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ON SOME PETERSEN GRAPHS HAVING LOCATING CHROMATIC NUMBER FOUR OR FIVE

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Abstract

In this paper, we determine some Petersen graphs having locating chromatic number four or five. Moreover, we give some conjecture to characterize some Petersen graphs having locating chromatic number four.

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1. Introduction

The concept of locating chromatic number of a graph is the combination between the partition dimension [5] and vertex coloring of a graph. The following definition of locating chromatic number of a graph is taken from [6].

Let $G = (V, E)$ be a connected graph. Let c be a proper coloring of G using the colors $1, 2, ..., k$ for some positive integer k. Let $\Pi = \{C_1, C_2, ..., C_k\}$ be a partition of $V(G)$, where C_i is a set of vertices colored i, for $i \in [1, k]$. The color code $c_\Pi(v)$ of vertex v in G is the ordered k-tuple $(d(v, C_1), ..., d(v, C_k))$, where $d(v, C_i) = \min\{d(v, x) | x \in C_i\}$ for $i \in [1, k]$. For every pair of vertices u and v, if $c_\Pi(u) \neq c_\Pi(v)$, then c is called a *locating coloring* of G using k colors. The locating chromatic number, denoted by $\chi_L(G)$, is the smallest k such that G has a locating coloring.

The following theorem taken from [6] is a fundamental theorem about the locating chromatic number of a graph. The neighborhood of a vertex u in a connected graph G, denoted by $N(u)$, is the set of vertices adjacent to u.

Theorem 1.1 [6]**.** *Let c be a locating coloring in a connected graph G. If s and t are distinct vertices of G such that* $d(s, u) = d(t, u)$ *for all* $u \in V(G) - \{s, t\}$, *then* $c(s) \neq c(t)$. *In particular, if s and t are non-adjacent vertices of G such that* $N(s) = N(t)$, *then* $c(s) \neq c(t)$.

Theorem 1.2 [6]**.** *The locating chromatic number of a cycle* C_n *is* 3 *for odd n and* 4 *for otherwise*.

Determining the locating chromatic number of a graph is an interesting topic because there is no algorithm that can be used to determine the locating chromatic number for any graph. Chartrand et al. [6] discussed the locating chromatic number for some classes of graphs such as paths, cycles, bipartite graphs, caterpillars, double stars, and certain trees. Moreover, Asmiati et al.

[1, 2] determined the locating chromatic number for amalgamation of stars and firecracker graphs.

In 2003, Chartrand et al. [6] characterized a graph having locating chromatic number $(n-1)$ or $(n-2)$. Asmiati and Baskoro [3] characterized a graph containing cycles with locating chromatic number three. Next, in 2013, Baskoro and Asmiati [4] characterized the trees with locating chromatic number three. In this paper, we determine the locating chromatic number of some Petersen graphs having locating chromatic number four or five.

The following definition of a Petersen graph is taken from [7]. Let $\{u_1, u_2, ..., u_n\}$ be some vertices on the outer cycle and $\{v_1, v_2, ..., v_n\}$ be some vertices on the inner cycle, for $n \geq 3.$ The Petersen graph, denoted by $P_{n,k}$, $n \geq 3,$ $1 \leq k \leq \left\lfloor \frac{n-1}{2} \right\rfloor,$ $1 \leq i \leq n$ is a graph that has $2n$ vertices $\{u_i\} \cup \{v_i\},$ and edges $\{u_i u_{i+1}\} \cup \{v_i v_{i+k}\} \cup \{u_i v_i\}.$

In this paper, we determine some Petersen graphs having locating chromatic number four or five. Moreover, we give some conjecture to characterize some Petersen graphs having locating chromatic number four.

2. Main Results

In this section, we determine Petersen graph $P_{n,1},$ $n \geq 3$ having locating chromatic number four or five. Besides, we discuss Petersen graphs $P_{4,2}$ and $P_{5,2}$ having locating chromatic number 4 or 5, respectively.

