The Use of Fractional Integral and Fractional Derivative α=5/2 in the 5th Order Function and Exponential Function Using the Riemann-Liouville Method

Dorrah Azis, Agus Sutrisno[*], Desfan Hafifullah, Subian Saidi

Department of Mathematics, University of Lampung, Lampung, Indonesia

Abstract Generally, the order of integral and derivative are connected with the real numbers, such as the first, second, third and more order of integral and derivative. This study aims to develop a theory of an integral or derivative which has order in a 5th order function and exponential function by using Riemann and Liouville method. The result of this study showed that the fractional derivative of order α=5/2 in the 5th Order Function using Riemann-Liouville Method is the same as the form of a third derivative, which means that the value of this derivative will be the same as the result of three times the fractional integral or vice versa. In addition the fractional derivative of the exponential function using the Riemann-Liouville Method is equal to in the form of multiplication of the incomplete gamma function upper limit with exponential function.

Keywords Fractional Integral, Fractional Derivative, 5th Order Function, Exponential Function, Riemann-Liouville Method

1. Introduction

Calculus is a branch of mathematics that studies the concept of calculating limits, changes in functions, derivatives, integrals and infinite series. In calculus, a function can undergo integrals and derivatives either once, twice, and so on with natural number orders. Then a question arises related to the order of fractions, for example, how are the integral and intermediate derivatives of the function, so that a material development is born, namely fractional calculus. Fractional calculus appeared almost simultaneously with classical calculus set. This is because in theory the fractional calculus is the basis for the expansion of the gamma and beta functions. On September 30, 1695, fractional calculus was first introduced in the writings of Leibniz who at that time sent a letter to L'hopital about how a derivative of a function that has a fraction order [DS]. The fractional calculus provides the answer to the question whether the operation of derivatives of integers of order α and α is not an integer? Many mathematicians participated in his contributions such as Abel, Riemann, Liouville, Euler, Laplace, Lacroix, Fourier. In 1819, Lacroix became the first mathematician to write a paper on the definition of fractional derivatives, he started from the function $y=x^m$, where m is a positive integer.

The functions used in fractional derivatives are factorial functions using the gamma and beta function approaches, expressed in the Legendre symbol (Γ). Not only for power functions, fractional calculus can also be developed in other functions such as trigonometry, Laplace, exponential, and exponential algebra with various working methods. The working method in fractional calculus has versions including Riemann-Liouville, Grundwald-Letnikov, M. Caputo (1967), Oldham and Spanier (1974), K.S. Miller and B. Ross (1993), Kolwankar and Gangal (1993). The method that is often used in fractional calculus is the Riemann-Liouville and Caputo method. Based on the above background and problems, the author will examine more deeply about fractional integrals and fractional derivatives using the Riemann-Liouville method with the order $\alpha=\frac{5}{2}$ on the 5th power and exponential function.

2. Material and Methods

2.1. Polynomial Methods

According to [CS], polynomial functions are functions that have many terms in the independent variable.

The form of the polynomial function equation is as follows:

$$y=a_0 +a_1x+a_2x^2+\ldots+a_nx^n. \tag{1}$$

2.2. Exponential Functions

According to [JM], exponential function is a function

with the independent variable being the power of a constant, inverse ln is an exponential function and is represented by exp with the following definition:

$$y= \ln(x) \tag{2}$$

with inverse

$$x= e(y) \tag{3}$$

2.3. Fractional Integral

Definition 2.3

According to [HR], based on its work, the fractional integral has various methods, such as Riemann, Liouville, Riemann-Liouville, Caputo and others. Fractional integrals are defined as follows:

$$D_x^\alpha f(x)=\frac{1}{\Gamma(\alpha)} \int_c^x (x-t)^{\alpha-1} f(t)dt \ \alpha>0. \tag{4}$$

if c=0 in Equation (4), then the Riemann-Liouville fractional integral is obtained which is defined as:

$$D_x^\alpha f(x)=\frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)dt \ \alpha>0. \tag{5}$$

