Variations of Graceful Labelling of Subgraph of Millipede Graph

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Abstract. In graph theory, there is the topics name by labelling. In 1967, Alex Rosa introduced the theory of labelling. Furthermore, Alex Rosa initiate the β-labelling known as the graceful labelling that Golomb introduced. There is various variations in graceful labelling, some of them are $\hat{\rho}$ labelling and odd-even graceful labelling. A Millipede graph $L_n \odot \overline{K}_r$ is a modification from ladder graph L_n by adding r number of path graph which has length 1 at each vertex of the ladder graph so that the result of the graph resembles a millipede. We show that the subgraph of the millipede graph that is $L_2 \odot \overline{K}_r$ graph and $L_3 \odot \overline{K}_r$ graph is graceful, have $\hat{\rho}$ labelling, and also an odd-even graceful graph.

INTRODUCTION

One of the topics in graph theory is labelling. Labelling is a valuation of a graph, that is a mapping of vertices, arcs, or even both (vertices and arcs) to positive integers. A Label is a positive integer that satisfy some properties based on the category of labelling which we considered [1]. According to Gallian in 2019, in these 50 years, there are more than 200 graph labeling types had studied in more than 3000 papers [2].

In 1967, Alex Rosa introduced the theory of labelling. He introduced the β-labelling subsequently known as the graceful labelling that Golomb introduced. The definition of graceful labelling is a mapping of its vertex f: $V(G) \rightarrow \{0,1,...,n\}$ where n is a number of vertices and the created function is an injective, so each edge $pq \in E$ is labelled by $|f(p) - f(q)|$ so the value is distinct [3].

The variations of Graceful labelling have many various types. One of the variations of graceful labelling is $\hat{\rho}$ labelling. The definition of $\hat{\rho}$ labelling is a mapping of its vertex f:$V(G) \rightarrow \{0,1,...,n+1\}$ where n is the number of edges that the function is injective with the result that its edge has a bijective function f^*:$V(E) \rightarrow \{1,2,...,n\}$ or f^*:$V(E) \rightarrow \{1,2,...,n-1,n+1\}$ so the edge $pq \in E$ and vertices $p,q \in V$ has $f^* = |f(p) - f(q)|$ [3].

Another types from graceful labelling is the odd-even graceful labelling. Sridevi, Navaneethakrishnan, Nagarajan and Nagarajanin in 2012 introduced the odd-even graceful labelling. The definition of odd-even graceful graph is if its graph has an odd-even graceful labelling. Odd-even graceful labelling is a mapping of its vertex f:$V(G) \rightarrow \{1,3,5,...,2n+1\}$ where n is the number of its vertex, so that the function is injective which was the result of edge $pq \in E$ is defined by $|f(p)-f(q)|$, so that the the label of the edge is $\{2,4,6,...,2n\}$[4].

In 2012 Haryono constructed a millipede graph as a graph that has a modification from ladder graph L_n which was the result resembles a millipede [5]. A Millipede graph $L_n \odot \overline{K}_r$ is a modification from ladder graph L_n by adding r number of path graph which has length 1 at each vertex of the ladder graph so that the result of the graph resembles a millipede [6].

The studies in graceful labelling on the modification of ladder graph is graceful labelling on bat $B_i(n,r,s)$ graph [7], graceful labeling of some new graphs [8], and graceful labeling on a new family of graphs [9].

- **Definition 1.** Millipede graph $L_n \odot \overline{K}_r$ is a graph made from ladder graph L_n by adding r number of path graph which has length 1 at each vertex of the ladder graph.

MAIN RESULT

In this section, it would be shown that $L_2 \odot \overline{K}_r$ and $L_3 \odot \overline{K}_r$ graph had various types from graceful labelling, there is graceful labelling, $\hat{\rho}$ labelling, and odd-even graceful labelling.

- **Theorem 1.** An $L_2 \odot \overline{K}_r$ is graceful.

Proof. Define the notation of vertices of $L_2 \odot \overline{K}_r$ graph such that we could see in Figure 1 below.

