

# Subdivision of Certain Barbell Operation of Origami Graphs has Locating-Chromatic Number Five

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## Abstract

The locating-chromatic number denote by  $\chi_L(G)$ , is the smallest  $t$  such that  $G$  has a locating  $t$ -coloring. In this research, we determined locating-chromatic number for subdivision of certain barbell operation of origami graphs.

## Key words:

locating-chromatic number, sudivision, certain barbell operation, origami graphs.

## 1. Introduction

The concept of partition dimension was introduced by Chartrand et al. [1] as the development of the concept of metric dimension. The application of metric dimension can be found in robotic navigation [2], chemical data classification [3], and the optimization of threat detecting sensors [4]. The locating-chromatic number was first discovered by Chartrand et al. [5] in 2002, with obtained two graph concepts, coloring vertices and partition dimension of a graph. The locating-chromatic number denote by  $\chi_L(G)$ , is the smallest  $t$  such that  $G$  has a locating  $t$ -coloring. Next, investigated the locating-chromatic number for a path graph  $P_n$ , a cycle graph  $C_n$ , and double star graph  $S_{a,b}$ . Furthermore, Chartrand et al. [6] characterized all graphs of order  $n$  with locating-chromatic number  $n - 1$ . Baskoro and Asmiati [7] characterized all trees with locating-chromatic number 3.

The locating-chromatic number of the join of graphs was introduced by Behtoei and Anbarloei [8]. Purwasih et al. [9], obtained locating-chromatic number for a subdivision of a graph on one edge. For graph with dominant vertices have been studied in [10]. In [11], Asmiati found the locating-chromatic number of non-homogeneous caterpillar and firecrackers graph, [12] certain barbell graphs  $B_{m,n}$  and  $B_{P(n,1)}$ . In 2019, Irawan et al. [13] obtained the locating-chromatic number for certain operation of generalized Petersen graphs  $sP(4,2)$ . Furthermore, in [14] determined the locating-chromatic number for  $sP(n,1)$ , origami graphs [15] and certain barbell origami graphs [16]. The locating-chromatic

number of a graph is a newly interesting topic to study because there is no general theorem for determining the locating-chromatic number of any graph. In this research, we specifying about locating-chromatic number for subdivision of certain barbell operation of origami graphs, called  $B_{O_n}^s$ . This study is a continuation of previous research.

The following definition of the locating-chromatic number of a graph, dominant vertices, origami graph, and certain barbell origami graphs is taken from [5, 17, 18, 16]. We use some theorems that is basics to work out a lower bound of the locating-chromatic number of a graph is taken from [5, 15]. The set of neighbours of a vertex  $l$  in  $G$ , denoted by  $N(l)$ .

**Theorem 1.1.** [5] *Let  $c$  be a locating coloring in a connected graph  $G$ . If  $k$  and  $l$  are distinct vertices of  $G$  such that  $d(k,w) = d(l,w)$  for all  $w \in V(G) - \{k,l\}$ , then  $c(k) \neq c(l)$ . In particular, if  $k$  and  $l$  are non-adjacent vertices of  $G$  such that  $N(k) \neq N(l)$ , then  $c(k) \neq c(l)$ .*

**Theorem 1.2.** [15] *Let  $O_n$  be an origami graph for  $n \geq 3$ . The locating chromatic number of an origami graphs  $O_n$  is 4 for  $n=3$  and 5 otherwise.*

## 2. Results and Discussion

In this section, we will discuss the locating-chromatic number for subdivision of certain barbell operation of origami graphs, denoted by  $B_{O_n}^s$ .