Theorem 2.1. *The locating chromatic number of Petersen graph* $P_{n,1}$ *is* 4 *for odd* $n \geq 3$ *or* 5 *for even* $n \geq 4.$

Proof. To prove this theorem, we consider the following cases:

Case 1. $\chi_L(P_{n,1}) = 4,$ for odd $n \geq 3.$

First, we determine the lower bound of $\chi_L(P_{n,1})$ for odd $n \geq 3$. Because the Petersen graph $P_{n,1}$ contains some even cycles, then by Theorem 1.2, $\chi_L(P_{n,1}) \geq 4$. Thus, $\chi_L(P_{n,1}) \geq 4$ for odd $n \geq 3$.

Let c be a coloring of Petersen graph $P_{n,1}$, for odd $n \geq 3$. We make the partition of the vertices of $V(P_{n,1}) : C_1 = \{u_1\}$; $C_2 = \{u_{2j}, v_{2j-1}\}$; $C_3 = \{u_{2j+1}, v_{2j}\}$; $C_4 = \{v_n\}$, for $j > 0$. Therefore, the color codes of all the vertices of G are:

(a) $c_\Pi(u_1) = (0, 1, 1, 2)$.

(b) $C_2 = \{u_{2j}, v_{2j-1}\}$. Let u_i, $2 \leq i \leq n-1$; $i = 2j$; $1 \leq j \leq \left\lfloor \frac{n}{2} \right\rfloor$ and v_i, $1 \leq i \leq n-2$; $i = 2j-1$; $1 \leq j \leq \left\lfloor \frac{n}{2} \right\rfloor$.

For $i < \left\lceil \frac{n}{2} \right\rceil$, we define

$$c_\Pi(u_i) = (i-1, 0, 1, i+1),$$

$$c_\Pi(v_i) = (i, 0, 1, i).$$

For $i = \left\lceil \frac{n}{2} \right\rceil$, we define

$$c_\Pi(u_i) = c_\Pi(u_{n-2j+1}) = (i-1, 0, 1, 2j),$$

$$c_\Pi(v_i) = c_\Pi(v_{n-2j}) = (i, 0, 1, 2j).$$

For $i > \left\lceil \frac{n}{2} \right\rceil$,

$$c_\Pi(u_i) = c_\Pi(u_{n-2j+1}) = (2j, 0, 1, 2j),$$

$$c_\Pi(v_i) = c_\Pi(v_{n-2j}) = (2j+2, 0, 1, 2j).$$

(c) $C_3 = \{u_{2j+1}, v_{2j}\}$. Let u_i, $3 \le i \le n$; $i = 2j + 1$; $1 \le j \le \left\lceil \frac{n}{2} \right\rceil$ and v_i, $2 \le i \le n - 1$; $i = 2j$; $1 \le j \le \left\lfloor \frac{n}{2} \right\rfloor$.

For $i < \left\lceil \frac{n}{2} \right\rceil$,

$$c_\Pi(u_i) = (i - 1, 1, 0, i + 1),$$

$$c_\Pi(v_i) = (i, 1, 0, i).$$

For $i = \left\lceil \frac{n}{2} \right\rceil$,

$$c_\Pi(u_i) = c_\Pi(u_{n-2j+2}) = (i - 1, 0, 1, 2j - 1),$$

$$c_\Pi(v_i) = c_\Pi(v_{n-2j+1}) = (i, 1, 0, 2j - 1).$$

For $i > \left\lceil \frac{n}{2} \right\rceil$,

$$c_\Pi(u_i) = c_\Pi(u_{n-2j+2}) = (2j - 1, 1, 0, 2j - 1),$$

$$c_\Pi(v_i) = c_\Pi(v_{n-2j+1}) = (2j + 1, 1, 0, 2j - 1).$$

(d) $c_\Pi(v_n) = (2, 1, 1, 0)$.

Since the color codes of each vertex in $V(P_{n,1})$ is different, c is a locating coloring of $P_{n,1}$ for odd $n \ge 3$. Therefore, $\chi_L(P_{n,1}) \le 4$.

Case 2. $\chi_L(P_{n,1}) = 5$, for even $n \ge 4$.

The Petersen graph $P_{n,1}$, for even $n \ge 4$, contains some even cycles. Then, by Theorem 1.2, $\chi_L(P_{n,1}) \ge 4$. Suppose that c is a locating coloring of $P_{n,1}$, for even $n \ge 4$. Let $C_1 = \{u_1\}$; $C_2 = \{u_{2i}, v_{2i-1}\}$; $C_3 = \{u_{2i+1}, v_{2i}\}$ and $C_4 = \{u_n\}$ for $i > 0$. Then there are two vertices with the same color codes, $c_{\Pi}(u_2) = c_{\Pi}(v_1)$, a contradiction. Thus, we need at least five colors. Therefore, $\chi_L(P_{n,1}) \geq 5$ for even $n \geq 4$.