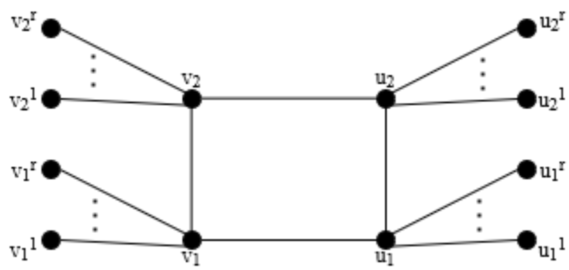

FIGURE 1. Notation of vertices of $L_2 \odot \overline{K}_r$ graph

From the notation in Figure 1, the set of vertices that we have is $V(L_2 \odot \overline{K}_r) = \{v_1, v_2, u_1, u_2, v_1^1, v_1^2, \dots, v_1^r, v_2^1, v_2^2, \dots, v_2^r, u_1^1, u_1^2, \dots, u_1^r, u_2^1, u_2^2, \dots, u_2^r\}$, and we have the set of edges $E(L_2 \odot \overline{K}_r) = \{v_1v_1^1, \dots, v_1v_1^r, v_2v_2^1, \dots, v_2v_2^r, u_1u_1^1, \dots, u_1u_1^r, u_2u_2^1, \dots, u_2u_2^r, v_1u_1, v_1v_2, v_2u_2, u_1u_2\}$. So that, we have the number of elements V and E were $4r + 4$, or $|V| = |E| = 4r + 4$, where r was the number of path graph which has length 1 at each vertex of $L_2 \odot \overline{K}_r$ graph.

Denote the function f for vertices of $L_2 \odot \overline{K}_r$ graph as the function below:

$$f(v_1) = 0 \quad (1)$$
$$f(v_2) = r + 2 \quad (2)$$
$$f(u_1) = 3r + 4 \quad (3)$$
$$f(u_2) = r + 1 \quad (4)$$
$$f(v_1^i) = 3r + 4 + i; i = 1,2, \dots, r \quad (5)$$
$$f(v_2^i) = r + 3 + i; i = 1,2, \dots, r \quad (6)$$
$$f(u_1^i) = r + 1 - i; i = 1,2, \dots, r \quad (7)$$
$$f(u_2^i) = 2r + 3 + i; i = 1,2, \dots, r \quad (8)$$

As we could see from each function f above, that give the label to each vertex, all vertices have different label and has a mapping to $\{0, 1, 2, \dots, |E|\}$. The function f which is construct at equation (1) $-$ (8), are injective and has a mapping from V to $\{0, 1, 2, \dots, |E|\}$. All edges $pq \in E$ have different value from f as $f'(pq) = |f(p) - f(q)|$.

So, function f' at each edge $L_2 \odot \overline{K}_r$ graph could be defined as the function below:

$$f'(v_1u_1) = |0 - (3r + 4)| = 3r + 4 \quad (9)$$
$$f'(v_1v_2) = |0 - (r + 2)| = r + 2 \quad (10)$$
$$f'(u_1u_2) = |(3r + 4) - (r + 1)| = 2r + 3 \quad (11)$$
$$f'(u_2v_2) = |(r + 1) - (r + 2)| = 1 \quad (12)$$
$$f'(v_1v_1^i) = |0 - (3r + 4 + i)| = 3r + 4 + i; i = 1,2, \dots, r \quad (13)$$
$$f'(v_2v_2^i) = |(r + 2) - (r + 3 + i) = 1 + i; i = 1,2, \dots, r \quad (14)$$

$$f'(u_1u_1^i) = |(3r+4) - (r+1-i)| = 2r+3+i; i = 1,2,\dots,r \quad (15)$$
$$f'(u_2u_2^i) = |(r+1) - (2r+3+i)| = r+2+i; i = 1,2,\dots,r \quad (16)$$

As we could see, the equation in (9)−(16) give each edge different label. The labels construct a set $\{0,1,2,\dots,|E|\}$. So that, the function f and f' have different label for all edges as well the labelling for each vertex set had different values in $\{0,1,2,\dots,|E|\}$. In addition, we can assume that $L_2 \odot \overline{K}_r$ graph was graceful. ■

The graph that shown in Figure 2 is one of graceful labelling of $L_2 \odot \overline{K}_2$ graph.

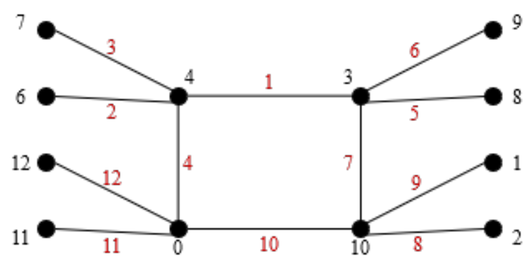

FIGURE 2. Graceful labelling of $L_2 \odot \overline{K}_2$ graph

- **Theorem 2.** An $L_2 \odot \overline{K}_r$ graph has $\hat{\rho}$ labelling.

Proof. Define the notation of vertices of $L_2 \odot \overline{K}_r$ graph as in Figure 1.