According to [BN], for $\alpha \geq 0$ and $\beta \geq 0$, fractional integral has the following properties:

$$J^\alpha kf(x)=k \ J^\alpha f(x) \tag{6}$$

$$J^\alpha J^\beta f(x)=J^{\alpha+\beta} f(x) \tag{7}$$

$$J^\alpha J^\beta f(x)= J^\beta J^\alpha f(x). \tag{8}$$

2.4. Fractional Derivative

Definition 2.4

According to [KM], the Riemann-Liouville fractional derivative is defined as:

$$D_{x_0}^\alpha f(x)=\frac{d^n(J^{n-\alpha}f(x))}{dx^n} \tag{9}$$

or

$$D_{x_0}^\alpha \ f(x)=\frac{1}{\Gamma(n-\alpha)} \frac{d^n(\int_{x_0}^x (x-t)^{n-\alpha-1} f(t)dt)}{dx^n} \tag{10}$$

with α of any order, n-1 $\leq$ α <n,n$\in$Z$^+$, $x_0 < x$, x > 0 and D^α operator of the fractional derivative of order α. The properties of the fractional derivative by [JG] is:

1. $D^n\big(af(x)\big)=a \ D^n\big(f(x)\big)$ (11)

2. $D^\alpha\big(af(x)+bf(x)\big)=aD^\alpha f(x)+bD^\alpha f(x).$ (12)

The method used in this research is carried out by primary literature studies, namely developing theories that have been worked on by previous researchers, literature study, namely studying text books in the Mathematics Department library and the University of Lampung library, journals and internet access that support the research process.

The procedure in the Riemann-Liouville method is:

1. The form of the equation for the powers of five and the exponential.

2. Choose $\alpha=\frac{5}{2}$ for the respective order integrals and derivatives.

3. Formulate fractional integrals and fractional derivatives in the quadrangle and exponential equation.
4. Perform operations for each term in the equation.
5. After obtaining the results of the integral and its derivatives substitute it to the initial equation form.
6. Replace the value in each of the equation variables with the number one (option).

3. Results and Discussion

3.1. General Fractional IntegralForm of 5th Order Function α Order

Fractional integral order α from the polynomial function according to Riemann-Liouville can be stated as the multiplication form of gamma function and polynomial function. The general fractional integral form α order with the polynomial function f(x)=x^m for α>0 and m>-1 is:

$$D^{-\alpha}f(x)=\frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)dt$$

$$D^{-\alpha}x^m=\frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} t^m dt$$

For example:

$$u=\frac{t}{x} \Rightarrow t = ux \text{ then } du=\frac{dt}{x} \Rightarrow xdu=dt$$

when t=0 $\Rightarrow$ u=0, t=x $\Rightarrow$ u=1

So, the general form of fractional integral α order from the polynomial function $f(x) = x^m$ is:

$$D^{-\alpha}x^m=\frac{\Gamma(m+1)}{\Gamma(\alpha+m+1)} x^{\alpha+m}.$$

So that the fractional integral with α order from the polynomial function which form is f(x)=x^m can be stated as a multiplication of gamma function and polynomial function which is stated in the Theorem 3.1.

Theorem 3.1

Fractional integral with α order from the polynomial function form of f(x)=x^m is

$$D^{-\alpha}x^m=\frac{\Gamma(m+1)}{\Gamma(\alpha+m+1)} x^{\alpha+m} \tag{13}$$

for α>0, m>-1 and x>0.

3.2. General Fractional Integral Form of Exponential Function α Order

Fractional integral α orderfrom the exponential function according to Riemann-Liouville can be stated as a multiplication of gamma function and polynomial function. The general form of fractional integral α order with the exponential function f(x)=e^{ax} as follows:

$$D^{-\alpha}e^{ax}=\frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} e^{at} \ dt$$

For example:

$$u=x-t \Rightarrow t=x-u \text{ then } du=-dt \Rightarrow -du=dt$$

when t=0 ⇒ u=x, t=x ⇒ u=0

then the general form of fractional integral α order with $f(x)=e^{ax}$ is

$$D^{-\alpha}e^{ax}=\frac{e^{ax}a^{-\alpha}}{\Gamma(\alpha)}\gamma(\alpha,ax).$$

So that the fractional integral α orderfrom the exponential function which form is $f(x)=e^{ax}$ can be stated as a multiplication form of an incomplete gamma function with the exponential function which can be stated in the theorem 3.2.

