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THE COMPARISON OF THE EFFICIENCY OF THE METHODS OF PARAMETERS ESTIMATION FOR GENERALIZED BETA OF THE SECOND KIND (GB2) DISTRIBUTION

Dian Kurniasari[1], Warsono[1], Widiarti[1], Siti U. Nabila[1], Nourma Indryani[1], Mustofa Usman[1] and Sutopo Hadi[2]
[1]Department of Mathematics, Faculty of Mathematics and Natural Sciences, Universitas Lampung, Gedongmeneng Bandar Lampung, Indonesia
[2]Department of Chemistry, Faculty of Mathematics and Natural Sciences, Universitas Lampung, Gedongmeneng Bandar Lampung, Indonesia
E-Mail: dian.kurniasari@fmipa.unila.ac.id

ABSTRACT

The generalize distribution from a classical distribution is performed by adding more parameters to the distribution that makes the distribution more flexible in analyzing empirical data and able to adjust the shape of empirical data. The generalization of this distribution produces a Beta Generalized of the first kind distribution or a GB2 distribution with three and four parameters. This paper will discuss the GB2 distribution with four parameters namely α, p and q as shape parameters while parameter b is the scale parameter. In statistical inference, especially parameter estimation, is needed in analyzing empirical data with this distribution. Obviously the estimation results obtained are expected to be a sound estimator, namely to meet the criteria of unbiasedness and minimum variance. The estimation results of the GB2 distribution parameters through simulations using the methods of moment, the Maximum Likelihood Estimation, and the Probability Weighted Moment. Based on the results from the simulation of the three estimation methods that the estimation of parameters by using the Maximum Likelihood Estimation is better than the method of Probability Weighted Moment and the method of moment where in a larger sample size gives a smaller bias and MSE value.

Keywords: generalized beta 2 distribution; method of moment; maximum likelihood estimation; method of probability weighted moment.

1. INTRODUCTION

A probability distribution is a distribution (or graph) of the results that might occur from an experiment [1]. In its application, probability distribution is used for modeling in various fields of studies or to analyze natural phenomena. One of the distribution is a Beta distribution with two parameters which is widely used in various fields of science, such as statistics, economics, insurance, health, industry, and microeconomics. However, the application of this distribution is limited in some ways. Therefore, it is necessary to generalize the distribution so that the distribution can be applied more flexible in analyzing empirical data. The results of this generalization distribution are Generalized Beta of the First Kind or Generalized Beta of the Second Kind with three and four parameters, respectively. This study will discuss the distribution of Generalized Beta of the Second Kind with four parameters.

The four parameters in Generalized Beta of the Second Kind distribution are α, b, p, and q. Where parameters α, p, q are the shape parameters while parameter b is a scale parameter [2]. The parameters of a distribution are numerical characteristics or characteristics of a population. The shape parameter is a numeric parameter that shows the shape of the curve while the scale parameter is a numeric parameter that shows the distribution of the data.

In the analysis of empirical data, this parameter cannot be measured directly but rather in a predictable way by taking samples from a population. Furthermore, the results from these samples are used to estimate the parameters of population. Therefore, we need a statistical inference, which provide the estimation methods of parameters from the information available from the sample. Estimation and application of the Beta Generalized of the first kind distribution have been discussed by several researchers [3-11].

There are several methods of estimation that can be used to estimate parameters: (1) method of moment, the basic idea of this method is to equate population moments with sample moments. (2) The method of Maximum Likelihood Estimation (MLE) is used to estimate the parameter values if the distribution of the population is known. The basic idea is to maximize the likelihood function. (3) The method of Probability Weighted Moment (PWM) is a modification of the "conventional" method of moment [12].

It is expected that the results of the information obtained from the sample represent information about the population parameters. In parameter estimation, an estimator is said to be a sound estimator if it meets the criteria of unbiased and minimum variance. This study is comparing the three methods of estimating the parameters and determining the best method for estimating the parameters of Generalized Beta of the Second Kind distribution.

In this paper the probability density function, mean and variance will be presented in the second section. In the third section we will discuss mathematical procedures by using mathematical functions and formulas to estimate parameters either by the methods of Moment [13], Maximum Likelihood or Probability Weighted

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Moment and some good estimating criteria. In the fourth section a simulation study will be conducted to see the goodness of the parameter estimation results. Indicators that are used to fulfil the nature of unbiased and minimum variance are bias and Mean Square. Bias equal to zero indicates that the results of the estimation are unbiased predictors. Mean square error summarizes information about bias and variance. In the fifth Section is a conclusion.

2. GENERALIZED BETA 2 DISTRIBUTION

A random variable is said to have a Generalized Beta 2 (GB2) distribution with parameters (α, b, p, q) if the probability density function is as follows:

$$f(x) = \frac{\alpha x^{\alpha p-1}}{b^{\alpha p}\, B(p,q)\left(1+\left(\frac{x}{b}\right)^{\alpha}\right)^{p+q}}\ ,\ x > 0 \quad (1)$$

where b is scale parameter and α, p, q are shape parameters, B(p,q) is beta function [7].

The mean and variance of the *Generalized* Beta 2 distribution is:

1. $E(X) = \frac{b\,\Gamma\left(p+\frac{1}{\alpha}\right)\Gamma\left(q-\frac{1}{\alpha}\right)}{\Gamma(p)\,\Gamma(q)}$ (2)

2. $Var\,(X) = \frac{b^2[\Gamma(p)\,\Gamma(q)\Gamma\left(p+\frac{2}{\alpha}\right)\Gamma\left(q-\frac{2}{\alpha}\right)]-b^2\left(\Gamma\left(p+\frac{1}{\alpha}\right)\right)^2\left(\Gamma\left(q-\frac{1}{\alpha}\right)\right)^2}{[\Gamma(p)]^2[\Gamma(q)]^2}$ (3)

3. THE ESTIMATION OF GENERALIZED BETA 2 DISTRIBUTION

3.1 Method of Moment

Estimation method using the method of moment is conducted by equating the first k moment of the sample with the first k moment of the population and solve the system of equation simultaneously [14].

The four moments of population obtained at the GB2 distribution are equated with the four moments of sample so that we have four system of equation that will be solved simultaneously to obtain the predicted values of α, b, p and q. The four equations are as follows:

$$M_1 = \mu_1(\theta_1,\theta_2,\ldots,\theta_k)\ ;\quad M_1 = \frac{1}{n}\sum_{i=1}^{n} X_i,\quad \mu_1 = E(X)$$

$$M_2 = \mu_2(\theta_1,\theta_2,\ldots,\theta_k)\ ;\quad M_2 = \frac{1}{n}\sum_{i=1}^{n} X_i^2,\quad \mu_2 = E(X^2)$$

$$M_k = \mu_k(\theta_1,\theta_2,\ldots,\theta_k)\ ;\quad M_k = \frac{1}{n}\sum_{i=1}^{n} X_i^k,\quad \mu_k = E(X^k)$$

where, $E(X^n) = \frac{b^n\,\Gamma(p+\frac{n}{\alpha})\Gamma(q-\frac{n}{\alpha})}{\Gamma(p)\Gamma(q)}$

Such that we have:

$$\frac{b\,\Gamma(p+\frac{1}{\alpha})\Gamma(q-\frac{1}{\alpha})}{\Gamma(p)\Gamma(q)} = \frac{\sum x_i}{n} = M_1 \quad (4)$$

$$\frac{b^2\,\Gamma(p+\frac{2}{\alpha})\Gamma(q-\frac{2}{\alpha})}{\Gamma(p)\Gamma(q)} = \frac{\sum {x_i}^2}{n} = M_2 \quad (5)$$

$$\frac{b^3\,\Gamma(p+\frac{3}{\alpha})\Gamma(q-\frac{3}{\alpha})}{\Gamma(p)\Gamma(q)} = \frac{\sum {x_i}^3}{n} = M_3 \quad (6)$$

$$\frac{b^4\,\Gamma(p+\frac{4}{\alpha})\Gamma(q-\frac{4}{\alpha})}{\Gamma(p)\Gamma(q)} = \frac{\sum {x_i}^4}{n} = M_4 \quad (7)$$

From the four central moment equations of the population, it can be seen that in the gamma function there are still two parameters whose values are unknown, namely α and p. To obtain the estimated value of the parameter, one of the parameter values must be assumed. In this case, the value of $\hat{\alpha}$ is assumed by $\hat{\alpha} = 1$ so the equation becomes:

$$\frac{bp}{q-1} = \frac{\sum x_i}{n} = M_1 \quad (8)$$

$$\frac{b^2 p(p+1)}{(q-1)(q-2)} = \frac{\sum {x_i}^2}{n} = M_2 \quad (9)$$

$$\frac{b^3 p(p+1)(p+2)}{(q-1)(q-2)(q-3)} = \frac{\sum {x_i}^3}{n} = M_3 \quad (10)$$

$$\frac{b^4 p(p+1)(p+2)(p+3)}{(q-1)(q-2)(q-3)(q-4)} = \frac{\sum {x_i}^4}{n} = M_4 \quad (11)$$