**Theorem 2.1.** *Let  $B_{O_n}^s$  be a subdivision of certain barbell operation of origami graphs for  $n \geq 3, s \geq 1$ . Then the locating-chromatic number of  $B_{O_n}^s$  is five,  $\chi_L(B_{O_n}^s) = 5$ .*

**Proof.** Let  $B_{O_n}^s$  be a subdivision of certain barbell operation of origami graphs for  $n \geq 3, s \geq 1$ , with  $V(B_{O_n}^s) = \{u_i, u_{n+i}, v_i, v_{n+i}, w_i, w_{n+i} : i \in \{1, \dots, n\}\} \cup \{x_i :$

$i \in \{1, \dots, s\}$  and  $E(B_{O_n}^s) = \{u_i w_i, u_i v_i, v_i w_i, u_i u_{i+1}, w_i u_{i+1}; i \in \{1, \dots, n\}\} \cup \{u_{n+i} w_{n+i}, u_{n+i} v_{n+i}, v_{n+i}, w_{n+i}, u_{n+i} u_{n+i+1}, w_{n+i} u_{n+i+1}; i \in \{1, \dots, n-1\}\} \cup \{u_n x_1, x_s u_{n+1}\} \cup \{x_i x_{i+1}; i \in \{1, \dots, s-1\}\}$ .

To prove the theorem, we will be divided into two cases :

**Cases 1. For  $n = 3$**

First, we determine lower bound of  $\chi_L(B_{O_3}^s)$ . Since subdivision of certain barbell operation of origami graphs, containing origami graphs  $O_3$ , then by Theorem 1.2.  $\chi_L(B_{O_3}^s) \geq 4$ . Next, we will show that 4 colors are not enough. Origami graph  $B_{O_3}^s$  there are six complete graph with four vertices, denote by  $K_4$ . Without loss of generality, we assign three colors for any  $K_4$  in  $B_{O_3}^s$ , and then the six vertices are dominant vertices. As a result, if we use four colors it is not enough because there are more than one  $K_4$  in  $B_{O_3}^s$ . So  $\chi_L(B_{O_3}^s) \geq 5$ .

Next, we determined the upper bound of  $\chi_L(B_{O_3}^s) \leq 5$ . To show that  $\chi_L(B_{O_3}^s) \leq 5$ , consider the 5-coloring  $c$  on  $B_{O_3}^s$  as follow,

$$\begin{aligned} C_1 &= \{u_1, w_2, u_6, v_5\}; \\ C_2 &= \{u_4, w_1, w_5\}; \\ C_3 &= \{u_2, v_1, w_3, u_5, v_4, v_6\} \cup \{x_i | \text{for odd } i, i \geq 1\}; \\ C_4 &= \{u_3, v_2, w_4, w_6\} \cup \{x_i | \text{for even } i, i \geq 2\}; \\ C_5 &= \{v_3\}; \end{aligned}$$

