The Implementation of Rough Set on a Group Structure

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Abstract. Let $U$ be a non-empty set and $R$ an equivalence relation on $U$. Then, $(U, R)$ is an approximation space. The equivalence relation on $U$ forms disjoint equivalence classes. If $X \subseteq U$, we can form a lower approximation and an upper approximation of $X$. If $X \subseteq U$, then we can form a lower approximation and an upper approximation of $X$. In this research, rough group and rough subgroups are constructed in the approximation space $(U, R)$ for commutative and non-commutative binary operations.

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1. Introduction

Zdzislaw Pawlak [1] first introduced the rough set theory in 1982 as a mathematical technique to deal with vagueness and uncertainty problems. Various studies have discussed this theory and the possibility of its applications, for example, in data mining [2] and some algebraic structures. In [3], Biswaz and Nanda introduce the rough group and rough ring. Furthermore, Miao et al. [4] improve definitions of a rough group and rough subgroup and prove their new properties. In [5], Jesmalar investigates the homomorphism and isomorphism of the rough group. Furthermore, in [6], Bagirmaz and Ozcan give the concept of rough semigroups on approximation spaces. Then, Kuroki in [7] gives some results about the rough ideal of semigroups. In [8], Davvaz investigates roughness in the ring, and in [9], Davvaz and Mahdavipour give a roughness in modules. In [10], Isaac and Neelima introduce the concept of the rough ideal. Moreover, in [11], Zhang et al. give some properties of rough modules. Davvaz and Malekzadeh give roughness in modules [12]. They use the notion of reference points. Furthermore, Ozturk and Eren give the multiplicative rough modules [13]. Then, Sinha and Prakash introduce the rough exact sequence of rough modules[14]. They also give the injective module based on rough set theory [15]. In [16], Kazanci and Davvaz give the rough prime in a ring. Jun in [17] investigate the roughness of ideals in BCK-algebras. Moreover, Dubois and Prade [18] define the rough fuzzy sets.

This research focuses on the algebraic aspects by applying a rough set theory to construct a rough group and its subgroups on an approximation space. Moreover, in this research, we discuss the centralizer and the center of a rough group.

2. Preliminaries

In this section, there will be several definitions and theorems that can be helpful for this article. Those definitions are written as follows:

**Definition 1** [19] Define \( C_G(A) = \{ g \in G \mid gag^{-1} = a \text{ for all } a \in A \} \). This subset of \( G \) is called the centralizer of \( A \) in \( G \). Since \( gag^{-1} = a \) if and only if \( ga = ag \), \( C_G(A) \) is the set of elements of \( G \) which commute with every element of \( A \).

**Definition 2** [19] Define \( Z(G) = \{ g \in G \mid gx = xg \text{ for all } x \in G \} \), the set of elements commuting with all the elements of \( G \). This subset of \( G \) is called the center of \( G \).

**Definition 3** [20] Let \( R \) be an equivalence relation on \( A \) and \( a \in A \). Then the equivalence class of \( a \) under \( R \) is \( [a]_R = \{ x : x \in A \text{ and } aRx \} \). In other words, the equivalence class of \( a \) under \( R \) contains all the elements in \( A \) to which \( a \) is related by \( R \).

**Definition 4** [3] Let \((U, R)\) be an approximation space and \( X \) be a subset of \( U \), the sets,
\[
\overline{X} = \{ x \mid [x]_R \cap X \neq \emptyset \} \quad (1)
\]
\[
\underline{X} = \{ x \mid [x]_R \subseteq X \} \quad (2)
\]
are called upper approximation and lower approximation of \( X \).

**Definition 5** [1] Let \( R \) be an equivalence relation on universe set \( U \), a pair \((U, R)\) is called an approximation space. A subset \( X \subseteq U \) can be defined if \( \overline{X} = \overline{X} \), in the opposite case, if \( \overline{X} - \underline{X} \neq \emptyset \) then \( X \) is called a rough set.
Definition 6 [3] Let $K = (U, R)$ be an approximation space and $\ast$ be a binary operation defined on $U$. A subset $G$ of universe $U$ is called a rough group if the following properties are satisfied:

i. $\forall x, y \in G, x \ast y \in \overline{G}$;

ii. Association property holds in $\overline{G}$;

iii. $\exists e \in \overline{G}$ such that $\forall x \in G, x \ast e = e \ast x = x$; $e$ is called the rough identity element of $G$;

iv. $\forall x \in G, \exists y \in G$ such that $x \ast y = y \ast x = e$; $y$ is called the rough inverse element of $x$ in $G$.

We will give the example of rough group in Section 3.

The following theorem gives the characteristics of a rough group.

Theorem 1. [3] A necessary and sufficient condition for a subset $H$ of rough group $G$ to be a rough subgroup is that:

(i) $\forall x, y \in H, x \ast y \in \overline{H}$;

(ii) $\forall x \in H, x^{-1} \in H$.