Based on the central moment of the population and the central moment of the sample of the Generalized Beta 2 distribution and solving the above equation, so that we have the results as follows:

$$b = \frac{2(\sum x_i)^2(\sum {x_i}^3)-n\,(\sum {x_i}^2)(\sum {x_i}^3)-(\sum x_i)(\sum {x_i}^2)^2}{2n\,(\sum {x_i}^2)^2-(\sum x_i)^2(\sum {x_i}^2)-n\,(\sum x_i)(\sum {x_i}^3)} \quad (12)$$

$$p = \frac{2\,(\sum x_i)(\sum {x_i}^2)^2-2(\sum x_i)^2(\sum {x_i}^3)}{2(\sum x_i)^2(\sum {x_i}^3)-n\,(\sum {x_i}^2)(\sum {x_i}^3)-(\sum x_i)(\sum {x_i}^2)^2} \quad (13)$$

$$q = \frac{-3n\,(\sum x_i)(\sum {x_i}^3)+4n(\sum {x_i}^2)^2-(\sum x_i)^2(\sum {x_i}^2)}{2n(\sum {x_i}^2)^2-(\sum x_i)^2(\sum {x_i}^2)-n\,(\sum x_i)(\sum {x_i}^3)} \quad (14)$$

3.2 Method of Maximum Likelihood Estimation (MLE)

Let $X = (X_1, X_2, \ldots, X_n)$ is a random sample of size n from the Generalized Beta 2 distribution. According to Hosking *et al*. [12], estimation of parameters by the Maximum Likelihood Estimation method can be done by first determining the join probability density function of X_1, X_2, ..., X_n namely $f(x_1;\theta)\, f(x_2;\theta)\ \ldots\ f(x_n;\theta)$ which is called as Likelihood Function of the distribution.

The likelihood function from the probability function of Generalized Beta 2 distribution is

$$L(\underline{x};\alpha,b,p,q) = \prod_{i=1}^{n} \frac{\alpha x^{\alpha p-1}}{b^{\alpha p}\, B(p,q)\left(1+\left(\frac{x}{b}\right)^{\alpha}\right)^{p+q}}$$

$$= \frac{\alpha^n\left(\prod_{i=1}^{n} {x_i}^{\alpha p-1}\right)}{b^{\alpha p n}\,(B(p,q))^n\left[\prod_{i=1}^{n}\left(1+\left(\frac{x_i}{b}\right)^{\alpha}\right)^{p+q}\right]} \quad (15)$$

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Furthermore, the estimating of the parameters α, b, p, and q are conducting by maximizing the likelihood functions above. In order to make the above functions easy to solve, the natural logarithm can be used as follows:

$$\ln L(\underline{x};\alpha,b,p,q) = ln\left\{\frac{\alpha^n\left(\prod_{i=1}^n {x_i}^{\alpha p-1}\right)}{b^{\alpha pn}\left(B(p,q)\right)^n\left[\prod_{i=1}^n\left(1+\left(\frac{x_i}{b}\right)^{\alpha}\right)^{p+q}\right]}\right\}$$
$$= n\ln\alpha + (\alpha p-1)\sum_{i=1}^n \ln(x_i) - \alpha pn\ln b - n\ln\Gamma(p) - n\ln\Gamma(q) + n\ln\Gamma(p+q) - (p+q)\sum_{i=1}^n \ln\left(1+\left(\frac{x_i}{b}\right)^{\alpha}\right) \quad (16)$$

So that the function can be maximized by applying differentiation with respect to all parameters and then set equal to zero.

a) Differentiation with respect to α and made it equal to zero, we have

$$\frac{\partial \ln L(\underline{x};\alpha,b,p,q)}{\partial\alpha} = 0$$
$$\hat{a} = \frac{n}{-\hat{p}\sum_{i=1}^n \ln(x_i) + n\hat{p}\ln\hat{b} + (\hat{p}+\hat{q})\sum_{i=1}^n \frac{\left(\frac{x_i}{\hat{b}}\right)^{\hat{\alpha}}\ln\left(\frac{x_i}{\hat{b}}\right)}{\left(1+\left(\frac{x_i}{\hat{b}}\right)^{\hat{\alpha}}\right)}} \quad (17)$$

b) Differentiation with respect to b and set it equal to zero, we have

$$\frac{\partial \ln L(\underline{x};\alpha,b,p,q)}{\partial b} = 0$$
$$\hat{b} = \frac{\hat{\alpha}\hat{p}n}{(\hat{p}+\hat{q})\sum_{i=1}^n \frac{\hat{\alpha}\, x_i\left(\frac{x_i}{b}\right)^{\alpha-1}}{\hat{b}^2\left(1+\left(\frac{x_i}{b}\right)^{\alpha}\right)}} \quad (18)$$

c) Differentiation with respect to p and made it equal to zero, we have

$$\frac{\partial \ln L(\underline{x};\alpha,b,p,q)}{\partial p} = 0$$
$$n\,\psi(\hat{p}) = \hat{\alpha}\sum_{i=1}^n \ln(x_i) - n\hat{\alpha}\ln\hat{b} + n\,\psi(\hat{p}+\hat{q}) - \sum_{i=1}^n \ln\left(1+\left(\frac{x_i}{\hat{b}}\right)^{\hat{\alpha}}\right) \quad (19)$$

In this case $\psi(p)$ is a function psi (Digamma).

d) Differentiation with respect to q and made it equal to zero, we have

$$\frac{\partial \ln L(\underline{x};\alpha,b,p,q)}{\partial q} = 0$$
$$n\,\psi(\hat{q}) = n\,\psi(\hat{p}+\hat{q}) - \sum_{i=1}^n \ln\left(1+\left(\frac{x_i}{\hat{b}}\right)^{\hat{\alpha}}\right) \quad (20)$$

In this case $\psi(q)$ is function psi (Digamma).

The equation obtained cannot be solved analytically, so that an iteration method is needed to get the estimated parameters a, b, p and q. The iteration method used is Newton-Raphson.

Newton Raphson method is a method for solving nonlinear equations iteratively. This method can be extended to solve systems of equations with more than one parameter [15]. The iteration is as follows:

$$\theta_{i+1} = \theta_i - [H^{-1}g]$$

Where $\hat{\theta}_{i+1} = \begin{bmatrix}\hat{\theta}_{i+1}\\ \vdots \\ \hat{\theta}_{p+1}\end{bmatrix}$ and $\hat{\theta}_i = \begin{bmatrix}\hat{\theta}_{1i}\\ \vdots \\ \hat{\theta}_{pi}\end{bmatrix}$

The first step is to determine the initial value of each parameter. Next, to determine the gradient vector or the first derivative vector of the natural logarithmic function of α, b, p, and q denoted by $g(\theta)$.

$$g(\theta) = \begin{bmatrix}\frac{\partial \ln L(\underline{x};\alpha,b,p,q)}{\partial\alpha}\\ \frac{\partial \ln L(\underline{x};\alpha,b,p,q)}{\partial b}\\ \frac{\partial \ln L(\underline{x};\alpha,b,p,q)}{\partial p}\\ \frac{\partial \ln L(\underline{x};\alpha,b,p,q)}{\partial q}\end{bmatrix}$$
$$= \begin{bmatrix}\frac{n}{\hat{\alpha}} + \hat{p}\sum_{i=1}^n \ln(x_i) - n\hat{p}\ln\hat{b} - (\hat{p}+\hat{q})\sum_{i=1}^n \frac{\left(\frac{x_i}{\hat{b}}\right)^{\hat{\alpha}}\ln\left(\frac{x_i}{\hat{b}}\right)}{\left(1+\left(\frac{x_i}{\hat{b}}\right)^{\hat{\alpha}}\right)}\\ \frac{-\hat{\alpha}\hat{p}n}{\hat{b}} + (\hat{p}+\hat{q})\sum_{i=1}^n \frac{\hat{\alpha}\, x_i\left(\frac{x_i}{b}\right)^{\alpha-1}}{\hat{b}^2\left(1+\left(\frac{x_i}{b}\right)^{\alpha}\right)}\\ \hat{\alpha}\sum_{i=1}^n \ln(x_i) - n\hat{\alpha}\ln\hat{b} - n\,\psi(\hat{p}) + n\,\psi(\hat{p}+\hat{q}) - \sum_{i=1}^n \ln\left(1+\left(\frac{x_i}{\hat{b}}\right)^{\hat{\alpha}}\right)\\ -n\,\psi(\hat{q}) + n\,\psi(\hat{p}+\hat{q}) - \sum_{i=1}^n \ln\left(1+\left(\frac{x_i}{\hat{b}}\right)^{\hat{\alpha}}\right)\end{bmatrix} \quad (21)$$

