closed under intersections [10]. We will use this property to show the following proposition.

**Proposition 2.4.** *Given a family $\mathscr{U}$ of R-modules. If $(X_\alpha, V)$ is a $\mathscr{U}$-basis of an R-module M, for every $\alpha \in A$, then $(\cap_\alpha X_\alpha, V)$ is $\mathscr{U}$-basis of M.*

**Proof.** Suppose that $(X_\alpha, V)$ is a $\mathscr{U}$-basis of an $R$-module $M$, for every $\alpha \in A$. Consequently, a family $\mathscr{U}$ is $X_\alpha$-sub-linearly independent to $M$, for every $\alpha \in A$. Hence, $X_\alpha \in \sigma(0, \oplus_\Lambda U_\lambda, M)$, for every $\alpha \in A$. Based on [10], the set $\sigma(0, \oplus_\Lambda U_\lambda, M)$ is closed under intersections. As a consequence, we have $(\cap_\alpha X_\alpha, V)$ is in $\sigma(0, \oplus_\Lambda U_\lambda, M)$. In other words, a family $\mathscr{U}$ is $\cap_\alpha X_\alpha$-sub-linearly independent to $M$ and hence we have $(\cap_\alpha X_\alpha, V)$ is a $\mathscr{U}$-basis of $M$. QED

Next, we will use the fact that the set $\sigma(0, \oplus_\Lambda U_\lambda, M)$ is closed under submodules to proof the following property of $\mathscr{U}$-basis of $M$.

**Proposition 2.5.** *Given a family $\mathscr{U}$ of R-modules. If $(X, V)$ is a $\mathscr{U}$-basis of R-module M, then a pair $(X', V)$ is a $\mathscr{U}$-basis of M, for every submodule $X'$ of X.*

**Proof.** Let a pair $(X, V)$ is a $\mathscr{U}$-basis of an $R$-module $M$. Then a family $\mathscr{U}$ is $X$-sub-linearly independent to $M$. This implies that $X \in \sigma(0, \oplus_\Lambda U_\lambda, M)$. Based on [10], $\sigma(0, \oplus_\Lambda U_\lambda, M)$ is closed under submodules. So, for every submodule $X'$ of $X$, $X'$ is in $\sigma(0, \oplus_\Lambda U_\lambda, M)$. Therefore, $(X', V)$ is a $\mathscr{U}$-basis of $M$, for every submodule $X'$ of $X$. QED

In the next proposition, we focus on the second component of a pair $(X, V)$ which is $\mathscr{U}$-basis of an $R$-module $M$. We will use the properties of the set $\mathscr{U}(M)$ to proof the next property of $\mathscr{U}$-basis.

**Proposition 2.6.** *Given a family $\mathscr{U}$ of R-modules. If $(X, V_\beta)$ is a $\mathscr{U}$-basis of R-module M, for every $\beta \in B$, then $(X, \oplus_B V_\beta)$ is a $\mathscr{U}$-basis of M.*

**Proof**. Suppose that $(X, V_\beta)$ is a $\mathscr{U}$-basis of $R$-module $M$, for every $\beta \in B$. Then $M$ is a $\mathscr{U}_{V_\beta}$-generated module. This implies $V_\beta \in \mathscr{U}(M)$, for every $\beta \in B$. Based in [11], we already know that the set $\mathscr{U}(M)$ is closed under direct sums. Therefore, we have $\oplus_B V_\beta \in \mathscr{U}(M)$. In other words, we can say $M$ is a $\mathscr{U}_{\oplus_B V_\beta}$-generated module. Hence, a pair $(X, \oplus_B V_\beta)$ is a $\mathscr{U}$-basis of $M$. QED

**Proposition 2.7.** *Given a family $\mathscr{U}$ of R-modules. If $(X_\gamma, V_\gamma)$ is a $\mathscr{U}$-basis of R-module $M_\gamma$, for every $\gamma \in \Gamma$, then $(\oplus_\Gamma X_\gamma, \oplus_\Gamma V_\gamma)$ is $\mathscr{U}$-basis of $\oplus_\Gamma M_\gamma$.*