Theorem 3.2

Fractional integral α orderfrom the exponential function form $f(x)=e^{ax}$ is

$$D^{-\alpha}e^{ax}=\frac{e^{ax}a^{-\alpha}}{\Gamma(\alpha)}\gamma(\alpha,ax) \qquad (14)$$

for $\alpha>0$, a∈R, $a\neq 0$ and $x>0$.

Fractional derivative function can be define using the definition of function integration, assuming $\alpha=n-u$ with $0<\alpha<1$ and n is the smallest integer which is bigger than u, so that the derivative function with $f(x)$ can be stated as:

$$D^{\alpha}f(x)=D^{n}[D^{-\alpha}f(x)] \qquad (15)$$

with α is any order, n-1≤α<n, and n∈Z⁺. Therefore, will be analyzed the general form of fractional derivative 5th order function and exponentialfunctionwith α order.

3.3. General Fractional Derivative Form of 5th Order Function α Order

Assume that to find the fractional derivative $f(x)=x^{m}\alpha$ order with m≥0. Based on equation (15) and Theorem 3.1, assumed that u=n-α, with 0<u<1 and n is the smallest integer larger than u, it means that n is the multiple order from that derivative. So that the definition will be:

$$D^{\alpha}f(x)=D^{n}[D^{-\alpha}f(x)]$$

Choose n=1 and u=n-α, then

$$D^{\alpha}x^{m}=D^{1}[D^{-(1-\alpha)}x^{m}]$$

By substituting equation (13) we get

$$D^{\alpha}x^{m}=D^{1}\left[\frac{\Gamma(m+1)}{\Gamma(m+(1-\alpha)+1)}x^{m+1-\alpha}\right]=\frac{\Gamma(m+1)}{\Gamma(m+1-\alpha)}x^{m-\alpha}. \qquad (16)$$

In equation (16), can be seen that the general form of fractional derivative is eligible for 0<α<1 or this can be meant that the derivative is defined for the first one that matches with the boundary of the α order. This is also applies for the other n, if we choose n=2 then the boundary of the fractional derivative will be changed to 1<α<2 which means that the value of this derivative will be the same with the result of the fractional integral two times or vice versa. So that the fractional integral with $\alpha=\frac{5}{2}$ order from the polynomial function which form is $f(x)=x^{m}$ can be made by choosing n=3 so that 2<α<3, then

$$D^{\alpha}x^{m}=D^{3}[D^{-(3-\alpha)}x^{m}]$$

By substituting equation (13) we get

$$D^{\alpha}x^{m}=D^{3}\left[\frac{\Gamma(m+1)}{\Gamma(m+(3-\alpha)+1)}x^{m+3-\alpha}\right]=\frac{\Gamma(m+1)}{\Gamma(m-\alpha+3)}x^{m-\alpha+2}. \qquad (17)$$

So that the fractional derivative $\alpha=\frac{5}{2}$ order can be defined as the third form of derivative which means that the value of this derivative will be the same with the fractional integer three times or vice versa, then the fractional derivative can be stated as the form of multiplication of gamma function and polynomial function which is explained in the theorem 3.3.

Theorem 3.3

Fractional derivative α order from the polynomial function form of $f(x)=x^{m}$ is

$$D^{\alpha}x^{m}=\frac{\Gamma(m+1)}{\Gamma(m-\alpha+3)}x^{m-\alpha+2} \qquad (18)$$

for m≥0, 2<α<3 and x>0.

3.4. General Fractional Derivative Form of Exponential Function α Order

Based on Theorem 3.2, the fractional derivative of order α with $n=3$ of the exponential function of the form $f(x)=e^{ax}$ is

$$D^{\alpha}f(x)=D^{n}[D^{-\alpha}f(x)]$$

$$D^{\alpha}e^{ax}=D^{3}[D^{-(3-\alpha)}e^{ax}]$$

by substituting equation (14), we obtained

$$D^{\alpha}e^{ax}=\frac{a^{\alpha-3}}{\Gamma(3-\alpha)}D\ [e^{ax}\gamma(3-\alpha,ax)]$$

based on the specific value equation for the incomplete derivative of the gamma function i.e.