Next, we determine the upper bound of $\chi_L(P_{n,1})$ for even $n \geq 4$. We assign the vertices like this:

$$C_1 = \{u_1\};\ C_2 = \{u_{2j}, v_{2j-1}\};\ C_3 = \{u_{2j+1}, v_{2j}\};\ C_4 = \{u_n\};\ C_5 = \{v_n\}.$$

So, we have the color codes of vertices of *G* as follows:

(a) $c_{\Pi}(u_1) = (0, 1, 2, 1, 2)$.

(b) $C_2 = \{u_{2j}, v_{2j-1}\}$. Let $u_i, 1 \leq i \leq n-2; i = 2j; 1 \leq j \leq \frac{n}{2} - 1$ and $v_i, 1 \leq i \leq n-1; i = 2j-1; 1 \leq j \leq \frac{n}{2}$.

For $i \leq \left\lceil \frac{n}{2} \right\rceil$,

$$c_{\Pi}(u_i) = (i-1, 0, 1, i, i+1),$$

$$c_{\Pi}(v_i) = (i, 0, 1, i+1, i).$$

For $i > \left\lceil \frac{n}{2} \right\rceil$,

$$c_{\Pi}(u_i) = c_{\Pi}(u_{n-2j}) = (2j+1, 0, 1, 2j, 2j+1),$$

$$c_{\Pi}(v_i) = c_{\Pi}(v_{n-2j+1}) = (2j+1, 0, 1, 2j, 2j-1).$$

(c) $C_3 = \{u_{2j+1}, v_{2j}\}$. Let $u_i, 1 \leq i \leq n-1; i = 2j+1; 1 \leq j \leq \frac{n}{2} - 1$ and $v_i, 1 \leq i \leq n-2; i = 2j; 1 \leq j \leq \frac{n}{2} - 1$.

For $i \leq \left\lceil \frac{n}{2} \right\rceil$,

$$c_{\Pi}(u_i) = (i-1, 1, 0, i, i+1),$$

$$c_{\Pi}(v_i) = (i,\ 1,\ 0,\ i+1,\ i).$$

For $i > \left\lceil \frac{n}{2} \right\rceil$,

$$c_{\Pi}(u_i) = c_{\Pi}(u_{n-2j+1}) = (2j,\ 1,\ 0,\ 2j-1,\ 2j),$$

$$c_{\Pi}(v_i) = c_{\Pi}(v_{n-2j}) = (2j+2,\ 1,\ 0,\ 2j+1,\ 2j).$$

(d) $C_4 = \{u_n\}$, then $c_{\Pi}(u_n) = (1,\ 2,\ 1,\ 0,\ 1)$.

(e) $c_{\Pi}(v_n) = (2,\ 1,\ 2,\ 1,\ 0)$.

Since all the vertices have different color codes, c is a locating coloring of Petersen graphs $P_{n,1}$ for even $n \geq 4$. Therefore, we have $\chi_L(P_{n,1}) = 5$, for even $n \geq 4$. □

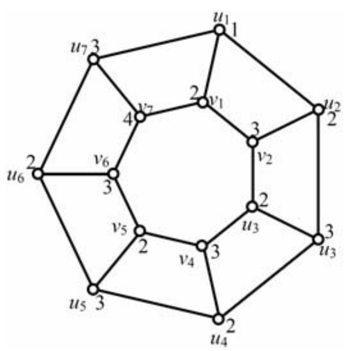

Figure 1. A minimum locating coloring of $P_{7,1}$.

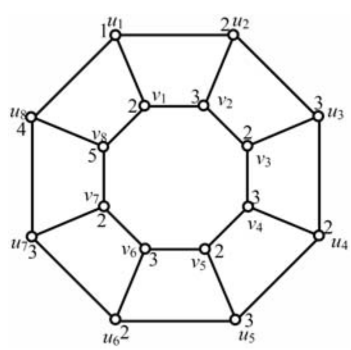

Figure 2. A minimum locating coloring of $P_{8,1}$.