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**Proof.** Suppose that $(X_\gamma, V_\gamma)$ is a $\mathcal{U}$-basis of $R$-module $M$, for every $\gamma \in \Gamma$. From this we have a family $\mathcal{U}$ is $X_\gamma$-sub-linearly independent to $M_\gamma$ and $M_\gamma$ is $\mathcal{U}_{V_\gamma}$ -generated. This implies $X_\gamma \in \sigma(0, \oplus_\Lambda U_\lambda, M)$ and $V_\gamma \in \mathcal{U}(M)$, for every $\gamma \in \Gamma$. Therefore, we have a monomorphism from $X_\gamma$ to $M$ and an epimorphism from $V_\gamma$ to $M_\gamma$, for every $\gamma \in \Gamma$. Clearly, we can construct a monomorphism from $\oplus_\Lambda X_\lambda$ to $\oplus_\Lambda M_\lambda$. Also, we can define an epimorphism from $\oplus_\Lambda V_\lambda$ to $\oplus_\Lambda M_\lambda$. Hence $\oplus_\Gamma X_\gamma \in \sigma(0, \oplus_\Lambda U_\lambda, \oplus_\Lambda M_\lambda)$ and $\oplus_\Gamma V_\gamma \in \mathcal{U}(\oplus_\Lambda M_\lambda)$. Therefore, $(\oplus_\Gamma X_\gamma, \oplus_\Gamma V_\gamma)$ is $\mathcal{U}$-basis of $\oplus_\Gamma M_\gamma$. QED

We can see from Example 2.2 that $\mathcal{U}$-basis of an $R$-module $M$ is not uniquely determined. From this fact, we will choose a maximal element in first part of $\mathcal{U}$-basis of $M$ and a minimal element in second part of $\mathcal{U}$-basis of $M$. In other words, we will find a maximal element of the set $\sigma(0, \oplus_\Lambda U_\lambda, M)$ and a minimal element of $\mathcal{U}(M)$. Based on [10], the set $\sigma(0, \oplus_\Lambda U_\lambda, M)$ always has a maximal element. We will denote $\mathcal{U}$-basis of $M$ which has a maximal element in the first component and a minimal element in the second component of a pair $(X, V)$, a $\mathcal{U}$-basis of $M$. Now, we give the formal definition of $\mathcal{U}$-basis of $M$ and $\mathcal{U}$-free module.

**Definition 2.8.** *Given a family $\mathcal{U}$ of R-modules. A pair (X, V) is said to be a $\mathcal{U}$-basis if (X, V) is $\mathcal{U}$-basis of M, X is a maximal element of $\sigma(0, \oplus_\Lambda U_\lambda, M)$ and V is a minimal element of $\mathcal{U}(M)$. An R-module M is called $\mathcal{U}$-free if M has $\mathcal{U}$-basis.*

**Example 2.9.** Given a family $\mathcal{U}$ of $R$-modules. Then a pair (0,0) is $\mathcal{U}$-basis of $R$-module 0.

**Example 2.10.** Given a family $\mathcal{U} = \{\mathbb{Z}_n \mid n \text{ prime}\}$ of $\mathbb{Z}$-modules. From Example 2.2, we have some $\mathcal{U}$-basis of $\mathbb{Z}$-module $\mathbb{Z}_6$, i.e. $(0, \mathbb{Z}_2\oplus\mathbb{Z}_3)$, $(\mathbb{Z}_2, \mathbb{Z}_2\oplus\mathbb{Z}_3)$, $(\mathbb{Z}_3, \mathbb{Z}_2\oplus\mathbb{Z}_3)$, $(\mathbb{Z}_2\oplus\mathbb{Z}_3, \mathbb{Z}_2\oplus\mathbb{Z}_3)$. Therefore, we have $\mathbb{Z}_2\oplus\mathbb{Z}_3$ is a maximal element in the first component of $\mathcal{U}$-basis and also a minimal element of the second component of $\mathcal{U}$–basis of $\mathbb{Z}_6$. Hence, a pair $(\mathbb{Z}_2\oplus\mathbb{Z}_3, \mathbb{Z}_2\oplus\mathbb{Z}_3)$ is a $\mathcal{U}$-basis of $\mathbb{Z}$-module $\mathbb{Z}_6$.