Next determine the Hessian matrix or the second derived matrix of natural logarithmic functions of the parameters α, b, p, and q denoted by $H(\theta)$.

$$H(\theta)=\begin{bmatrix}\frac{\partial^2 \ln L(\underline{x};\alpha,b,p,q)}{\partial\alpha\,\partial\alpha} & \cdots & \frac{\partial^2 \ln L(\underline{x};\alpha,b,p,q)}{\partial\alpha\,\partial q}\\ \vdots & \ddots & \vdots \\ \frac{\partial^2 \ln L(\underline{x};\alpha,b,p,q)}{\partial q\,\partial\alpha} & \cdots & \frac{\partial^2 \ln L(\underline{x};\alpha,b,p,q)}{\partial q\,\partial q}\end{bmatrix}$$

and the entry of the Hessian matrix is as follows:

- $$\frac{\partial^2 \ln L(\underline{x};\alpha,b,p,q)}{\partial\alpha\,\partial\alpha}=\frac{-n}{\hat{\alpha}^2}-(\hat{p}+\hat{q})\sum_{i=1}^{n}\left[\frac{(\ln(\frac{x_i}{\hat{b}}))^2\,(\frac{x_i}{\hat{b}})^{\hat{\alpha}}\,[1+(\frac{x_i}{\hat{b}})^{\hat{\alpha}}]-(\frac{x_i}{\hat{b}})^{2\hat{\alpha}}\,(\ln(\frac{x_i}{\hat{b}}))^2}{[1+({x_i}/{\hat{b}})^{\hat{\alpha}}]^2}\right] \quad (22)$$
- $$\frac{\partial^2 \ln L(\underline{x};\alpha,b,p,q)}{\partial\alpha\,\partial b}=\frac{-n\hat{p}}{\hat{b}}-(\hat{p}+\hat{q})\sum_{i=1}^{n}\left\{\left[\frac{-\hat{\alpha}\,x_i\left(\frac{x_i}{\hat{b}}\right)^{\hat{\alpha}-1}\ln\left(\frac{x_i}{\hat{b}}\right)-\hat{b}\left(\frac{x_i}{\hat{b}}\right)^{\hat{\alpha}}}{\hat{b}^2}\right]\left(1+(\frac{x_i}{\hat{b}})^{\hat{\alpha}}\right)+\left(\left(\frac{x_i}{\hat{b}}\right)^{\hat{\alpha}}\ln\left(\frac{x_i}{\hat{b}}\right)\right)\left[\frac{\hat{\alpha}\,x_i\left({x_i}/{\hat{b}}\right)^{\hat{\alpha}-1}}{b^2}\right]\right\}:\left(1+(\frac{x_i}{\hat{b}})^{\hat{\alpha}}\right)^2 \quad (23)$$
- $$\frac{\partial^2 \ln L(\underline{x};\alpha,b,p,q)}{\partial\alpha\,\partial p}=\sum_{i=1}^{n}\ln x_i-n\ln\hat{b}-\sum_{i=1}^{n}\frac{\left(\frac{x_i}{\hat{b}}\right)^{\hat{\alpha}}\ln\left(\frac{x_i}{\hat{b}}\right)}{\left(1+({x_i}/{\hat{b}})^{\hat{\alpha}}\right)} \quad (24)$$
- $$\frac{\partial^2 \ln L(\underline{x};\alpha,b,p,q)}{\partial\alpha\,\partial q}=-\sum_{i=1}^{n}\frac{\left(\frac{x_i}{\hat{b}}\right)^{\hat{\alpha}}\ln\left(\frac{x_i}{\hat{b}}\right)}{\left(1+({x_i}/{\hat{b}})^{\hat{\alpha}}\right)} \quad (25)$$
- $$\frac{\partial^2 \ln L(\underline{x};\alpha,b,p,q)}{\partial b\,\partial\alpha}=\frac{-n\hat{p}}{\hat{b}}+(\hat{p}+\hat{q})\sum_{i=1}^{n}\frac{1}{\hat{b}^2}\left\{\frac{x_i\,({x_i}/{\hat{b}})^{\hat{\alpha}-1}}{[1+({x_i}/{\hat{b}})^{\hat{\alpha}}]}+\frac{\hat{\alpha}\,x_i\,({x_i}/{\hat{b}})^{\hat{\alpha}-1}\ln({x_i}/{\hat{b}})}{[1+({x_i}/{\hat{b}})^{\hat{\alpha}}]}-\frac{\hat{\alpha}\,x_i\,({x_i}/{\hat{b}})^{2\hat{\alpha}-1}\ln({x_i}/{\hat{b}})}{[1+({x_i}/{\hat{b}})^{\hat{\alpha}}]^2}\right\} \quad (26)$$
- $$\frac{\partial^2 \ln L(\underline{x};\alpha,b,p,q)}{\partial b\,\partial b}=\frac{n\hat{\alpha}\hat{p}}{\hat{b}^2}+(\hat{p}+\hat{q})\sum_{i=1}^{n}\left\{\frac{\hat{\alpha}^2 x_i^2({x_i}/{\hat{b}})^{2\hat{\alpha}-2}}{\hat{b}^4[1+({x_i}/{\hat{b}})^{\hat{\alpha}}]^2}-\frac{\hat{\alpha}(\hat{\alpha}-1)({x_i}/{\hat{b}})^{\hat{\alpha}-2}x_i^2}{\hat{b}^4[1+({x_i}/{\hat{b}})^{\hat{\alpha}}]}-\frac{2\hat{\alpha}\,x_i({x_i}/{\hat{b}})^{\hat{\alpha}-1}}{\hat{b}^3[1+({x_i}/{\hat{b}})^{\hat{\alpha}}]}\right\} \quad (27)$$
- $$\frac{\partial^2 \ln L(\underline{x};\alpha,b,p,q)}{\partial b\,\partial p}=\frac{-\hat{\alpha}n}{\hat{p}}+\sum_{i=1}^{n}\frac{\hat{\alpha}\,x_i\,({x_i}/{\hat{b}})^{\hat{\alpha}-1}}{\hat{b}^2[1+({x_i}/{\hat{b}})^{\hat{\alpha}}]} \quad (28)$$
- $$\frac{\partial^2 \ln L(\underline{x};\alpha,b,p,q)}{\partial b\,\partial q}=\sum_{i=1}^{n}\frac{\hat{\alpha}\,x_i\,({x_i}/{\hat{b}})^{\hat{\alpha}-1}}{\hat{b}^2[1+({x_i}/{\hat{b}})^{\hat{\alpha}}]} \quad (29)$$
- $$\frac{\partial^2 \ln L(\underline{x};\alpha,b,p,q)}{\partial p\,\partial\alpha}=\sum_{i=1}^{n}\ln x_i-n\ln\hat{b}-\sum_{i=1}^{n}\frac{\left(\frac{x_i}{\hat{b}}\right)^{\hat{\alpha}}\ln\left(\frac{x_i}{\hat{b}}\right)}{\left[1+\left(\frac{x_i}{\hat{b}}\right)^{\hat{\alpha}}\right]} \quad (30)$$
- $$\frac{\partial^2 \ln L(\underline{x};\alpha,b,p,q)}{\partial p\,\partial b}=\frac{-\hat{\alpha}n}{\hat{b}}+\sum_{i=1}^{n}\frac{\hat{\alpha}\,x_i\,({x_i}/{\hat{b}})^{\hat{\alpha}-1}}{\hat{b}^2[1+({x_i}/{\hat{b}})^{\hat{\alpha}}]} \quad (31)$$
- $$\frac{\partial^2 \ln L(\underline{x};\alpha,b,p,q)}{\partial p\,\partial p}=-n\,\psi'(p)+n\,\psi'(p+q) \quad (32)$$
- $$\frac{\partial^2 \ln L(\underline{x};\alpha,b,p,q)}{\partial p\,\partial q}=n\,\psi'(p+q) \quad (33)$$
- $$\frac{\partial^2 \ln L(\underline{x};\alpha,b,p,q)}{\partial q\,\partial\alpha}=-\sum_{i=1}^{n}\frac{\left(\frac{x_i}{\hat{b}}\right)^{\hat{\alpha}}\ln\left(\frac{x_i}{\hat{b}}\right)}{\left[1+\left(\frac{x_i}{\hat{b}}\right)^{\hat{\alpha}}\right]} \quad (34)$$
- $$\frac{\partial^2 \ln L(\underline{x};\alpha,b,p,q)}{\partial q\,\partial b}=\sum_{i=1}^{n}\frac{\hat{\alpha}\,x_i\,({x_i}/{\hat{b}})^{\hat{\alpha}-1}}{\hat{b}^2[1+({x_i}/{\hat{b}})^{\hat{\alpha}}]} \quad (35)$$
- $$\frac{\partial^2 \ln L(\underline{x};\alpha,b,p,q)}{\partial q\,\partial p}=n\,\psi'(p+q) \quad (36)$$
- $$\frac{\partial^2 \ln L(\underline{x};\alpha,b,p,q)}{\partial q\,\partial q}=-n\,\psi'(q)+n\,\psi'(p+q) \quad (37)$$

The iteration process will be conducted by using program *software* R.