Thus $V(L_2 \odot \overline{K}_r) = \{v_1, v_2, u_1, u_2, v_1^1, v_1^2, \dots, v_1^r, v_2^1, v_2^2, \dots, v_2^r, u_1^1, u_1^2, \dots, u_1^r, u_2^1, u_2^2, \dots, u_2^r\}$, and the set of edges $E(L_2 \odot \overline{K}_r) = \{v_1v_1^1, \dots, v_1v_1^r, v_2v_2^1, \dots, v_2v_2^r, u_1u_1^1, \dots, u_1u_1^r, u_2u_2^1, \dots, u_2u_2^r, v_1u_1, v_1v_2, v_2u_2, u_1u_2\}$ and $|V| = |E| = 4r+4$, where r was the number of path graph which has length 1 at each vertex of $L_2 \odot \overline{K}_r$ graph.
Defined function f for vertices of $L_2 \odot \overline{K}_r$ graph as the function below:

$$f(v_1) = 1 \quad (17)$$
$$f(v_2) = r+3 \quad (18)$$
$$f(u_1) = 3r+5 \quad (19)$$
$$f(u_2) = r+2 \quad (20)$$
$$f(v_1^i) = 3r+5+i; i = 1,2,\dots,r \quad (21)$$
$$f(v_2^i) = r+4+i; i = 1,2,\dots,r \quad (22)$$
$$f(u_1^i) = r+2-i; i = 1,2,\dots,r \quad (23)$$
$$f(u_2^i) = 2r+4+i; i = 1,2,\dots,r \quad (24)$$

From the function f above that is construct by equation (17) − (24), we could see that all vertices have different label and have a mapping to a set $\{0,1,2,\dots,|E|+1\}$. So that, the function f of equation (17) − (24) is injective that has a mapping from V to $\{0,1,2,\dots,|E|+1\}$. All edges $pq \in E$ had different label, and the label from $f'(pq) = |f(p) - f(q)|$.

Function f' at each edge $L_2 \odot \overline{K}_r$ graph could be defined as the function below::

$$f'(v_1u_1) = |1 - (3r+5)| = 3r+4 \quad (25)$$
$$f'(v_1v_2) = |1 - (r+3)| = r+2 \quad (26)$$
$$f'(u_1u_2) = |(3r+5) - (r+2)| = 2r+3 \quad (27)$$
$$f'(u_2v_2) = |(r+2) - (r+3)| = 1 \quad (28)$$
$$f'(v_1v_1^i) = |1 - (3r+5+i)| = 3r+4+i; i = 1,2,\dots,r \quad (29)$$
$$f'(v_2v_2^i) = |(r+3) - (r+4+i) = 1+i; i = 1,2,\dots,r \quad (30)$$
$$f'(u_1u_1^i) = |(3r+5) - (r+2-i)| = 2r+3+i; i = 1,2,\dots,r \quad (31)$$
$$f'(u_2u_2^i) = |(r+2) - (2r+4+i)| = r+2+i; i = 1,2,\dots,r \quad (32)$$

We could see that the label of each edge that construct in equation (25)−(32) have distinct value and have a mapping to $\{0, 1, 2, \ldots, |E| + 1\}$. So that, the function f and the created function f' have different values for each edge as well the label for vertices have different values and has a mapping to $\{0, 1, 2, \ldots, |E| + 1\}$. So, we can concluded that $L_2 \odot \overline{K}_r$ graph has $\hat{\rho}$ labelling. ■

In Figure 3 below there is one of $\hat{\rho}$ labelling of $L_2 \odot \overline{K}_3$ graph.

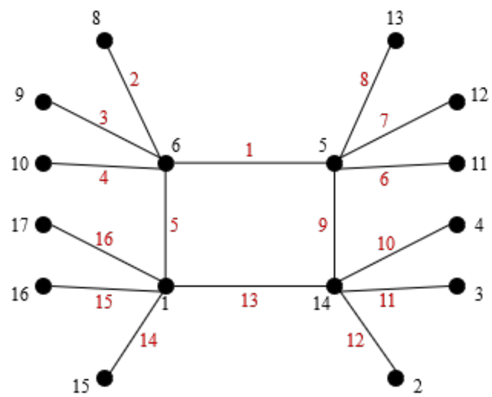

FIGURE 3. $\hat{\rho}$ labelling of $L_2 \odot \overline{K}_3$ graph

- **Theorem 3.** An $L_2 \odot \overline{K}_r$ graph is an odd-even graceful graph.

Proof. Denote the vertices of $L_2 \odot \overline{K}_r$ graph as in Figure 1.