**Example 2.11.** Given a family $\mathcal{U}$ of $\mathbb{Z}$-module. Based on Example 2.3, we have a pair $(0, \mathbb{Z})$ is a $\mathcal{U}$-basis of $\mathbb{Z}$-module $\mathbb{Z}_n$, where $n \geq 2$.

We already know that $\mathbb{Z}$-module $\mathbb{Z}_n$ is not a free module. But, from Example 2.11 we have $\mathbb{Z}$-module $\mathbb{Z}_n$ is a $\mathcal{U}$-free module relative to a family $\mathcal{U} = \{\mathbb{Z}\}$ of $\mathbb{Z}$-module, where $n \geq 2$.

In the Proposition 2.7, we have proved that a $\mathcal{U}$–basis of an $R$-module $M$ is closed under direct sums. A similar result holds for an $\mathcal{U}$-basis of an $R$-module $M$.

**Proposition 2.12.** *Given a family $\mathcal{U}$ of R-modules. If $(X_\gamma, V_\gamma)$ is a $\mathcal{U}$-basis of R-module $M_\gamma$, for every $\gamma \in \Gamma$, then $(\oplus_\Gamma X_\gamma, \oplus_\Gamma V_\gamma)$ is $\mathcal{U}$-basis of $\oplus_\Gamma M_\gamma$.*

In the previous examples, a submodule $X$ and $V$ of $\oplus_\Lambda U_\lambda$ which is a $\mathcal{U}$-basis of an $R$-module $M$ need not be isomorphic. In case $X$ is isomorphic to $V$, we will introduce a $\mathcal{U}$-strictly basis and a $\mathcal{U}$-strictly free module as follows:

**Definition 2.13.** *Given a family $\mathcal{U}$ of R-modules. A pair (X, V) is said to be a $\mathcal{U}$-strictly basis if (X, V) is $\mathcal{U}$-basis of M and X is isomorphic to V. An R-module M is called a $\mathcal{U}$-strictly free if M has $\mathcal{U}$-strictly basis.*

Since $X$ is isomorphic to $V$, we simply write $X$ instead $(X, V)$ as a $\mathcal{U}$-strictly basis of an $R$-module $M$. We will determine a family $\mathcal{U}$ of $R$-modules to regard a free module as a $\mathcal{U}$-strictly free module. We already know that if a free module $F$ has a basis $X$, then $F \cong \oplus_{x\in X} R_x$ with each $R_x \cong R$. We can choose $\mathcal{U} = \{R\}$ as a family of $R$-module. Hence, we have $\oplus_{x\in X} R_x = R^{(X)}$ is a $\mathcal{U}$-strictly basis of $F$. This implies $F$ is $\mathcal{U}$-strictly free. From this fact, we can say that every free $R$-module $F$ is a $\mathcal{U}$-strictly free module. Based on Example 2.9, a pair (0,0) is $\mathcal{U}$-basis of $R$-module 0, for any family $\mathcal{U}$ of $R$-modules. As a consequence, $R$-module 0 is $\mathcal{U}$-strictly free.

Moreover, in case $X$ is an element of $\mathcal{U}$, $X$ is $\mathcal{U}$-strictly free. Furthermore, we consider the result of Proposition 2.12. If $M_i$ is a $\mathcal{U}$-strictly free module for every $i = 1, 2, \ldots, n$, then $\oplus_{i=1}^{n} M_i$ is also a $\mathcal{U}$-strictly free module. Now, we will give some examples of $\mathcal{U}$-strictly free modules.