Several steps will be taken to achieve the objectives of this research. Those steps are written as follows:

1. Determine a set $U$, where $U \neq \emptyset$.
2. Define a relation $R$ on $U$.
3. Shows that a relation $R$ is the equivalence relation on $U$.
4. Determine equivalence classes on $U$.
5. Determine a set $G$, where $G \subseteq U$ and $G \neq \emptyset$.
6. Determine the approximation space, lower approximation on $G (G)$, and upper approximation on $G (G)$.
7. Determine a rough set $Apr(G) = (G, G)$.
8. Determine a binary operation $\ast$ on the set $G$.
9. Shows that $(G, \ast)$ is a rough group in the approximation space that has been constructed.
10. Determine a rough subgroup $(H, \ast)$ from a rough group $(G, \ast)$.

3. Rough Group Construction

3.1 Commutative Rough Group Construction

In this section, we will give the construction of commutative rough group.

Example 3.1. Given a non-empty set $U = \{0,1,2,3, ..., 99\}$. We define a relation $R$ on the set $U$, that is, for every $a, b \in U$ apply $aRb$ if and only if $a - b = 7k$ where $k \in \mathbb{Z}$. Furthermore, it can be shown that relation $R$ is reflexive, symmetrical, and transitive. So, relation $R$ is an equivalence relation on $U$. As a result, relation $R$ produces some disjoint partitions called equivalence classes. The equivalence classes are written as follows:

$E_1 = [1] = \{1,8,15,22,29,36,43,50,57,64,71,78,85,92,99\}$;
$E_2 = [2] = \{2,9,16,23,30,37,44,51,58,65,72,79,86,93\}$;
$E_3 = [3] = \{3,10,17,24,31,38,45,52,59,66,73,80,87,94\}$;
$E_4 = [4] = \{4,11,18,25,32,39,46,53,60,67,74,81,88,95\}$;
$E_5 = [5] = \{5,12,19,26,33,40,47,54,61,68,75,82,89,96\}$;
$E_6 = \{6\} = \{6,13,20,27,34,41,48,55,62,69,76,83,90,97\};$
$E_7 = \{0\} = \{0,7,14,21,28,35,42,49,56,63,70,77,84,91,98\}.$

Given a non-empty subset $X \subseteq U$ that is $X = \{10,20,30,40,50,60,70,80,90\}$. Because the set $U \neq \emptyset$ and $R$ is an equivalence relation on $U$, a pair $(U, R)$ is the approximation space. Furthermore, it can be obtained the lower approximation and upper approximation of $X$, that is:

\[ X = \emptyset. \]
\[ \bar{X} = E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5 \cup E_6 \cup E_7 = U. \]

After determining the lower approximation and upper approximation of $X$, then given a binary operation $+_\{100\}$ on $X$. Here is given Table Cayley of $X$ with the operation $+_\{100\}$.

**Table 1.** Table Cayley of $X$ with the operation $+_\{100\}$

<table>
<thead>
<tr>
<th>$+_{100}$</th>
<th>10</th>
<th>20</th>
<th>30</th>
<th>40</th>
<th>50</th>
<th>60</th>
<th>70</th>
<th>80</th>
<th>90</th>
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</thead>
<tbody>
<tr>
<td>10</td>
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<td>20</td>
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<td>60</td>
<td>70</td>
<td>80</td>
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<td>0</td>
<td>10</td>
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<td>30</td>
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<td>70</td>
<td>80</td>
<td>90</td>
<td>0</td>
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<td>30</td>
<td>40</td>
<td>50</td>
<td>60</td>
<td>70</td>
<td>80</td>
</tr>
</tbody>
</table>

i. Based on Table 1, it is proved that for each $x, y \in X$, apply $x(+\{100\})y \in \bar{X}$.

ii. For each $x, y, z \in X$, the associative property that is $(x(+\{100\})y)(+\{100\})z = x(+\{100\})(y(+\{100\})z)$ holds in $\bar{X}$. The operation $+_\{100\}$ is associative in $\bar{X}$.

iii. There is a rough identity element $e \in \bar{X}$ that is $0 \in \bar{X}$ such that for each $x \in X$, $x(+\{100\})e = e(+\{100\})x = x$.

**Table 2.** Table of element inverse of the set $X$

<table>
<thead>
<tr>
<th>$x$</th>
<th>10</th>
<th>20</th>
<th>30</th>
<th>40</th>
<th>50</th>
<th>60</th>
<th>70</th>
<th>80</th>
<th>90</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x^{-1}$</td>
<td>90</td>
<td>80</td>
<td>70</td>
<td>60</td>
<td>50</td>
<td>40</td>
<td>30</td>
<td>20</td>
<td>10</td>
</tr>
</tbody>
</table>

iv. For each $x \in X$, there is a rough inverse element of $x$ that is $x^{-1} \in X$ such that $x(+\{100\})x^{-1} = x^{-1}(+\{100\})x = e$. Based on Table 2, it can be seen that each element $x$ in the set $X$, then the inverse element $x^{-1}$ is also in $X$.