From the notation in Figure 1, we have $V(L_2 \odot \overline{K}_r) = \{v_1, v_2, u_1, u_2, v_1^1, v_1^2, \ldots, v_1^r, v_2^1, v_2^2, \ldots, v_2^r, u_1^1, u_1^2, \ldots, u_1^r, u_2^1, u_2^2, \ldots, u_2^r\}$, and also we have the set of edges $E(L_2 \odot \overline{K}_r) = \{v_1v_1^1, \ldots, v_1v_1^r, v_2v_2^1, \ldots, v_2v_2^r, u_1u_1^1, \ldots, u_1u_1^r, u_2u_2^1, \ldots, u_2u_2^r, v_1u_1, v_1v_2, v_2u_2, u_1u_2\}$ thus $|V| = |E| = 4r + 4$, where r was the number of path graph which has length 1 at each vertex of $L_2 \odot \overline{K}_r$ graph.
The labels of the vertices of $L_2 \odot \overline{K}_r$ is defined by function f as the function below:

$$f(v_1) = 1 \quad (33)$$
$$f(v_2) = 2r + 5 \quad (34)$$
$$f(u_1) = 6r + 9 \quad (35)$$
$$f(u_2) = 2r + 3 \quad (36)$$
$$f(v_1^i) = 6r + 9 + 2i; i = 1,2, \ldots, r \quad (37)$$
$$f(v_2^i) = 2r + 7 + 2i; i = 1,2, \ldots, r \quad (38)$$
$$f(u_1^i) = 2r + 3 - 2i; i = 1,2, \ldots, r \quad (39)$$
$$f(u_2^i) = 4r + 7 + 2i; i = 1,2, \ldots, r \quad (40)$$

By equation (33) − (40), we could see that the label for vertices has distinct value and has a mapping to $\{1, 3, 7, \ldots, 2|E| + 1\}$. So that, the function f is injective and has a mapping from V to $\{1, 3, 7, \ldots, 2|E| + 1\}$. Each edge in graph G $pq \in E$ has different value from function f by $f'(pq) = |f(p) - f(q)|$.

So that, the function f' that is the label at edges $L_2 \odot \overline{K}_r$ graph could be defined as the function below:

$$f'(v_1u_1) = |1 - (6r + 9)| = 6r + 8 \quad (41)$$
$$f'(v_1v_2) = |1 - (2r + 5)| = 2r + 4 \quad (42)$$
$$f'(u_1u_2) = |(6r + 9) - (2r + 3)| = 4r + 6 \quad (43)$$
$$f'(u_2v_2) = |(2r + 3) - (2r + 5)| = 2 \quad (44)$$
$$f'(v_1v_1^i) = |1 - (6r + 9 + 2i)| = 6r + 8 + 2i; i = 1,2, \ldots, r \quad (45)$$
$$f'(v_2v_2^i) = |(2r + 5) - (2r + 7 + 2i) = 2 + 2i; i = 1,2, \ldots, r \quad (46)$$
$$f'(u_1u_1^i) = |(6r + 9) - (2r + 3 - 2i)| = 4r + 6 + 2i; i = 1,2, \ldots, r \quad (47)$$

$$f'(u_2u_2^i) = |(2r+3) - (4r+7+2i)| = 2r+4+2i; i = 1,2,\dots,r \quad (48)$$

From function f' that construct in equation (41)−(48), we could see all edges have different value and has a mapping to $\{2,4,6,\dots,2|E|\}$. We could see that the function f' give different values at each edge and has a mapping to $\{2,4,6,\dots,2|E|\}$. So that, the function f and the created function f' give different values for each edge as well the labels for each vertex had different values and has a mapping to $\{2,4,6,\dots,2|E|\}$. So, $L_2 \odot \overline{K}_r$ graph is an odd-even graceful graph. ■

In Figure 4 we could see one of an odd-even graceful labelling of $L_2 \odot \overline{K}_3$ graph.

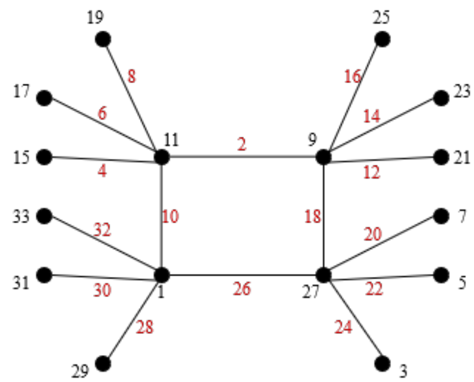

FIGURE 4. Odd-even labelling of $L_2 \odot \overline{K}_3$ graph

- **Theorem 4.** An $L_3 \odot \overline{K}_r$ graph is graceful.

Proof. Define the notation of vertices of $L_3 \odot \overline{K}_r$ graph as shown in Figure 5.