$$\frac{d}{dx}(\gamma(\alpha,x))=-\frac{d}{dx}(\Gamma(\alpha,x))=x^{\alpha-1}e^{-x}$$

then

$$D^{\alpha}e^{ax}=\frac{a^{\alpha-3}}{\Gamma(3-\alpha)}D\ [e^{ax}\Gamma(3-\alpha,ax)]$$

$$=\frac{a^{\alpha-2}e^{ax}\Gamma(3-\alpha,ax)}{\Gamma(3-\alpha)}-\frac{x^{2}}{\Gamma(3-\alpha)x^{\alpha}}.$$

So that the fractional derivative of order $\alpha=\frac{5}{2}$ can be interpreted as containing in the form of a third derivative, which means that the value of this derivative will be the same as the result of three times the fractional integral or vice versa, then the fractional derivative of the exponential function can be expressed in the form of multiplication of the incomplete gamma function upper limit with exponential function described in the following theorem:

Theorem 3.4

Fractional derivative α order from the exponential function form of $f(x)=e^{ax}$ is

$$D^{\alpha}e^{ax}=\frac{a^{\alpha-2}e^{ax}\Gamma(3-\alpha,ax)}{\Gamma(3-\alpha)}-\frac{x^{2}}{\Gamma(3-\alpha)x^{\alpha}} \qquad (19)$$

for 2<α<3, a∈R, and x>0.

3.5. Integral Fractional of 5th Order Function and Exponential $\frac{5}{2}$ Order

Looking for the derivative of the function $f(x) = x^m$ which is a 5th order function with $f(x)=ax^0+bx^1+cx^2+dx^3+ex^4+fx^5$ $\alpha=\frac{5}{2}$ order as follows:

$$D^{\frac{5}{2}}x^0=\frac{1}{\Gamma\left(\frac{5}{2}\right)}\int_0^x (x-t)^{\frac{5}{2}-1}t^0dt$$

for example:

$$t=xu \Rightarrow dt=xdu$$

$$t=0 \Rightarrow u=0, t=x \Rightarrow u=1$$

then

$$D^{\frac{5}{2}}x^0=\frac{1}{\Gamma\left(\frac{5}{2}\right)}\int_0^x (x-t)^{\frac{5}{2}-1}t^0dt=\frac{1}{\Gamma\left(\frac{5}{2}\right)}\int_0^x (x-xu)^{\frac{5}{2}-1}xdu=\frac{8}{15}\sqrt{\frac{x^5}{\pi}},$$

in the same way, the fractional integral of each term is obtained by

$$D^{\frac{5}{2}}x^1=\frac{16}{105}\sqrt{\frac{x^7}{\pi}}, D^{\frac{5}{2}}x^2=\frac{64}{945}\sqrt{\frac{x^{11}}{\pi}}, D^{\frac{5}{2}}x^3=\frac{384}{10395}\sqrt{\frac{x^7}{\pi}},$$

$$D^{\frac{5}{2}}x^4=\frac{3072}{135135}\sqrt{\frac{x^7}{\pi}}, D^{\frac{5}{2}}x^5=\frac{30720}{2027025}\sqrt{\frac{x^7}{\pi}}$$

Choose a,b,c,d,e,f,x=1, then we get

$$D^{\frac{5}{2}}\sum_{m=0}^5 x^m=\frac{111.928}{135.135\sqrt{\pi}}.$$

Next, looking for the fractional integral with the order $\alpha=\frac{5}{2}$ from the exponential function $f(x)=e^{ax}$ as follows:

$$D^{\frac{5}{2}}e^{ax}=\frac{e^{ax}a^{-\frac{5}{2}}}{\Gamma\left(\frac{5}{2}\right)}\gamma\left(\frac{5}{2},ax\right)=\frac{4e^{ax}}{3}\sqrt{\frac{1}{a^5\pi}}\gamma\left(\frac{5}{2},ax\right)$$

Choose any a=-1 for obtain

$$D^{\frac{5}{2}}e^{-x}=\frac{4e^{-x}}{3}\sqrt{\frac{1}{-\pi}}\gamma\left(\frac{5}{2},-x\right).$$