Theorem 2.2. $\chi_L(P_{4,2}) = 4.$

Proof. Since Petersen graph $P_{4,2}$ containing some even cycles, by Theorem 2.1, $\chi_L(P_{4,2}) \geq 4.$ Next, we determine the upper bound of $\chi_L(P_{4,2}).$ Let c be a coloring of Petersen graph $P_{4,2}$ using 4 colors, by assigning the colors like this: $c(u_1) = 1,$ $c(u_2) = 2,$ $c(u_3) = 3,$ $c(u_4) = 4,$ $c(v_1) = 2,$ $c(v_2) = 4,$ $c(v_3) = 1,$ $c(v_4) = 3.$ The color codes of the vertices:

$$c_\pi(u_1) = \{0, 1, 2, 1\};\ c_\pi(u_2) = \{1, 0, 1, 1\};\ c_\pi(u_3) = \{1, 1, 0, 1\};$$

$$c_\pi(u_4) = \{1, 2, 1, 0\};\ c_\pi(v_1) = \{1, 0, 2, 2\};\ c_\pi(v_2) = \{2, 1, 1, 0\};$$

$$c_\pi(v_3) = \{0, 1, 1, 2\};\ c_\pi(v_4) = \{2, 2, 0, 1\}.$$

Since all the vertices have distinct color codes, c is a locating coloring of $P_{4,2},$ so $\chi_L(P_{4,2}) \leq 4.$ □

Theorem 2.3. $\chi_L(P_{5,2}) = 5.$

Proof. Since Petersen graph $P_{5,2}$ containing some even cycles, by Theorem 2.1, $\chi_L(P_{5,2}) \geq 4.$ Suppose that the locating chromatic number of $P_{5,2}$ is 4. Without loss of generality, we assign $c(u_i) = 1;$ $c(u_i) = 2$ for $i = 2, 4;$ $c(u_i) = 3$ for $i = 3, 5$ and $c(v_1) = 2.$ Then $c(v_3) = 1$ or 4. If $c(v_3) = 1,$ then $c(v_5) = 2$ or 4. Set $c(v_2) = 4$ and $c(v_4) = 3,$ then $c_\Pi(u_3) = c_\Pi(u_5),$ a contradiction. So, we need at least 5 colors. As a result, $\chi_L(P_{5,2}) \geq 5.$

Next, we determine the upper bound of $\chi_L(P_{5,2}).$ Let c be a coloring of vertices of $P_{5,2}$ such that the color classes are as follows:

$$C_1 = \{u_1\};\ C_2 = \{u_2, u_4, v_1, v_5\};\ C_3 = \{u_3, u_5, v_2\};$$

$$C_4 = \{v_3\};\ C_5 = \{v_4\}.$$

So, we have the color codes of vertices are:

$c_\Pi(u_1) = (0, 1, 1, 2, 2)$; $c_\Pi(u_2) = (1, 0, 1, 2, 2)$; $c_\Pi(u_4) = (2, 0, 1, 2, 1)$;

$c_\Pi(v_1) = (1, 0, 2, 1, 1)$; $c_\Pi(v_5) = (2, 0, 1, 1, 2)$; $c_\Pi(u_3) = (2, 1, 0, 1, 2)$;

$c_\Pi(u_5) = (1, 1, 0, 2, 2)$; $c_\Pi(v_2) = (2, 1, 0, 2, 1)$; $c_\Pi(v_3) = (2, 1, 1, 0, 2)$;

$c_\Pi(v_4) = (2, 1, 1, 2, 0)$.

Since the color codes of all the vertices in $P_{5,2}$ are different, c is a locating coloring. So, $\chi_L(P_{5,2}) \leq 5$. □

Next, we give some conjecture related to the locating chromatic number of Petersen graph.

Conjecture. The locating chromatic number of Petersen graph P is four if and only if $P = P_{n,1}$, for odd $n \geq 3$ or $P_{4,2}$.

3. Conclusion

Based on the results, locating chromatic number of Petersen graph $P_{n,1}$ is 4 for odd $n \geq 3$ and 5 for even $n \geq 4$; $\chi_L(P_{4,2}) = 4$; $\chi_L(P_{5,2}) = 5$. Then we have a conjecture to characterize Petersen graph having locating chromatic number four.

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