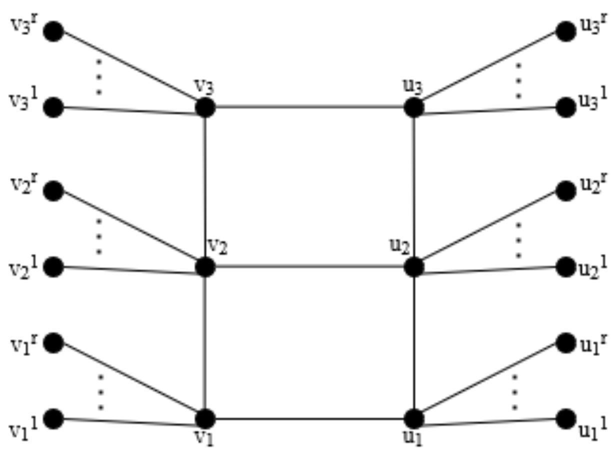

FIGURE 5. Notation of vertices of $L_3 \odot \overline{K}_r$ graph

From the notation in Figure 5, we have the set of vertices $V(L_3 \odot \overline{K}_r) = \{v_1, v_2, v_3, u_1, u_2, u_3, v_1^1, v_1^2, \dots, v_1^r, v_2^1, v_2^2, \dots, v_2^r, v_3^1, v_3^2, \dots, v_3^r, u_1^1, u_1^2, \dots, u_1^r, u_2^1, u_2^2, \dots, u_3^r, u_3^1, u_3^2, \dots, u_3^r\}$, and the set of edges $E(L_3 \odot \overline{K}_r) = \left\{ \begin{matrix} v_1v_1^1, \dots, v_1v_1^r, v_2v_2^1, \dots, v_2v_2^r, v_3v_3^1, \dots, v_3v_3^r, u_1u_1^1, \dots, u_1u_1^r, u_2u_2^1, \dots, \\ u_2u_2^r, u_3u_3^1, \dots, u_3u_3^r, v_1u_1, v_1v_2, v_2u_2, v_2v_3, v_3u_3, u_1u_2, u_2u_3 \end{matrix} \right\}$. So the number elements of V and E is $6r+6$ and $6r+7$ or $|V| = 6r+6$ and $|E| = 6r+7$ and r is the number of path graph which has length 1 at each vertex of $L_3 \odot \overline{K}_r$ graph.

Construct function f for each vertice of $L_3 \odot \overline{K}_r$ graph as the function below:

$$f(v_1) = 0 \tag{49}$$
$$f(v_2) = 2r + 3 \tag{50}$$
$$f(v_3) = 2r + 2 \tag{51}$$
$$f(u_1) = 5r + 7 \tag{52}$$
$$f(u_2) = r + 1 \tag{53}$$
$$f(u_3) = 4r + 6 \tag{54}$$
$$f(v_1^i) = 5r + 7 + i; i = 1,2, \dots , r \tag{55}$$
$$f(v_2^i) = 3r + 5 + i; i = 1,2, \dots , r \tag{56}$$
$$f(v_3^i) = 2r + 3 + i; i = 1,2, \dots , r \tag{57}$$
$$f(u_1^i) = r + 1 - i; i = 1,2, \dots , r \tag{58}$$
$$f(u_2^i) = 4r + 6 + i; i = 1,2, \dots , r \tag{59}$$
$$f(u_3^i) = 2r + 2 - i; i = 1,2, \dots , r \tag{60}$$

From function f that gives value to each vertex, each vertices have different label and has a mapping to $\{0, 1, 2, \dots , |E|\}$. So, the function f that is construct by equation (49) − (60), is injective and has a mapping from V to $\{0, 1, 2, \dots , |E|\}$. Edge $pq \in E$ have label from function f as $f'(pq) = |f(p) - f(q)|$.

So that, the label f' at edges $L_3 \odot \overline{K}_r$ graph could be define as the function below:

$$f'(v_1u_1) = |0 - (5r + 7)| = 5r + 7 \tag{61}$$
$$f'(v_1v_2) = |0 - (2r + 3)| = 2r + 3 \tag{62}$$
$$f'(v_2u_2) = |(2r + 3) - (r + 1)| = r + 2 \tag{63}$$
$$f'(v_2v_3) = |(2r + 3) - (2r + 2)| = 1 \tag{64}$$
$$f'(v_3u_3) = |(2r + 2) - (4r + 6)| = 2r + 4 \tag{65}$$
$$f'(u_1u_2) = |(5r + 7) - (r + 1)| = 4r + 6 \tag{66}$$
$$f'(u_2u_3) = |(r + 1) - (4r + 6)| = 3r + 5 \tag{67}$$
$$f'(v_1v_1^i) = |0 - (5r + 7 + i)| = 5r + 7 + i; i = 1,2, \dots , r \tag{68}$$
$$f'(v_2v_2^i) = |(2r + 3) - (3r + 5 + i) = r + 2 + i; i = 1,2, \dots , r \tag{69}$$
$$f'(v_3v_3^i) = |(2r + 2) - (2r + 3 + i) = 1 + i; i = 1,2, \dots , r \tag{70}$$
$$f'(u_1u_1^i) = |(5r + 7) - (r + 1 - i)| = 4r + 6 + i; i = 1,2, \dots , r \tag{71}$$
$$f'(u_2u_2^i) = |(r + 1) - (4r + 6 + i)| = 3r + 5 + i; i = 1,2, \dots , r \tag{72}$$
$$f'(u_3u_3^i) = |(4r + 6) - (2r + 2 - i)| = 2r + 4 + i; i = 1,2, \dots , r \tag{73}$$