3.6. Derivative Fractional 5th Order Function and Exponential $\frac{5}{2}$ Order

Looking for the derivative of the function $f(x)=x^m$ which is a 5th order function with $f(x)=ax^0+bx^1+cx^2+dx^3+ex^4+fx^5$ orde $\alpha=\frac{5}{2}$ as follows:

$$D^{\frac{5}{2}}x^0=\frac{d}{dx}\left[\frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_0^x (x-t)^{-\frac{1}{2}}dt\right]$$

for example:

$$t=xu \Rightarrow dt=xdu$$

and

$$t=0 \Rightarrow u=0, t=x \Rightarrow u=1$$

then

$$D^{\frac{5}{2}}x^0=\frac{d}{dx}\left[\frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_0^x (x-t)^{-\frac{1}{2}}dt\right]=\frac{d}{dx}\left[\frac{1}{\Gamma\left(\frac{1}{2}\right)}x^{\frac{1}{2}}\beta\left(\frac{1}{2},1\right)\right]=\sqrt{\frac{1}{\pi x}}$$

In the same way, you get the fractional derivative of each term

$$D^{\frac{5}{2}}x^1=2\sqrt{\frac{x^1}{\pi}}, D^{\frac{5}{2}}x^2=\frac{8}{3}\sqrt{\frac{x^3}{\pi}}, D^{\frac{5}{2}}x^3=\frac{16}{5}\sqrt{\frac{x^5}{\pi}},$$

$$D^{\frac{5}{2}}x^4=\frac{128}{35}\sqrt{\frac{x^7}{\pi}}, D^{\frac{5}{2}}x^5=\frac{256}{63}\sqrt{\frac{x^5}{\pi}}$$

Choose a,b,c,d,e,f,x=1, then

$$D^{\frac{5}{2}}\sum_{m=0}^5 x^m=\frac{21.945}{1.323\sqrt{\pi}}.$$

Next, looking for the fractional derivative with the order $\alpha=\frac{5}{2}$ from the exponential function $f(x)=e^{ax}$ as follows:

$$D^{\alpha}e^{ax}=\frac{a^{\alpha-2}e^{ax}\Gamma(3-\alpha,ax)}{\Gamma(3-\alpha)}-\frac{x^2}{\Gamma(3-\alpha)x^{\alpha}}$$

Thus $D^{\frac{5}{2}}e^{ax}=\sqrt{\frac{a}{\pi}}e^{ax}\Gamma\left(\frac{1}{2},ax\right)-\sqrt{\frac{1}{\pi x}}$

Choose any $a=-1$, we get

$$D^{\frac{5}{2}}e^{-x}=\sqrt{\frac{-1}{\pi x}}\frac{\Gamma\left(\frac{1}{2},-x\right)}{e^x}-\sqrt{\frac{1}{\pi x}}.$$

4. Conclusions

We can conclude that the fractional derivative of order $\alpha=\frac{5}{2}$ in the 5th Order Function using Riemann-Liouville Method is the same as the form of a third derivative, which means that the value of this derivative will be the same as the result of three times the fractional integral or vice versa. In addition the fractional derivative of the exponential function using the Riemann-Liouville Method is equal to in the form of multiplication of the incomplete gamma function upper limit with exponential function.

REFERENCES

[1]   Badruzzaman, F.H., Nahar, J., dan Johansyah, M.D. 2017 [BN]. Analisis Turunan dan Integral Fraksional Fungsi Pangkat Tigadan Fungsi Eksponensial. *Jurnal Matematika*. 16(2): 1-9.

[2]   Conte, S.D. 1980 [CS]. *Dasar-Dasar Analisis Numerik Suatu Pendekatan Algoritma.* Edisi ketiga. Erlangga, Jakarta.

[3] Das, S. 2008 [DS]. *Functional Fractional Calculus for System Identification and Controls.* Springer, London.

[4] Herrmann, R. 2011 [HR]. *Fractional Calculus an Introduction for Physicists.* World Scientific, New Jersey.

[5] Jameson, G.J.O. 2017 [JG] The Incomplete Gamma Function. *Gazette Article.* 100(2006): 298-306.

[6] Johansyah, M.D., Badruzzaman, F.H., danNahar, J. 2017 [JM] Kajian Dasar Integral danTurunan Fraksional. *IRONS.* 16(2): 204 209.

[7] Kilbas, A.A., and Marichev, O.I. Samko, S.G., 2006 [KM]. *Theory and Aplications Fractional Differential Equation.* Elsevier, Amsterdam.