The coloring  $c$  will create partition  $\Pi$  on  $V(B_{O_3}^s)$ . We shall show that the color codes of all vertices in  $B_{O_3}^s$  are different. We have  $c_\Pi(u_1) = (0, 2, 1, 1, 1)$ ;  $c_\Pi(u_2) = (1, 1, 0, 1, 2)$ ;  $c_\Pi(u_3) = (1, 2, 1, 0, 1)$ ;  $c_\Pi(u_4) = (1, 0, 1, 1, s + 3)$ ;  $c_\Pi(u_5) = (1, 1, 1, 0, s + 4)$ ;  $c_\Pi(u_6) = ((0, 1, 1, 1, s + 4)$ ;  $c_\Pi(v_1) = (1, 3, 2, 0, 1)$ ;  $c_\Pi(v_2) = (1, 3, 0, 1, 2)$ ;  $c_\Pi(v_3) = (2, 0, 1, 1, 3)$ ;  $c_\Pi(v_4) = (2, 1, 1, 0, s + 4)$ ;  $c_\Pi(v_5) = (0, 1, 2, 1, s + 5)$ ;  $c_\Pi(v_6) = (1, 2, 1, 0, s + 5)$ ;  $c_\Pi(w_1) = (1, 3, 2, 1, 0)$ ;  $c_\Pi(w_2) = (0, 2, 1, 1, 2)$ ;  $c_\Pi(w_3) = (1, 1, 1, 0, 2)$ ;  $c_\Pi(w_4) = (2, 1, 0, 1, s + 4)$ ;  $c_\Pi(w_5) = (1, 0, 2, 1, s + 5)$ ;  $c_\Pi(w_6) = (1, 1, 0, 1, s + 4)$ . For  $s = 1$ , we have  $c_\Pi(x_i) = (i + 1, 1, 1, 0, i + 2)$ . For  $i$  odd,  $i \leq \lfloor \frac{s}{2} \rfloor$ ,  $s \geq 2$ , we have  $c_\Pi(x_i) = (i + 1, i + 1, 1, 0, i + 2)$ . For  $i$  even,  $i \leq \lfloor \frac{s}{2} \rfloor$ ,  $s \geq 2$ , we have  $c_\Pi(x_i) = (i + 1, i + 1, 0, 1, i + 2)$ . For  $i$  odd,  $i > \lfloor \frac{s}{2} \rfloor$ ,  $s \geq 2$ , we have  $c_\Pi(x_i) = (s - i + 2, s - i + 1, 1, 0, i + 2)$ . For  $i$  even,  $i > \lfloor \frac{s}{2} \rfloor$ ,  $s \geq 2$ , we have  $c_\Pi(x_i) = (s - i + 2, s - i + 1, 0, 1, i + 2)$ .

Since the color codes of all vertices  $B_{O_3}^s$  are different, thus  $c$  is a locating coloring. So  $\chi_L(B_{O_3}^s) \leq 5$ .

**Case 2. For  $n \geq 4$**

First, we determine lower bound of  $\chi_L(B_{O_n}^s)$  for  $n \geq 4$ . Since subdivision of certain barbell operation of origami graphs, containing origami graphs  $O_n$ , then by Theorem 1.2 it is clear that  $\chi_L(B_{O_n}^s) \geq 5$ .

To show the upper bound for the locating-chromatic number for subdivision of certain barbell operation of origami graphs  $\chi_L(B_{O_n}^s) \geq 5$  for  $n \geq 4$ . Let us diferent some subcases.

**Subcase 2.1. (odd  $n$ ), for  $\lfloor \frac{n}{2} \rfloor$  odd,  $n \geq 5$**

Let  $c$  be a coloring for subdivison of certain barbell operation of origami graph  $B_{O_n}^s$ , for  $\lfloor \frac{n}{2} \rfloor$  odd,  $n \geq 5$  we make the partition  $\Pi$  of  $V(B_{O_n}^s)$  :

$$\begin{aligned} C_1 &= \{w_1 | 1 \leq i \leq n\} \cup \{u_{n+1}\}; \\ C_2 &= \{u_i | \text{for odd } i, 3 \leq i \leq n\} \cup \{v_i | \text{for even } i, 2 \leq i \leq n-1\} \cup \{u_{n+i} | \text{for odd } i, 3 \leq i \leq \lfloor \frac{n}{2} \rfloor - 2\} \cup \{u_{n+i} | \text{for odd } i, \lfloor \frac{n}{2} \rfloor + 2 \leq i \leq n\} \cup \{v_{n+i} | \text{for even } i, 2 \leq i \leq n-1\} \cup \{x_i | \text{for even } i, i \geq 2\}; \\ C_3 &= \{u_i | \text{for even } i, 2 \leq i \leq \lfloor \frac{n}{2} \rfloor - 1\} \cup \{u_i | \text{for even } i, \lfloor \frac{n}{2} \rfloor + 3 \leq i \leq n-1\} \cup \{v_i | \text{for odd } i, 1 \leq i \leq n\} \cup \{u_{n+i} | \text{for even } i, 2 \leq i \leq n-1\} \cup \{v_{n+i} | \text{for odd } i, 1 \leq i \leq n\}; \\ C_4 &= \{u_1\} \cup \{w_{n+i} | 1 \leq i \leq n\} \cup \{x_i | \text{for odd } i, i \geq 1\}; \\ C_5 &= \{u_{\lfloor \frac{n}{2} \rfloor + 1}\} \cup \{u_{n + \lfloor \frac{n}{2} \rfloor}\}. \end{aligned}$$