By equation (61)−(73) we could see that the function have different value and has a mapping to $\{0, 1, 2, \dots , |E|\}$. So, the function f and the created function f' have different label for all edges as well the label for all vertices have different label and has a mapping to $\{0, 1, 2, \dots , |E|\}$. So that, we can assume that $L_3 \odot \overline{K}_r$ graph was graceful. ■

The graph in Figure 6 is one of graceful labelling of $L_3 \odot \overline{K}_1$ graph.

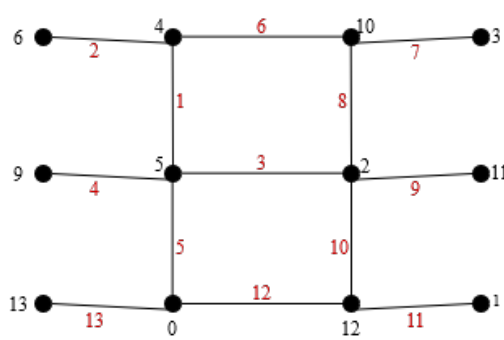

FIGURE 6. Graceful labelling of $L_3 \odot \overline{K}_1$ graph

- **Theorem 5.** An $L_3 \odot \overline{K}_r$ graph has graceful labelling.

Proof. Define the notation of each vertex of $L_3 \odot \overline{K}_r$ graph as shown in Figure 5

Thus $V(L_3 \odot \overline{K}_r) = \left\{ \begin{matrix} v_1, v_2, v_3, u_1, u_2, u_3, v_1^1, v_1^2, \dots, v_1^r, v_2^1, v_2^2, \dots, v_2^r, \\ v_3^1, v_3^2, \dots, v_3^r, u_1^1, u_1^2, \dots, u_1^r, u_2^1, u_2^2, \dots, u_3^r, u_3^1, u_3^2, \dots, u_3^r \end{matrix} \right\}$, and the set of edges $E(L_3 \odot \overline{K}_r) = \left\{ \begin{matrix} v_1v_1^1, \dots, v_1v_1^r, v_2v_2^1, \dots, v_2v_2^r, v_3v_3^1, \dots, v_3v_3^r, u_1u_1^1, \dots, u_1u_1^r, u_2u_2^1, \\ \dots, u_2u_2^r, u_3u_3^1, \dots, u_3u_3^r, v_1u_1, v_1v_2, v_2u_2, v_2v_3, v_3u_3, u_1u_2, u_2u_3 \end{matrix} \right\}$ and $|V| = 6r + 6$ and $|E| = 6r + 7$ where r was the number of path graph which has length 1 at each vertex of $L_3 \odot \overline{K}_r$ graph.

Defined function f for vertices of $L_3 \odot \overline{K}_r$ graph as follows:

$$f(v_1) = 1 \tag{74}$$
$$f(v_2) = 2r + 4 \tag{75}$$
$$f(v_3) = 2r + 3 \tag{76}$$
$$f(u_1) = 5r + 8 \tag{77}$$
$$f(u_2) = r + 2 \tag{78}$$
$$f(u_3) = 4r + 7 \tag{79}$$
$$f(v_1^i) = 5r + 8 + i; i = 1,2, \dots, r \tag{80}$$
$$f(v_2^i) = 3r + 6 + i; i = 1,2, \dots, r \tag{81}$$
$$f(v_3^i) = 2r + 4 + i; i = 1,2, \dots, r \tag{82}$$
$$f(u_1^i) = r + 2 - i; i = 1,2, \dots, r \tag{83}$$
$$f(u_2^i) = 4r + 7 + i; i = 1,2, \dots, r \tag{84}$$
$$f(u_3^i) = 2r + 3 - i; i = 1,2, \dots, r \tag{85}$$

From function f that is defined in equation $(74) - (85)$, we have all vertices had different label and construct a set $\{0, 1, 2, \dots, |E| + 1\}$. So, the function f is injective and has a mapping from V to $\{0, 1, 2, \dots, |E| + 1\}$. The value that we have for edge $pq \in E$ is from function f' as $f'(pq) = |f(p) - f(q)|$.