3.3 Method of Probability Weighted Moment (PWM)

The PWM method is a modification of the "conventional" method of moments and was first proposed by Hosking et al. [12]. The PWM function of the random variable X with the cumulative distribution function (CDF), F(x) is defined as follows:

$$M_{r,s,t}=E[(X(F))^r\left(F(x)\right)^s\left(1-F(x)\right)^t] \quad (38)$$

Where r, s and t are real number. If s = t = 0 and r is a positive integer, then $M_{r,s,t}$ will become $M_{r,0,0}$ which is a conventional probability moment. Furthemore, the subclass function of PWM is $M_{1,s,t}\,(r=1, s=0,1,2,\dots, t=0,1,2,\dots)$, where X(F) in the inverse of CDF. $M_{1,s,t}$ ca be decomposed into two part as follows:

a) If s=0, then $M_{1,0,t}=E[(X(F)\left(1-F(x)\right)^t]$

where $M_{1,0,t}=\int_0^1[(X(F)\left(1-F(x)\right)^t]\,dt$

b) If t=0, then $M_{1,s,0}=E[(X(F)\left(F(x)\right)^s]$

where $M_{1,s,0}=\int_0^1[(X(F)\left(F(x)\right)^s]\,dt$

By solving M_t we will obtain the estimation of parameters in the form of M_t. The unbiased estimation of M_t is obtained based on order statistic: $x_{(1)}<x_{(2)}<\cdots<x_{(n)}$ from sample of size n and t is positive number, by soving the equation:

$$\widehat{M}_t = \frac{1}{n}\sum_{i=1}^{n-t}\frac{\binom{n-i}{t}}{\binom{n-1}{t}}x_{(i)} = \frac{1}{n}\sum_{i=1}^{n-t}\frac{(n-i)\ldots(n-i-t+1)}{(n-1)\ldots(n-t)}x_{(i)} \quad (39)$$

Next by substitution of M_t by $\widehat{M}_t$ we will have the estimation of parameters of the distribution [16].

The first step is to determine the cumulative distribution function (CDF) of the GB2 distribution and then used to find $M_{r,s,t}$ which is the basis for applying the Probability Weighted Moment method and then determining the estimator of the parameters α, b, p and q . The cumulative distribution function of the GB2 distribution is as follows:

$$F(x) = \frac{1}{B(p,q)}\int_0^z \frac{u^{p-1}}{(1+u)^{p+q}}du \; ; z > 0 \text{ and } z = \left(\frac{x}{b}\right)^{\alpha} \quad (40)$$

The cumulative distribution function of the GB2 distribution is an incomplete beta function, so the inverse of the cumulative function cannot be solved analytically. The inverse of the cumulative distribution function of the GB2 distribution can be found using R software by using the quantile function:

X(F) =qgb2(x, α, b, p, q) (41)

Furthermore, to find the weighted probability form of the GB2 distribution, the numerical integral technique Richardson method is used. After obtaining the population weighted probability moment form, the next step is to find the weighted probability moment form for the sample. To get the estimated value of the parameters, it is assumed that the population weighted probability moment is equal to the sample weighted probability moment such that the difference is zero. The estimated value will be obtained when the difference gets closer to zero with a predetermined error value of 0.00001.

3.3 Iteration of Richardson Method

The Richardson method is a method that uses two estimates of an integral to calculate a more accurate third estimator. Estimates and errors associated with multiple-application trapezoidal rules can be generally described as follows:

I = I(h) + E(h) (42)

where I is the true value of the integral, I(h) is the estimation by using trapezoidal rule with an n segments and h = (b-a) / n and E(h) is a truncation error. If we make two different estimates using the step width h_1 and h_2 we get the following equation:

$I(h_1) + E(h_1) = I(h_2) + E(h_2)$ (43)

error from the trapezoidal rule multi-application can be estimated as

$$E \cong -\frac{(b-a)^3}{12\,n^2}f''$$

Since the value n=(b-a)/h so that the equation above can be written as:

$$E \cong -\frac{(b-a)}{12}h^2 f'' \quad (44)$$

If it is assumed that f" is constant which means that it is not affected by the width of the steps, then the value of E can be used to determine the ratio of the two errors

$$\frac{E(h_1)}{E(h_2)} \cong \frac{{h_1}^2}{{h_2}^2}$$

This calculation has an important effect in removing f" from the calculation. Furthermore, we can change the ratio equation above

$$E(h_1) \cong E(h_2)\left(\frac{h_1}{h_2}\right)^2 \quad (45)$$

Then we substitute the equation (45) into (43)

$$I(h_1) + E(h_2)\left(\frac{h_1}{h_2}\right)^2 = I(h_2) + E(h_2)$$

$$E(h_2) \cong \frac{I(h_1)-I(h_2)}{1-\left(\frac{h_1}{h_2}\right)^2} \quad (46)$$

As a result, we have developed an estimate of truncation error in terms of estimating the integral and width of the step. This prediction can be substituted into

$I = I(h_2) + E(h_2)$ (47)

To produce an integral estimation that has been improved:

$$I \cong I(h_2) + \frac{I(h_2)-I(h_1)}{\left(\frac{h_1}{h_2}\right)^2-1} \quad (48)$$

This shows that the *error* from the estimation is (h^4) . For the special case where the interval ($h_2 = h_1/2$), the equation become:

$$I \cong I(h_2) + \frac{I(h_2)-I(h_1)}{2^2-1}$$

or

$$I \cong \frac{4}{3}I(h_2) - \frac{1}{3}I(h_1) \quad (49)$$

This approach is a subset of a more general method of combining integrals to produce an error that is $O(h^4)$ which can then be used to find even smaller errors namely $O(h^6)$, $O(h^8)$, and so on [17].

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3.4 Characteristics of Estimation

3.4.1 Unbiased

One of the properties that must be possessed by a good parameter estimator of a distribution is unbiasedness. An estimator $U(\boldsymbol{X}) = U(X_1, X_2, \dots, X_n)$ satisfied

$$E\big(U(\boldsymbol{X})\big) = g(\theta)$$

Is called unbiased estimator of $g(\theta)$ [14].

3.4.2 Minimum variance

An estimator is said to be a good estimator if in addition to having an unbiased property, it also has a minimum variance. Bain & Engelhardt stated that for example U(X) is an unbiased estimator for g (θ) with minimum variance [18], if for any other unbiased estimator $U_1(X)$ for g(θ), Var (U(X)) ≤Var (U_1(X)) for each θ∈Ω, where

$$Var\big(U_1(\boldsymbol{X})\big) \geq \frac{\left(\frac{\partial}{\partial\theta} g(\boldsymbol{\theta})\right)^2}{n\,.\,E\left[\frac{\partial}{\partial\theta} ln\, f(\boldsymbol{X};\boldsymbol{\theta})\right]^2}.$$

is the inequality of the minimum variance. According to Hogg and Craig [19], $E\left[\frac{\partial}{\partial\theta} ln\, f(\boldsymbol{X};\boldsymbol{\theta})\right]^2$ is called Fisher Information and denoted by $I(\theta)$. Such that the equation can be simplify as:

$$Var\big(U_1(\boldsymbol{X})\big) \geq \frac{[g'(\theta)]^2}{nI(\theta)}$$

and is called Cramer-Rao Lower Bound [9].