For  $\lfloor \frac{n}{2} \rfloor$  odd  $n \geq 5$ , the color codes of all the vertices of  $V(B_{O_n}^s)$  are :

$$c_\Pi(u_i) = \begin{cases} 0, & \text{for } 2^{nd} \text{ ordinate, even } i, 3 \leq i \leq n, n \geq 5 \\ & \text{for } 3^{rd} \text{ ordinate, even } i, 2 \leq i \leq \lfloor \frac{n}{2} \rfloor - 1, n \geq 5 \\ & \text{for } 3^{rd} \text{ ordinate, even } i, \lfloor \frac{n}{2} \rfloor + 3 \leq i \leq n-1, n \geq 9 \\ & \text{for } 4^{th} \text{ ordinate, } i = 1 \\ & \text{for } 5^{th} \text{ ordinate, } i = \lfloor \frac{n}{2} \rfloor + 1 \\ 2, & \text{for } 3^{rd} \text{ ordinate, } i = \lfloor \frac{n}{2} \rfloor + 1 \\ i - 1, & \text{for } 4^{th} \text{ ordinate, } 2 \leq i \leq \lfloor \frac{n}{2} \rfloor, n \geq 5 \\ n - i + 1, & \text{for } 4^{th} \text{ ordinate, } \lfloor \frac{n}{2} \rfloor + 1 \leq i \leq n, n \geq 5 \\ i - \lfloor \frac{n}{2} \rfloor - 1, & \text{for } 5^{th} \text{ ordinate, } \lfloor \frac{n}{2} \rfloor + 1 \leq i \leq n, n \geq 5 \\ \lfloor \frac{n}{2} \rfloor - i + 1, & \text{for } 5^{th} \text{ ordinate, } 2 \leq i \leq \lfloor \frac{n}{2} \rfloor, n \geq 5 \\ \lfloor \frac{n}{2} \rfloor - i, & \text{for } 5^{th} \text{ ordinate, } i = 1 \\ 1, & \text{otherwise.} \end{cases}$$

$$c_{\Pi}(v_i) = \begin{cases} 0, & \text{for } 2^{nd} \text{ ordinate, even } i, 2 \leq i \leq n-1, n \geq 5 \\ & \text{for } 3^{rd} \text{ ordinate, odd } i, 1 \leq i \leq n, n \geq 5 \\ 2, & \text{for } 2^{nd} \text{ ordinate, } i = 1 \\ 3, & \text{for } 3^{rd} \text{ ordinate, } i = \lfloor \frac{n}{2} \rfloor + 1 \\ i, & \text{for } 4^{th} \text{ ordinate, } 2 \leq i \leq \lfloor \frac{n}{2} \rfloor, n \geq 5 \\ n-i+2, & \text{for } 4^{th} \text{ ordinate, } \lfloor \frac{n}{2} \rfloor + 1 \leq i \leq n, n \geq 5 \\ \lfloor \frac{n}{2} \rfloor, & \text{for } 5^{th} \text{ ordinate, } i = 1 \\ \lfloor \frac{n}{2} \rfloor - i + 2, & \text{for } 5^{th} \text{ ordinate, } 2 \leq i \leq \lfloor \frac{n}{2} \rfloor, n \geq 5 \\ i - \lfloor \frac{n}{2} \rfloor, & \text{for } 5^{th} \text{ ordinate, } \lfloor \frac{n}{2} \rfloor + 1 \leq i \leq n, n \geq 5 \\ 1, & \text{otherwise.} \end{cases}$$