**Example 2.14.** Let $R$ be a commutative ring with unit and $\mathscr{U} = \{U_\lambda\}_\Lambda$ be a family of $R$-modules, where $U_\lambda = \mathrm{Hom}_R(R, M_\lambda)$ for every $\lambda \in \Lambda$. Based on [1], we can define a homomorphism $\varphi$ from $\mathrm{Hom}_R(R, M_\lambda)$ to $M_\lambda$, where $\varphi(f) := f(1)$. We can show that $\varphi$ is an isomorphism. This implies that a family $\mathscr{U}$ is $U_\lambda$-sub-linearly independent to $M_\lambda$ and $M_\lambda$ is $\mathscr{U}_{v_\gamma}$ -generated. Therefore, we can conclude that $M_\lambda$ is $\mathscr{U}$-strictly free.

**Example 2.15**. Given a family $\mathscr{U} = \{\mathbb{Z}_n \mid n \in \mathbb{Z}, n \geq 2\}$ of $\mathbb{Z}$-modules. Let $M = \mathbb{Z}_4^{(\mathbb{N})}$ and $N = \mathbb{Z}_2 \oplus M$ be $\mathbb{Z}$-modules. Since a family $\mathscr{U}$ is $M$-sub-linearly independent to $M$ and $M$ is $\mathscr{U}_N$-generated, $(M, N)$ is $\mathscr{U}$-basis of $M$. We will show that $M$ is not isomorphic to $N$. Assume that there is an isomorphism $f$ from $M$ to $N$. Since $(1,0,0,\ldots) \in N$, there is $0 \neq (a_i) \in M$ such that $f((a_i)) = (1,0,0,..)$. Then $f(2(a_i)) = 2f((a_i)) = 2(1,0,0,\ldots) = 0$. By hypothesis, $f$ is a monomorphism. So, we have $2(a_i) = 0$. Therefore, $a_1 = 0$ or 1 and $a_i = 0$ or 2, for $i \geq 2$. So, there is $(b_i) \in M$ such that $(a_i) = 2(b_i)$. This implies $b_1 = 0$ or 1 and $b_i = 0$, 1 or 3, for $i \geq 2$. Hence, $0 = 2f((b_i)) = f(2(b_i)) = f((a_i))$. But $1 = f((a_i))_1 = 0$, a contradiction. We can conclude that $M$ is not isomorphic to $N$ and hence $(M, N)$ is not $\mathscr{U}$-strictly basis of $M$.

Now, we consider the following properties of $\mathbb{Z}_n$ as an Abelian group.

**Theorem 2.16.** [15] *Let m and n be positive integers. If gcd(m,n)=1 (i.e. m and n are relative prime), then* $\mathbb{Z}_m \times \mathbb{Z}_n$ *is cyclic and is isomorphic to* $\mathbb{Z}_{mn}$ *and* (1,1) *is a generator of* $\mathbb{Z}_m \times \mathbb{Z}_n$.

**Theorem 2.17**. [15] *The group* $\Pi^n_{i=1}\mathbb{Z}_{mi}$ *is cyclic and isomorphic to* $\mathbb{Z}_{m_1 m_2 \ldots m_n}$ *if and only if the numbers* $m_i$, *for* $i = 1, \ldots, n$ *are pairwise relative prime, that is, the gcd of two of them is* 1.

Therefore, by using Theorem 2.16 and 2.17, we can determine some $\mathscr{U}$-strictly free modules as follows.

**Proposition 2.18.** *Given a family* $\mathscr{U} = \{\mathbb{Z}_p \mid p \text{ prime}\}$ *of* $\mathbb{Z}$*-modules and q, r be two distinct primes. Then* $\mathbb{Z}$*-module* $\mathbb{Z}_{qr}$ *is* $\mathscr{U}$*-strictly free.*

**Proof**. Since $q$ and $r$ are relative primes, $\mathbb{Z}_q \oplus \mathbb{Z}_r$ is $\mathscr{U}$-strictly basis of $\mathbb{Z}_{qr}$. Hence, $\mathbb{Z}$-modules $\mathbb{Z}_{qr}$ is a $\mathscr{U}$-strictly free. QED

**Proposition 2.19.** *Given a family* $\mathscr{U} = \{\mathbb{Z}_{p^n} \mid p \text{ prime}, n \in \mathbb{N}\}$ *of* $\mathbb{Z}$*-modules. Then* $\mathbb{Z}_n$ *is a* $\mathscr{U}$*-strictly free module, for every positive integer* $n \geq 2$.