The Fisher Information Matrix of the GB 2 distribution is as follows:

$$I_n(\alpha, b, p, q) = \begin{bmatrix} \frac{n}{\alpha^2} & 0 & 0 & 0 \\ 0 & \frac{n\alpha^2 pq}{b^2(p+q+1)} & \frac{n\alpha q}{b(p+q)} & -\frac{n\alpha p}{b\,(p+q)} \\ 0 & \frac{n\alpha q}{b(p+q)} & n\psi'(p) - n\psi'(p+q) & -n\psi'(p+q) \\ 0 & -\frac{n\alpha p}{b\,(p+q)} & -n\psi'(p+q) & n\psi'(q) - n\psi'(p+q) \end{bmatrix} \quad (50)$$

The matrix is nonsingular, so that it has inverse. So that the Cramer-Rao inequality is as the following form:

$$Var\big(\hat{\alpha}, \hat{b}, \hat{p}, \hat{q}\big) \geq I_n^{\;-1}(\alpha, b, p, q) \quad (51)$$

Based on the previous discussion, it is known that the parameter estimator (α, b, p, q) of the GB2 distribution is an unbiased estimator and based on the Cramer-Rao lower bound, the variance of the parameter estimator (α, b, p, q) attains the limit under Cramer-Rao lower bound, so that the estimator is the estimator with minimum variance.

4. SIMULATION AND APPLICATION

4.1 Simulation

In this section, a simulation will be conducted to estimate the parameters a, b, p and q. After the estimated parameter values are obtained, a simulation will then be performed to evaluate the properties of each estimator. A good estimator is an estimator that has the properties: an unbiasedness, efficient, and consistent. From the simulation by using the method of moment, Maximum Likelihood estimation, and Probability Weighted Moment the results will be compared to find the best estimator based on unbiasedness and minimum mean square error (MSE). The parameter values that will be assumed are obtained from the export data of four or more motorized vehicles according to the main destination countries from Indonesia in 2012-2017. The data are obtained from the Central Bureau of Statistics Indonesia with parameter values α = 1, b = 153.07, p = 6.3066 and q = 2.3 where the parameter values are obtained based on the easy-fit output.

The simulation stages are started by generating data from generalized Beta of the Second Kind distribution based on predetermined parameters values each with a sample size of 25, 50, 100, 200, 500 and 1000, respectively. In addition to this simulation scenario is also made based on the changes in scale parameter values, by setting the scale parameter of 25, 50, 75, 100, 125, 150, 175 and 200 sequentially. Then the estimating parameters is carried out on each sample size by the method of moment, Maximum Likelihood, and Probability Weighted Moment with 100 replications. Bias and Mean Square Error are compared to evaluate which one is the best method base on the minimum biased and minimum variance.

The Bias and MSE values presented in the graph are the average values of each parameter. The sample size, the unbiasedness and the mean square error (MSE) arethat will be used for comparison and the effect of the sample size on each estimator. The best estimator is the estimator which has a bias around zero which means that the estimation results are very close to the parameter values. Mean Square Error is a good measure of estimation because there is information about bias and variance in it, so that a good estimator is the estimator that has the smallest MSE. The following graph is the result of a simulation estimate for each parameter that has been carried out with a scenario as discussed earlier.

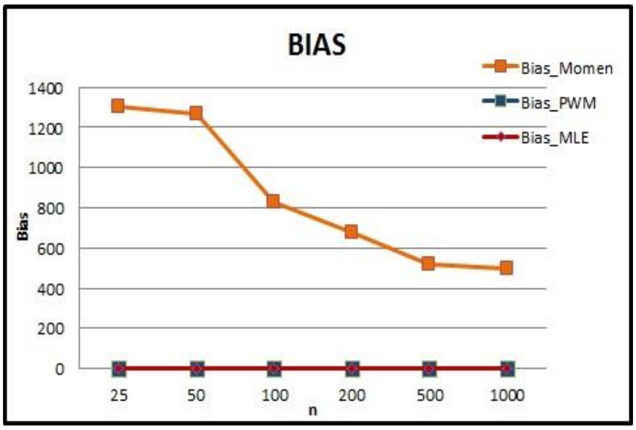

Figure-1. The average bias of estimation by method of moment, MLE and PWM for all values of n.

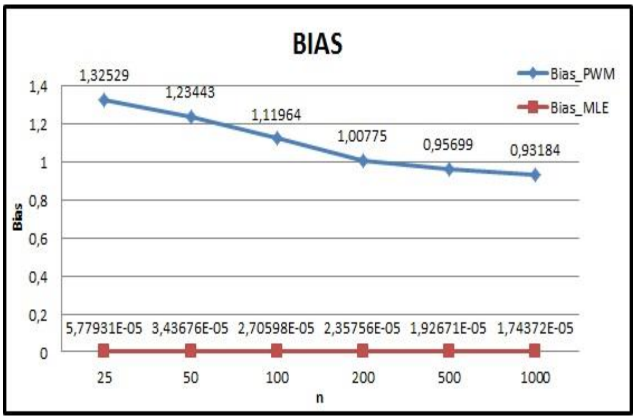

Figure-2. The average bias of estimation by methods of MLE and PWM for all values of n.

Figure-1 and Figure-2 are the average values of the estimation for bias by using the methods of moment, PWM and MLE. Based on Figure-1. the average bias value of the PWM and MLE methods looks very close each others and coincide because the average values of bias for the method of moment are overposition when compared with the results of the methods of PWM and MLE. So that the estimation with the method of moment can said to be biased compared to the other two methods.

To be able to see the difference in the value of the bias between the methods of PWM and MLE more clearly, it can be seen in Figure-2. The result is the bias value of the MLE estimator is around zero for any sample size. So these results indicate that the MLE estimator for each parameter is good and unbiased estimator. While the PWM estimator shows the average value of bias which gets smaller when the sample size is getting larger, because the bias value is positive then there is overestimating for the PWM estimator for each parameter. Information on the MSE values from the estimation of the moment method, PWM and MLE is presented in Figure-3 and Figure-4.

Based on Figure-3, the average value of the MSE method of moment is higher than the methods of PWM and MLE for each value of n and decreases with increasing value of n.

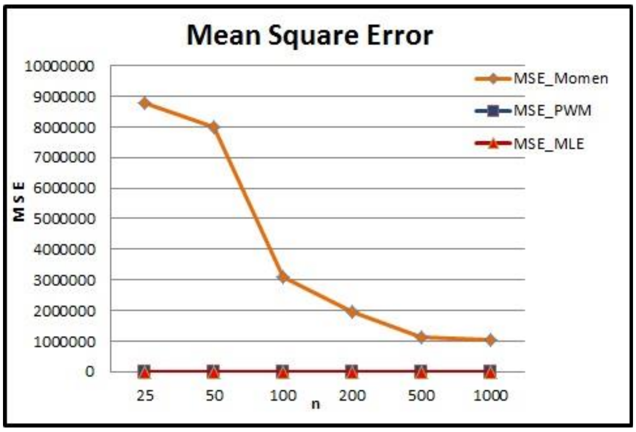

Figure-3. The average MSE estimation by methods of moment, MLE and PWM for all values of n.

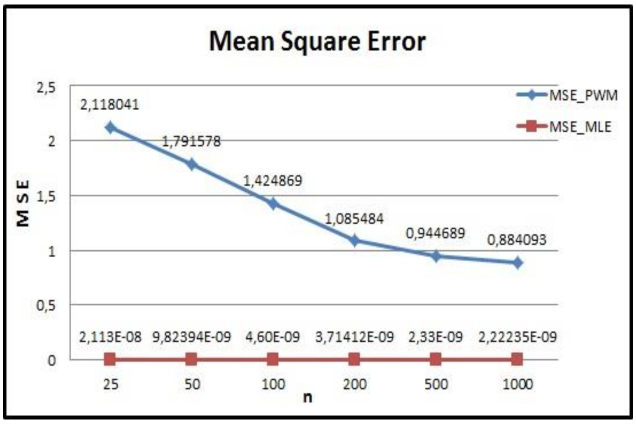

Figure-4. The average MSE estimation by methods of, MLE and PWM for all values of n.

In Figure-3, the average value of MSE in the methods of PWM and MLE looks coincide and seems to be in a line, this is because the mean value of the MSE method of moment is very large and overposition compared to the average value of the MSE of the other two methods. To be able to see the mean value of the estimated MSE results by using the methods of PWM and the MLE more clearly can be seen in Figure-4. The results are the average value of the MSE estimator for the method of PWM shows the average value of the MSE that gets smaller as the sample size gets larger. In addition, the mean value of MSE also decreased dramatically when n = 50 to n = 100. In addition, the estimated MSE value using PWM is closer to zero compared to the method of moment which gives a high MSE value for each n. The average MSE estimator value of MLE is around zero for any sample size used where the method of MLE is indeed very good when the size of data is large. This shows that the MLE method gives better results compared to the methods of moment and PWM.