Function f' at edges $L_3 \odot \overline{K}_r$ graph could be defined as the function below:

$$f'(v_1u_1) = |1 - (5r + 8)| = 5r + 7 \tag{86}$$
$$f'(v_1v_2) = |1 - (2r + 4)| = 2r + 3 \tag{87}$$
$$f'(v_2u_2) = |(2r + 4) - (r + 2)| = r + 2 \tag{88}$$
$$f'(v_2v_3) = |(2r + 4) - (2r + 3)| = 1 \tag{89}$$
$$f'(v_3u_3) = |(2r + 3) - (4r + 7)| = 2r + 4 \tag{90}$$
$$f'(u_1u_2) = |(5r + 8) - (r + 2)| = 4r + 6 \tag{91}$$
$$f'(u_2u_3) = |(r + 2) - (4r + 7)| = 3r + 5 \tag{92}$$
$$f'(v_1v_1^i) = |1 - (5r + 8 + i)| = 5r + 7 + i; i = 1,2, \dots, r \tag{93}$$
$$f'(v_2v_2^i) = |(2r + 4) - (3r + 6 + i) = r + 2 + i; i = 1,2, \dots, r \tag{94}$$
$$f'(v_3v_3^i) = |(2r + 3) - (2r + 4 + i) = 1 + i; i = 1,2, \dots, r \tag{95}$$
$$f'(u_1u_1^i) = |(5r + 8) - (r + 2 - i)| = 4r + 6 + i; i = 1,2, \dots, r \tag{96}$$
$$f'(u_2u_2^i) = |(r + 2) - (4r + 7 + i)| = 3r + 5 + i; i = 1,2, \dots, r \tag{97}$$
$$f'(u_3u_3^i) = |(4r + 7) - (2r + 3 - i)| = 2r + 4 + i; i = 1,2, \dots, r \tag{98}$$

From the function above, the label for each edge that construct by equation $(86)-(98)$ are distinct and has a mapping to $\{0, 1, 2, \dots, |E| + 1\}$. So, the function f and f' have different label for each edge as well the label for each vertex have different label and has a mapping to $\{0, 1, 2, \dots, |E| + 1\}$. So that, $L_3 \odot \overline{K}_r$ graph has $\hat{\rho}$ labelling. ■

In Figure 7 there is one of $\hat{\rho}$ labelling of $L_3 \odot \overline{K}_2$ graph.

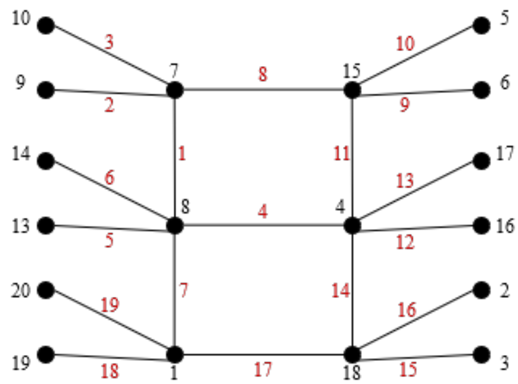

FIGURE 7. $\hat{\rho}$ labelling of $L_3 \odot \overline{K}_2$ graph

- **Theorem 6.** An $L_3 \odot \overline{K}_r$ graph is an odd-even graceful graph.

Proof. Define the notation of vertices of $L_3 \odot \overline{K}_r$ graph as shown in Figure 5.

From the notation in Figure 5, we have that $V(L_3 \odot \overline{K}_r) = \left\{ \begin{matrix} v_1, v_2, v_3, u_1, u_2, u_3, v_1^1, v_1^2, \dots, v_1^r, v_2^1, v_2^2, \dots, v_2^r, v_3^1, v_3^2, \dots, v_3^r, u_1^1, \\ u_1^2, \dots, u_1^r, u_2^1, u_2^2, \dots, u_3^r, u_3^1, u_3^2, \dots, u_3^r \end{matrix} \right\}$, and $E(L_3 \odot \overline{K}_r) = \left\{ \begin{matrix} v_1v_1^1, \dots, v_1v_1^r, v_2v_2^1, \dots, v_2v_2^r, v_3v_3^1, \dots, v_3v_3^r, u_1u_1^1, \dots, u_1u_1^r, u_2u_2^1, \dots, \\ u_2u_2^r, u_3u_3^1, \dots, u_3u_3^r, v_1u_1, v_1v_2, v_2u_2, v_2v_3, v_3u_3, u_1u_2, u_2u_3 \end{matrix} \right\}$ so $|V| = 6r + 6$ and $|E| = 6r + 7$, where r was the number of path graph which has length 1 at each vertex of $L_3 \odot \overline{K}_r$ graph.