Since those four conditions have been satisfied, then $(X, +_{\{100\}})$ is a rough group.

**3.2 Non-Commutative Rough Group Construction**

In this section, we will give the construction of non-commutative rough group.

**Example 3.2.** Given a permutation group $S_3$ to the operation of permutation multiplication "o." For example, take a subgroup $G = \{(1), (12)\}$ of the group $S_3$. For $x, y \in S_3$, define a relation $R$ that is $xRy$ if and only if $x \circ y^{-1} \in G$. Furthermore, it can be shown that relation $R$ is reflexive, symmetrical, and transitive. So, relation $R$ is an equivalence relation on $S_3$. As a result, relation $R$ produces some disjoint partitions called equivalence classes. Suppose $a$ is the element in $S_3$, the equivalence class containing $a$ defined as follows:

$[a]_R = \{x \in S_3 \mid xRa\}$
$= \{x \in S_3 \mid x \circ a^{-1} \in G\}$
\[ x \in S_3 \mid x \circ a^{-1} = g, \; g \in G \]
\[ x \in S_3 \mid x = g \circ a, \; g \in G \]
\[ \{ g \circ a \mid g \in G \} \quad (3) \]

Based on the Equation (3), this is corresponding to the definition of the right coset of \( G \) in \( S_3 \) that is \( G a = \{ g \circ a \mid g \in G \} \). Thus, the right cosets of \( G \) in \( S_3 \) as follows:

\[
G \circ (1) = G \circ (12) = \{(1), (1 2)\};
G \circ (1 3) = G \circ (1 2 3) = \{(1 3), (1 2 3)\};
G \circ (2 3) = G \circ (1 3 2) = \{(2 3), (1 3 2)\}.
\]

Given a non-empty subset \( Y \subseteq S_3 \) that is \( Y = \{(1), (1 2), (1 2 3), (1 3 2)\} \). Furthermore, it can be obtained the lower approximation and upper approximation of \( Y \), that is:

\[
\underline{Y} = \{(1), (1 2)\}.
\]

\[
\overline{Y} = \{(1), (1 2)\} \cup \{(1 3), (1 2 3)\} \cup \{(2 3), (1 3 2)\} = S_3.
\]

After determining the lower approximation and upper approximation of \( Y \), then we give a permutation multiplication "\( \circ \)" on \( Y \). We give a Table Cayley of \( Y \) with the operation of permutation multiplication as follows.

| Table 3. Table Cayley of \( Y \) with the operation of permutation multiplication |
|---|---|---|---|---|
| \( \circ \) | (1) | (1 2) | (1 2 3) | (1 3 2) |
| (1) | (1) | (1 2) | (1 2 3) | (1 3 2) |
| (1 2) | (1 2) | (1) | (2 3) | (1 3) |
| (1 2 3) | (1 2 3) | (1 3) | (1 3 2) | (1) |
| (1 3 2) | (1 3 2) | (2 3) | (1) | (1 2 3) |

i. Based on Table 3, it is proved that for each \( x, y \in Y \), apply \( x \circ y \in \overline{Y} \).

ii. For each \( x, y, z \in Y \), the associative property that is \( (x \circ y) \circ z = x \circ (y \circ z) \) holds in \( \overline{Y} \). The operation \( \circ \) is associative in \( \overline{Y} \).

iii. There is a rough identity element \( e \in \overline{Y} \) that is \( (1) \in \overline{Y} \) such that for each \( y \in Y \),

\[ y \circ e = e \circ y = y. \]

\[
| Table 4. Table of inverse element of \( Y \) |
|---|---|---|---|---|
| \( y \) | (1) | (1 2) | (1 2 3) | (1 3 2) |
| \( y^{-1} \) | (1) | (1 2) | (1 3 2) | (1 2 3) |

iv. For each \( y \in Y \), there is a rough inverse element of \( y \) that is \( y^{-1} \in Y \) such that \( y \circ y^{-1} = y^{-1} \circ y = e. \) Based on Table 4, it can be seen that each element \( y \) in the set \( Y \), then the inverse element \( y^{-1} \) is also in the set \( Y \).

Since those four conditions have been satisfied, then \( \langle Y, \circ \rangle \) is a rough group.

4. Subgroup Construction of the Rough Group

After constructing a commutative rough group and a non-commutative rough group, we will construct subgroups of each of the previously constructed rough groups.