In this study was also conducted simulations with several scenarios of the scale parameter values to

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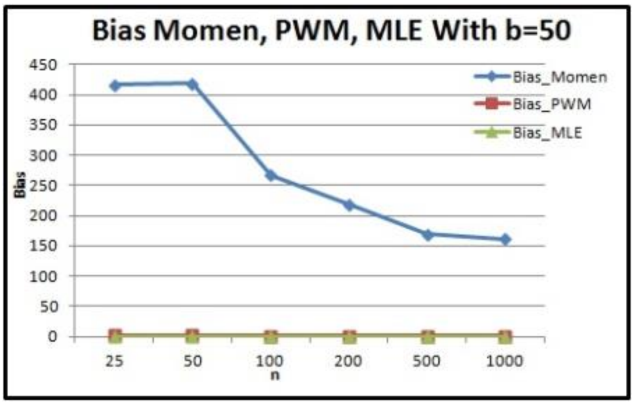
determine the effect of the scale parameters of the beta of the second kind distribution. The estimation results using the methods of moment, MLE and PWM, based on Figure-5 which contains graphs for the average bias values with several scale parameter values shows that each predetermined scale parameter value, the method of MLE gives a smaller average bias compared to the method of PWM and the method of moment where the average value of this bias decreases with increasing number of n. The bias value of the moment method gives the greatest value compared to the other two methods for each predetermined scale parameter value (Figure-5a, b, c, d, e, f, g, h).

Based on Figure-6 which is containing graphs for the average value of MSE based on the values of some predetermined shape parameters. The average value of the MSE method of moment is higher than the methods of PWM and MLE for each values of n and decreases with increasing values of n. In Figure-5 (a, b, c, d, e, f, g, h) the average value of the MSE methods of PWM and MLE looks coincide and like being in a line, this is because the mean value of the MSE method of moment is very large and overposition compared to the average MSE values of the other two methods. To be able to see the average value of the estimated MSE using the methods of PWM and MLE more clearly, it can be seen in Figure-6 (i, j, k, l, m, n, o, p). The results obtained from the average value of MSE estimator for method of PWM shows the value gets smaller when the sample size is getting larger. Besides the average value of MSE also decreases dramatically when n = 50 to n = 100.

In addition, the estimated MSE value by using method of PWM is closer to zero compared to the method of moment. The average value of the MSE estimator for the method of MLE is around zero for any sample size used where the MLE method is very good when the size of the data is large. This shows that the MLE method gives better results compared to the methods of moment and PWM. Based on Figure-5 and Figure-6 also show that for each values of n, the bias and MSE values decrease for each increase in the scale parameter values. Where the MLE method is the best method for estimating the parameters of beta of the second kind distribution, because it gives the lowest MSE and bias values compared to the methods of PWM and moment.

(a)



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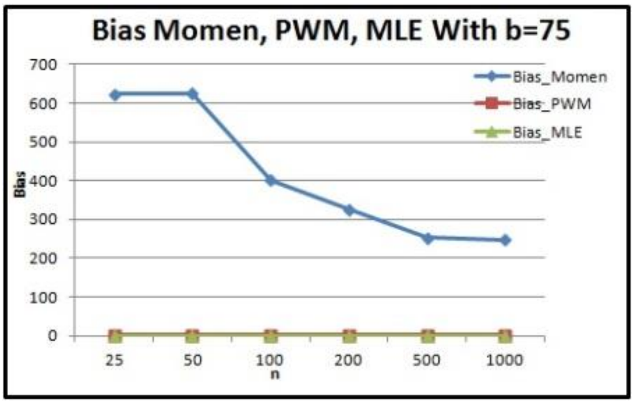

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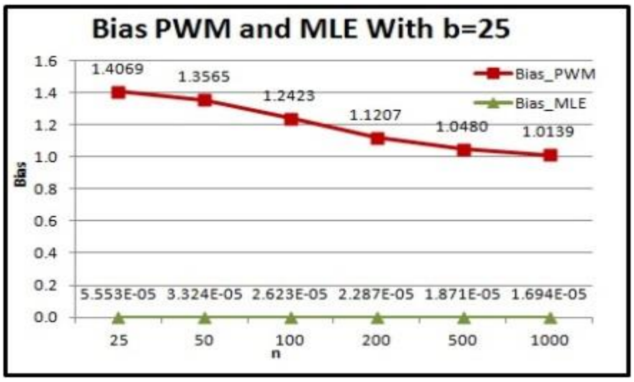

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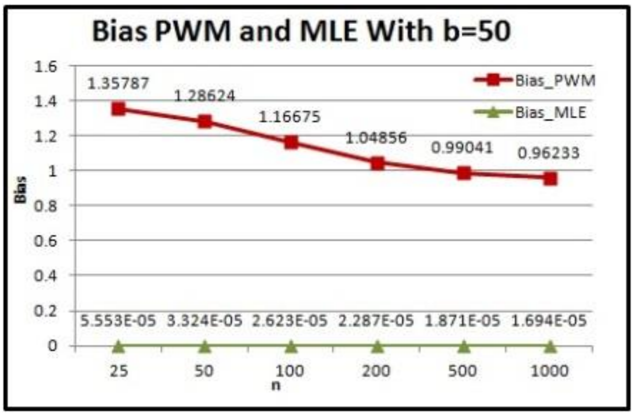

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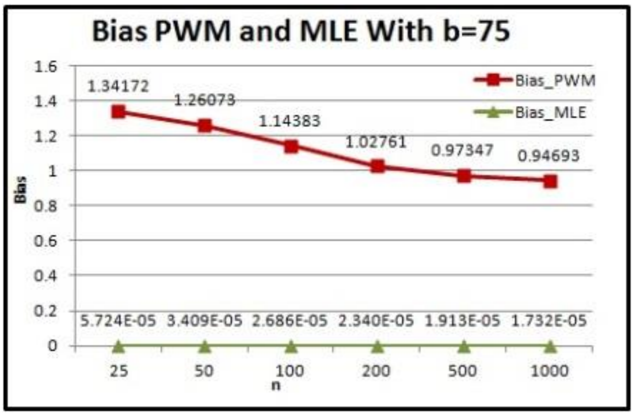

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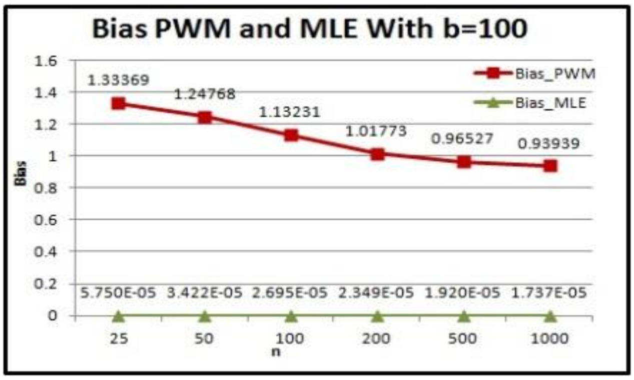

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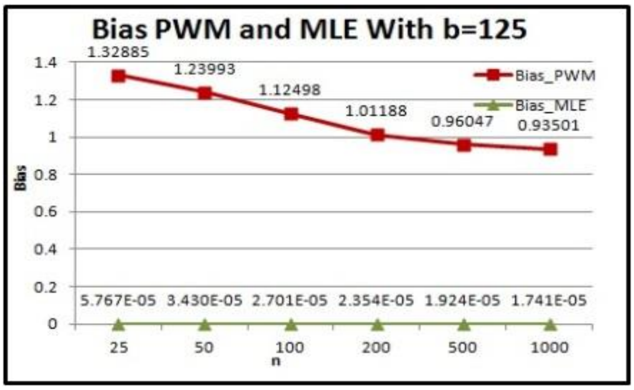

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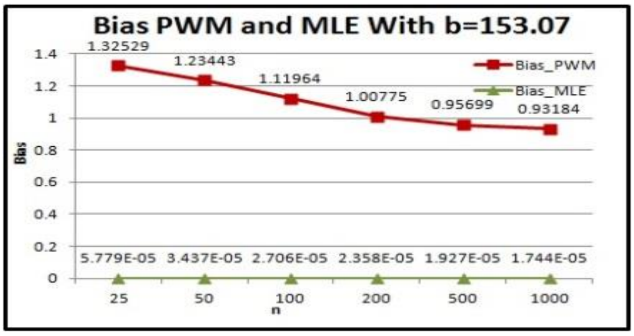