$$c_{\Pi}(w_i) = \begin{cases} 0, & \text{for } 1^{st} \text{ ordinate, } 1 \leq i \leq n, n \geq 5 \\ 2, & \text{for } 2^{nd} \text{ ordinate, } i = 1 \\ & \text{for } 3^{rd} \text{ ordinate, } i = \lfloor \frac{n}{2} \rfloor \\ i, & \text{for } 4^{th} \text{ ordinate, } 1 \leq i \leq \lfloor \frac{n}{2} \rfloor, n \geq 5 \\ n-i+1, & \text{for } 4^{th} \text{ ordinate, } \lfloor \frac{n}{2} \rfloor + 1 \leq i \leq n, n \geq 5 \\ \lfloor \frac{n}{2} \rfloor - i + 1, & \text{for } 5^{th} \text{ ordinate, } 1 \leq i \leq \lfloor \frac{n}{2} \rfloor, n \geq 5 \\ i - \lfloor \frac{n}{2} \rfloor, & \text{for } 5^{th} \text{ ordinate, } \lfloor \frac{n}{2} \rfloor + 1 \leq i \leq n, n \geq 5 \\ 1, & \text{otherwise.} \end{cases}$$

$$c_{\Pi}(u_{n+i}) = \begin{cases} i-1, & \text{for } 1^{st} \text{ ordinate, } 2 \leq i \leq \lfloor \frac{n}{2} \rfloor, n \geq 5 \\ n-i+1, & \text{for } 1^{st} \text{ ordinate, } \lfloor \frac{n}{2} \rfloor + 1 \leq i \leq n, n \geq 5 \\ 0, & \text{for } 1^{st} \text{ ordinate, } i = 1 \\ & \text{for } 2^{nd} \text{ ordinate, odd } i, 3 \leq i \leq \lfloor \frac{n}{2} \rfloor - 2, n \geq 9 \\ & \text{for } 2^{nd} \text{ ordinate, odd } i, \lfloor \frac{n}{2} \rfloor + 2 \leq i \leq n, n \geq 5 \\ & \text{for } 3^{rd} \text{ ordinate, even } i, 3 \leq i \leq n-1, n \geq 5 \\ & \text{for } 5^{th} \text{ ordinate, } i = \lfloor \frac{n}{2} \rfloor + 1 \\ \lfloor \frac{n}{2} \rfloor - 1, & \text{for } 1^{st} \text{ ordinate, } i = \lfloor \frac{n}{2} \rfloor \\ 2, & \text{for } 2^{nd} \text{ ordinate, } i = \lfloor \frac{n}{2} \rfloor \\ \lfloor \frac{n}{2} \rfloor - i, & \text{for } 5^{th} \text{ ordinate, } 1 \leq i \leq \lfloor \frac{n}{2} \rfloor - 1, n \geq 5 \\ i - \lfloor \frac{n}{2} \rfloor, & \text{for } 5^{th} \text{ ordinate, } \lfloor \frac{n}{2} \rfloor + 1 \leq i \leq n, n \geq 5 \\ 1, & \text{otherwise.} \end{cases}$$

$$c_{\Pi}(v_{n+i}) = \begin{cases} i, & \text{for } 1^{st} \text{ ordinate, } 2 \leq i \leq \lfloor \frac{n}{2} \rfloor, n \geq 5 \\ n-i+2, & \text{for } 1^{st} \text{ ordinate, } \lfloor \frac{n}{2} \rfloor + 1 \leq i \leq n, n \geq 5 \\ 0, & \text{for } 2^{nd} \text{ ordinate, even } i, 2 \leq i \leq n-1, n \geq 5 \\ & \text{for } 3^{rd} \text{ ordinate, odd } i, 1 \leq i \leq n, n \geq 5 \\ 2, & \text{for } 2^{nd} \text{ ordinate, } i = 1 \\ 3, & \text{for } 2^{nd} \text{ ordinate, } i = \lfloor \frac{n}{2} \rfloor \\ \lfloor \frac{n}{2} \rfloor - i + 1, & \text{for } 5^{th} \text{ ordinate, } 1 \leq i \leq \lfloor \frac{n}{2} \rfloor, n \geq 5 \\ i - \lfloor \frac{n}{2} \rfloor + 1, & \text{for } 5^{th} \text{ ordinate, } \lfloor \frac{n}{2} \rfloor + 1 \leq i \leq n, n \geq 5 \\ 1, & \text{otherwise.} \end{cases}$$