**Proof**. We already know that every positive integer $n$ can be uniquely factorized as a product of distinct prime number $n = p_1^{n_1} p_2^{n_2} \ldots p_r^{n_r}$ , where $p_i$ prime and $n_i \in \mathbb{N}$ for $i = 1, 2, \ldots, r$. By Theorem 2.17, we have:

$$\mathbb{Z}_n \cong \mathbb{Z}_{p_1^{n_1}} \times \mathbb{Z}_{p_2^{n_2}} \times \ldots \times \mathbb{Z}_{p_r^{n_r}}$$

Therefore, we have $\mathbb{Z}_{p_1^{n_1}} \times \mathbb{Z}_{p_2^{n_2}} \times \ldots \times \mathbb{Z}_{p_r^{n_r}}$ is a $\mathscr{U}$-strictly basis of $\mathbb{Z}_n$. This implies that $\mathbb{Z}_n$ is a $\mathscr{U}$-strictly free module, for every positive integer $n$. QED

From Proposition 2.18, we have some $\mathscr{U}$-strictly free modules, where $\mathscr{U}$ is a family of $\mathbb{Z}$-modules $\mathbb{Z}$ modulo $p$, $p$ prime. $\mathbb{Z}$-module $\mathbb{Z}_{qr}$ is $\mathscr{U}$-strictly free for every two distinct primes $q$ and $r$. Moreover, based on Proposition 2.19, we have $\mathbb{Z}$-modules $\mathbb{Z}_n$ are $\mathscr{U}$-strictly free module relative to a family $\mathscr{U}$ which contains all $\mathbb{Z}$-modules $\mathbb{Z}_{p^n}$ , $p$ prime, for every positive integer $n \geq 2$.

We already know that since $\mathbb{Z}$-module $\mathbb{Z}_n$ is not linearly independent, $\mathbb{Z}_n$ is not a free module, for every positive integer $n$ greater than 2. But, this module is $\mathscr{U}$-strictly free module relative to a family $\mathscr{U} = \{\mathbb{Z}_p \mid p \text{ prime}\}$ of $\mathbb{Z}$-modules. Consequently, $\mathscr{U}$-strictly free module is a generalization of a free module. If we take $\mathscr{U} = \{R\}$, where $R$ is a ring, then an $R$-module $M$ is $\mathscr{U}$-strictly free if and only if $R$-module $M$ is free. But, if $\mathscr{U}$ is another family of $R$-module, then not every $\mathscr{U}$-strictly free module is a free module.

## 4. Conclussions

A $\mathscr{U}$-basis and a $\mathscr{U}$-free modules are a basis and a free module relative to a family $\mathscr{U}$ of $R$-module. These notions are the generalization of the concept of a basis and a free module. Every free module $F$ is a $\mathscr{U}$-free module, where $\mathscr{U} = \{R\}$ as a family of $R$-module. But not every $\mathscr{U}$-free module is a free module. For example, $\mathbb{Z}$-module $\mathbb{Z}_n$ is a $\mathscr{U}$-strictly free module, but $\mathbb{Z}$-module $\mathbb{Z}_n$ is not a free module.

If $\mathscr{U}$ be a family of all $\mathbb{Z}$-module $\mathbb{Z}_p$, where $p$ prime, then $\mathbb{Z}$-module $\mathbb{Z}_{qr}$ is a $\mathscr{U}$-strictly free module, where $q$ and $r$ be distinct primes. Furthermore, if $\mathscr{U}$ be a family of all $\mathbb{Z}$ modulo $p$ power of $n$, where $p$ prime and $n$ positive integer larger than 2, $\mathbb{Z}$-module $\mathbb{Z}_n$ is a $\mathscr{U}$-strictly free module, for every positive integer $n \geq 2$.

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