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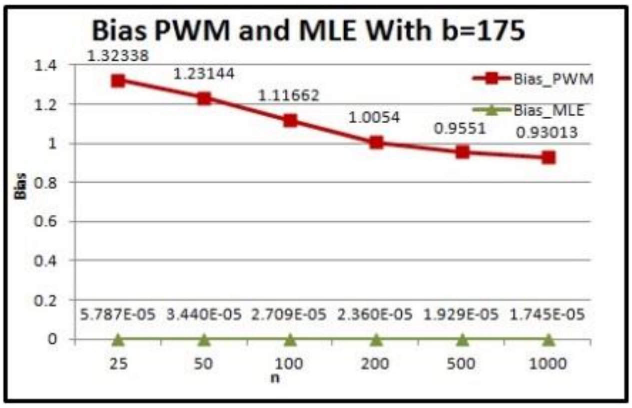

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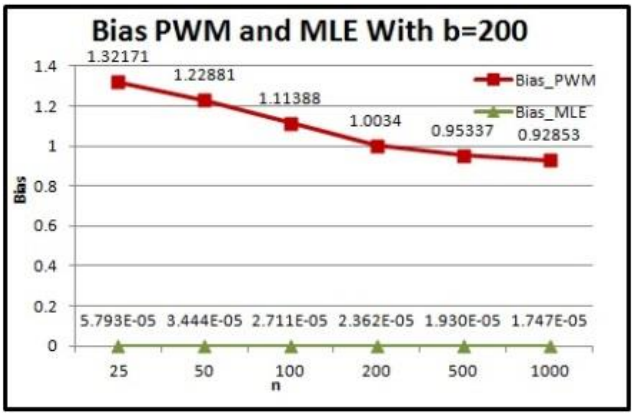

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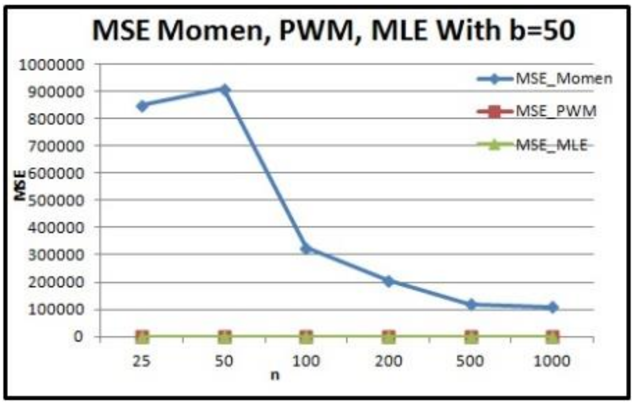
Figure-5. The average bias estimation by methods of moment, MLE and PWM for some different values of parameter b.

(a)



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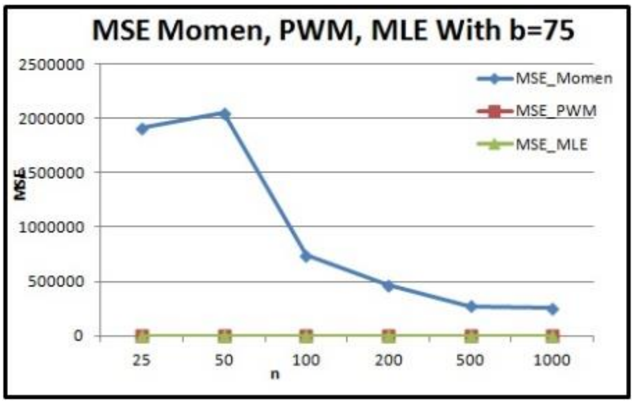

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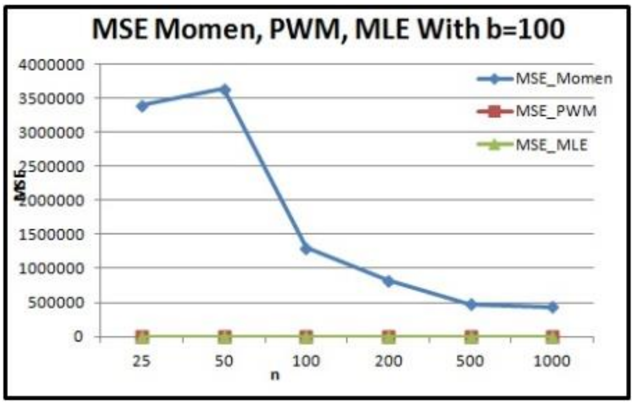

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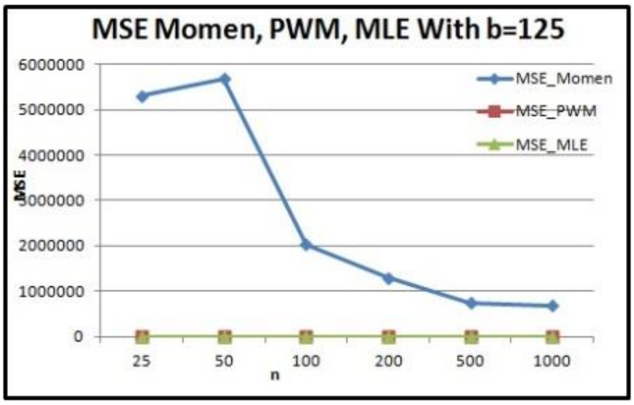

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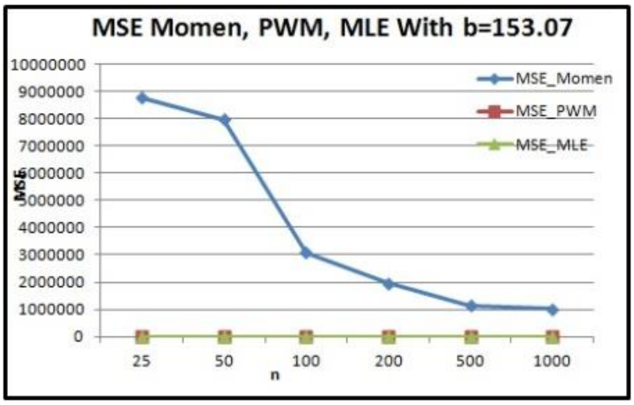

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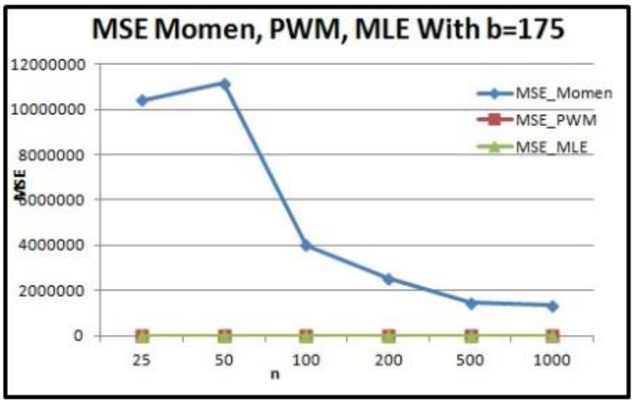

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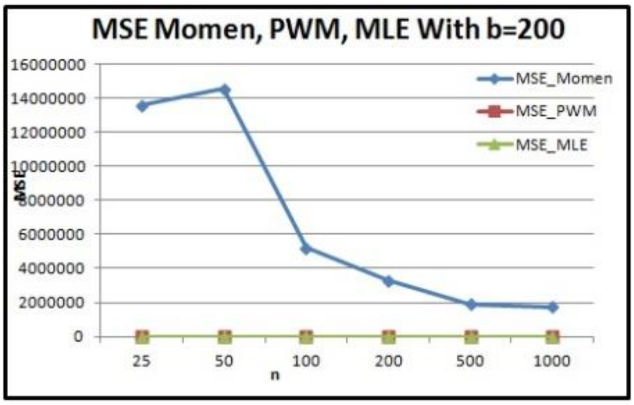

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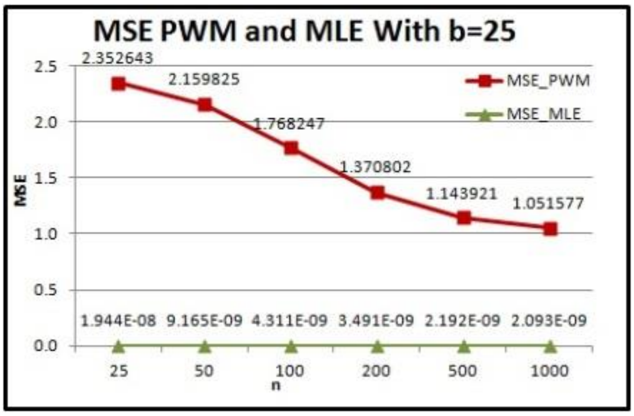