Defined the function f that give labels to all vertices of $L_3 \odot \overline{K}_r$ graph as the function below:

$$f(v_1) = 1 \tag{99}$$
$$f(v_2) = 4r + 7 \tag{100}$$
$$f(v_3) = 4r + 5 \tag{101}$$
$$f(u_1) = 10r + 15 \tag{102}$$
$$f(u_2) = 2r + 3 \tag{103}$$
$$f(u_3) = 8r + 13 \tag{104}$$
$$f(v_1^i) = 10r + 15 + 2i; i = 1,2, \dots, r \tag{105}$$
$$f(v_2^i) = 6r + 11 + 2i; i = 1,2, \dots, r \tag{106}$$
$$f(v_3^i) = 4r + 7 + 2i; i = 1,2, \dots, r \tag{107}$$
$$f(u_1^i) = 2r + 3 - 2i; i = 1,2, \dots, r \tag{108}$$
$$f(u_2^i) = 8r + 13 + 2i; i = 1,2, \dots, r \tag{109}$$
$$f(u_3^i) = 4r + 5 - 2i; i = 1,2, \dots, r \tag{110}$$

From the function f that is define at equation (99) $-$ (110), each vertex have distinct label and has a mapping to $\{1, 3, 7, \dots, 2|E| + 1\}$. So, the function f is injective and has a mapping from V to $\{1, 3, 7, \dots, 2|E| + 1\}$. Each edge $pq \in E$ have different label by $f'(pq) = |f(p) - f(q)|$.

So, the function f' is the label at each edges $L_3 \odot \overline{K}_r$ graph like the function below:

$$f'(v_1u_1) = |1 - (10r + 15)| = 10r + 14 \tag{111}$$
$$f'(v_1v_2) = |1 - (4r + 7)| = 4r + 6 \tag{112}$$
$$f'(v_2u_2) = |(4r + 7) - (2r + 3)| = 2r + 4 \tag{113}$$
$$f'(v_2v_3) = |(4r + 7) - (4r + 5)| = 2 \tag{114}$$
$$f'(v_3u_3) = |(4r + 5) - (8r + 13)| = 4r + 8 \tag{115}$$
$$f'(u_1u_2) = |(10r + 15) - (2r + 3)| = 8r + 12 \tag{116}$$

$$f'(u_2u_3) = |(2r+3) - (8r+13)| = 6r+10 \quad (117)$$

$$f'\left(v_1v_1^i\right) = |1 - (10r+15+2i)| = 10r+14+2i; i = 1,2,\dots,r \quad (118)$$

$$f'(v_2v_2^i) = |(4r+7) - (6r+11+2i) = 2r+4+2i; i = 1,2,\dots,r \quad (119)$$

$$f'(v_3v_3^i) = |(4r+5) - (4r+7+2i) = 2+2i; i = 1,2,\dots,r \quad (120)$$

$$f'\left(u_1u_1^i\right) = |(10r+15) - (2r+3-2i)| = 8r+12+2i; i = 1,2,\dots,r \quad (121)$$

$$f'\left(u_2u_2^i\right) = |(2r+3) - (8r+13+2i)| = 6r+10+2i; i = 1,2,\dots,r \quad (122)$$

$$f'\left(u_3u_3^i\right) = |(8r+13) - (4r+5-2i)| = 4r+8+2i; i = 1,2,\dots,r \quad (123)$$

By equation (111)−(123), all edges have different value and has a mapping to $\{2,4,6,\dots,2|E|\}$. Function f' have differentt values for each edge and has a mapping to $\{2,4,6,\dots,2|E|\}$. So, the value of f and f' construct different label for each edge as well the label for each vertex have different label and has a mapping to $\{2,4,6,\dots,2|E|\}$. So that, $L_3 \odot \overline{K}_r$ graph is an odd-even graceful graph. ■

In Figure 8 there is one of an odd-even graceful labelling of $L_3 \odot \overline{K}_2$ graph.

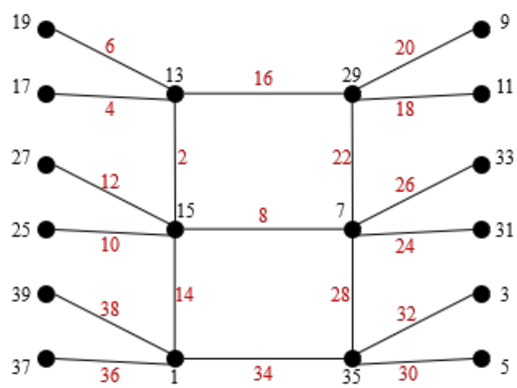

FIGURE 8. Odd-even graceful labelling of $L_3 \odot \overline{K}_2$ graph

CONCLUSION

$L_2 \odot \overline{K}_r$ graph and $L_3 \odot \overline{K}_r$ graph is graceful, have $\hat{\rho}$ labelling, and an odd-even graceful graph.

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