4.1 Subgroup Construction of Commutative Rough Group

Before it has been obtained, a commutative rough group \( X \) with the operation "\( +_{100} \)". Furthermore, we will construct several subgroups that can be formed from the rough group \( X \). Based on Theorem 1, we can obtain several subgroups from the rough group \( X \) that written as follows:
1. \( \langle \{20,30,40,50,60,70,80\}, +_{100}\rangle \);
2. \( \langle X, +_{100}\rangle \).

After determining several subgroups from the rough group \( X \) that is commutative, then we will determine the centralizer and the center of subgroups in rough group \( X \). Suppose all subgroups of rough group \( X \) above are denoted by \( A \). Based on Definition 1, the centralizer \( A \) in \( X \) is the set where is the element of \( X \) is commutative with each element of \( A \). Here is given the table that shows the centralizer of subgroups \( A \) in rough group \( X \).

| Table 5. Table of the centralizer of subgroups \( A \) in rough group \( X \) |
|------------------|-------------------|
| \( A \)          | \( C_x(A) \)      |
| \( \{20,30,40,50,60,70,80\} \) | \( X \) |
| \( X = \{10,20,30,40,50,60,70,80,90\} \) | \( X \) |

Since the operation \( +_{100} \) of rough group \( X \) is commutative, the centralizer of subgroups in rough group \( X \) is \( X \) itself.

Based on Definition 2, the center of \( X \) is the set of elements that is commutative with all elements of \( X \). Because rough group \( X \) using commutative operation, the center of rough group \( X \) is \( X \) itself, or it can be written as \( Z(X) = X \).

Using Theorem 1, we will show that the center of rough group \( X \) that is \( Z(X) = X \) is a rough subgroup of rough group \( X \).

i. Based on Table 1, it is proved that for each \( x, y \in Z(X) = X \), apply \( x(+_{100})y \in Z(X) = X \).

ii. For each \( x \in Z(X) = X \), there is an inverse element of \( x \) that is \( x^{-1} \in Z(X) = X \). Based on Table 2, it can be seen that if each element \( x \) in the set \( X \) then the inverse element of \( x \) also in the set \( X \).

Two conditions on Theorem 1 have been satisfied, so it is proved that the center of rough group \( X \) that is \( Z(X) = X \) is a rough subgroup of rough group \( X \).

### 4.2 Subgroup Construction of Non-Commutative Rough Group

Before it has been obtained a non-commutative rough group \( Y \) with the operation of permutation multiplication "\( \circ \)". Furthermore, we will construct several subgroups that can be formed from the rough group \( Y \). Based on Theorem 1, we can obtain several subgroups from the rough group \( Y \) that written as follows:

1. \( \langle \{(1)\}, \circ \rangle \);
2. \( \langle \{(1),(1,2)\}, \circ \rangle \);
3. \( \langle \{(1),(1,2,3),(1,3,2)\}, \circ \rangle \);
4. \( \langle \{(1,2),(1,2,3)\}, \circ \rangle \);
5. \( \langle Y, \circ \rangle \).

After determining several subgroups from the rough set \( Y \) that are non-commutative, then we will determine the centralizer and the center of subgroups in rough group \( Y \). Suppose all subgroups of rough group \( Y \) above are denoted by \( B \). Based on Definition 1, the centralizer \( B \) in \( Y \) is the set where is the element of \( Y \) is commutative with each element of \( B \). Here is given the table that shows the centralizer of subgroups \( B \) in rough group \( Y \).

| Table 6. Table of the centralizer of subgroups \( B \) in rough group \( Y \) |
Based on Definition 2, the center of $Y$ is the set of elements that is commutative with all elements of $Y$. From the Definition 2, the center of rough group $Y$ is an identity element, or it can be written as $Z(Y) = \{(1)\}$.

Using Theorem 1, we will show that the center of rough group $Y$ that is $Z(Y) = \{(1)\}$ is a rough subgroup of rough group $Y$. Previously, determine the upper approximation of $Z(Y)$ that is $\bar{Z}(Y) = \{(1), (1,2)\}$.

i. For $(1) \in Z(Y)$, apply $(1) \circ (1) = (1) \in \bar{Z}(Y)$.

ii. For $(1) \in Z(Y)$, there is an inverse element of $(1)$ that is $(1) \in Z(Y)$.

Based on Theorem 1, because the two conditions have been satisfied, it is proved that the center of rough group $Y$ that is $Z(Y) = \{(1)\}$ is a rough subgroup of rough group $Y$.

5 Conclusions

Based on the results, we construct a rough group, a rough subgroup in the case of the commutative and non-commutative binary operation. Furthermore, the centralizer of a commutative rough subgroup is also a rough group. In comparison, the centralizer of the subgroup of a non-commutative rough group must contain the identity element and the center. The center of each rough group, both commutative and non-commutative, are subgroups of each rough group.

References


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