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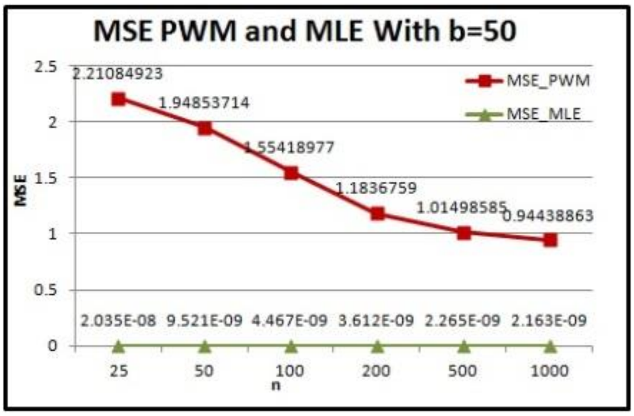

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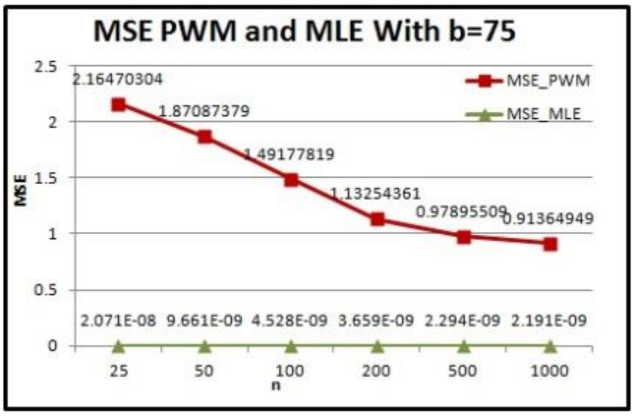

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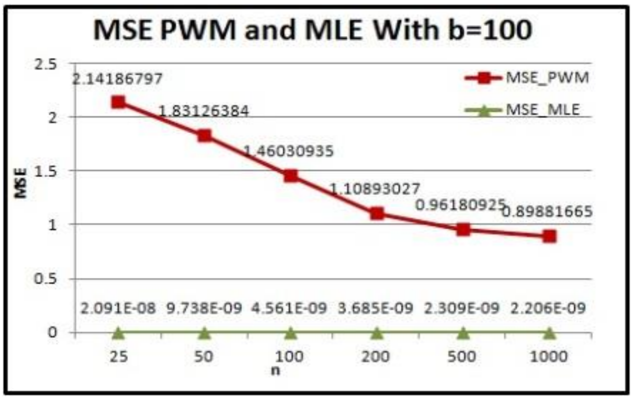

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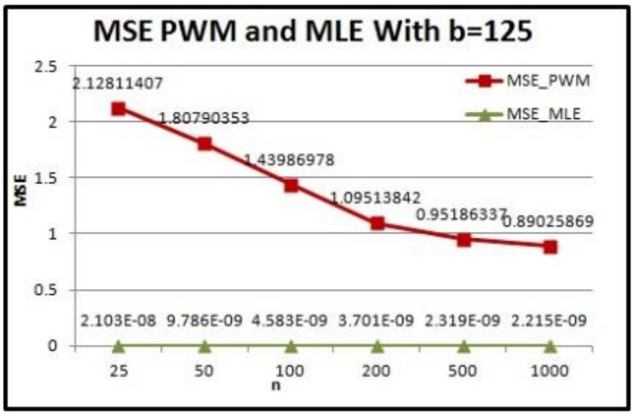

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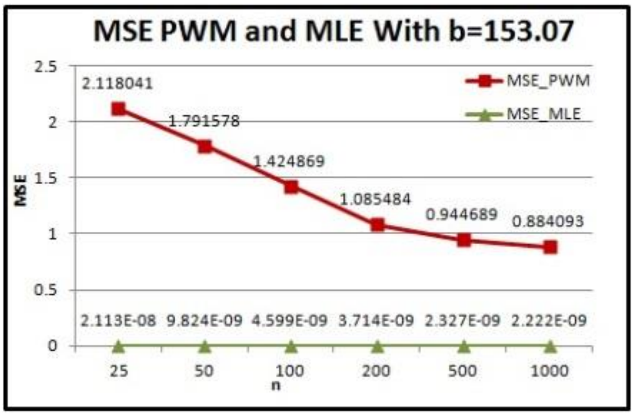

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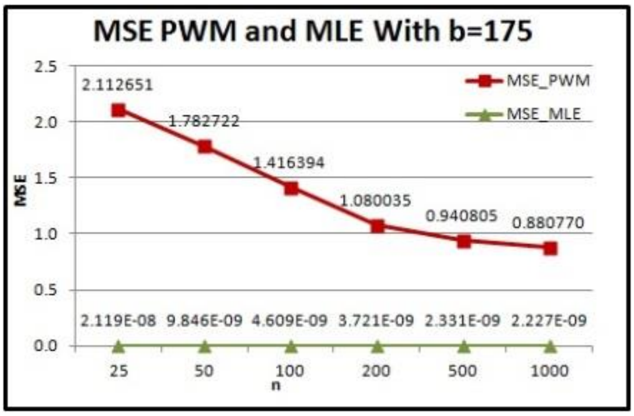

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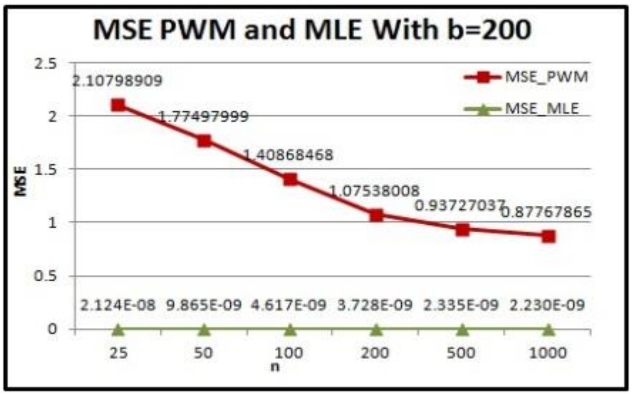

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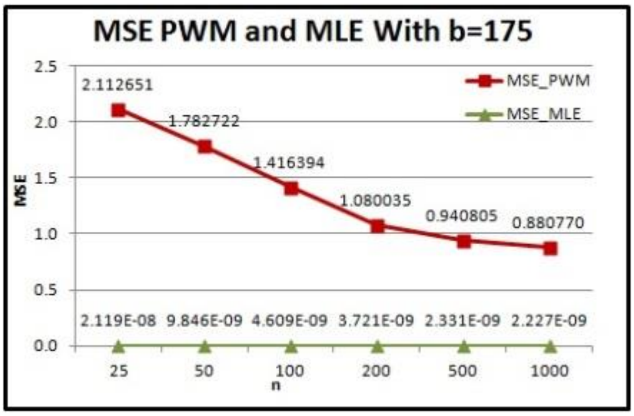

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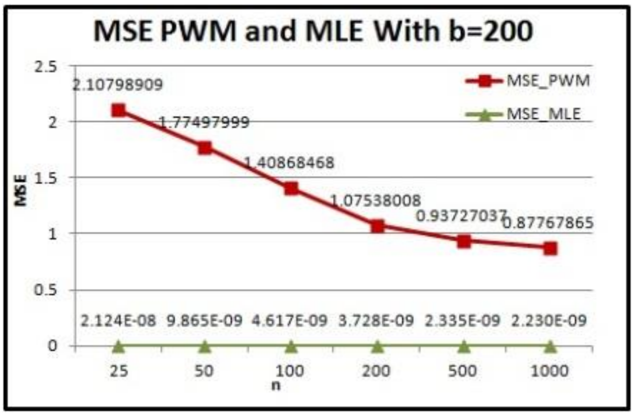

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Figure-6. The average MSE estimation by methods of moment, MLE and PWM for some different values of parameter b.

5. CONCLUSIONS

Based on the results and discussion, it can be concluded that the estimation of Generalized Beta 2 distribution parameters using Maximum Likelihood Estimation and Probability Weighted Moment produces estimators that cannot be solved analytically, so it needs to be solved numerically. From the three estimation methods of moment, MLE and PWM used, it can be concluded that the best method in estimating the parameters of Generalized Beta 2 distribution is the method of Maximum Likelihood Estimation (MLE) because it has the smallest mean square error value where the greater the sample size used, the estimated parameter value will get closer to the actual parameter. When viewed based on changes in the scale parameters that have been set estimates with the Maximum Likelihood Estimation method also gives the smallest average bias and MSE values. In addition, the PWM method can also be used as an alternative estimation parameters of generalized beta of the second kind distribution parameters because it has a bias value and MSE that is much smaller than the moment method.

ACKNOWLEDGEMENT

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