$$c_{\Pi}(w_{n+i}) = \begin{cases} i, & \text{for } 1^{st} \text{ ordinate, } 2 \leq i \leq \lfloor \frac{n}{2} \rfloor, n \geq 5 \\ n-i+1, & \text{for } 1^{st} \text{ ordinate, } \lfloor \frac{n}{2} \rfloor + 1 \leq i \leq n, n \geq 5 \\ 2, & \text{for } 2^{nd} \text{ ordinate, } i = 1 \text{ and } i = \lfloor \frac{n}{2} \rfloor \\ 0, & \text{for } 4^{th} \text{ ordinate, } 1 \leq i \leq n, n \geq 5 \\ \lfloor \frac{n}{2} \rfloor - i, & \text{for } 5^{th} \text{ ordinate, } 1 \leq i \leq \lfloor \frac{n}{2} \rfloor - 1, n \geq 5 \\ i - \lfloor \frac{n}{2} \rfloor + 1, & \text{for } 5^{th} \text{ ordinate, } \lfloor \frac{n}{2} \rfloor \leq i \leq n, n \geq 5 \\ 1, & \text{otherwise.} \end{cases}$$

$$c_{\Pi}(x_i) = \begin{cases} s-i+1, & \text{for } 1^{st} \text{ ordinate, } i > \lfloor \frac{s}{2} \rfloor, s \geq 2 \\ i+1, & \text{for } 1^{st} \text{ ordinate, } i \leq \lfloor \frac{s}{2} \rfloor, s \geq 2 \\ & \text{for } 3^{rd} \text{ ordinate, } i \leq \lfloor \frac{s}{2} \rfloor \\ 1, & \text{for } 1^{st} \text{ ordinate, } i = s \\ & \text{for } 2^{nd} \text{ ordinate, odd } i, i \geq 1 \\ & \text{for } 4^{th} \text{ ordinate, even } i, i \geq 2 \\ s-i+2, & \text{for } 3^{rd} \text{ ordinate, } i > \lfloor \frac{s}{2} \rfloor \\ i + \lfloor \frac{n}{2} \rfloor - 2, & \text{for } 5^{th} \text{ ordinate, } i < \lfloor \frac{s}{2} \rfloor \\ s-i + \lfloor \frac{n}{2} \rfloor + 1, & \text{for } 5^{th} \text{ ordinate, } i \geq \lfloor \frac{s}{2} \rfloor \\ 0, & \text{otherwise.} \end{cases}$$

Since for odd  $n$  all vertices have different color codes,  $c$  is a locating coloring for subdivision of certain barbell operation of origami graphs  $B_{O_n}^s$ , so that  $\chi_L(B_{O_n}^s) \leq 5$ , for  $\lfloor \frac{n}{2} \rfloor$  odd,  $n \geq 5$ .

**Subcase 2.2.** (odd  $n$ ), for  $\lfloor \frac{n}{2} \rfloor$  even,  $n \geq 7$

Let  $c$  be a coloring for subdivision of certain barbell operation of origami graph  $B_{O_n}^s$ , for  $\lfloor \frac{n}{2} \rfloor$  even,  $n \geq 7$  we make the partition  $\Pi$  of  $V(B_{O_n}^s)$ :

$$C_1 = \{w_i | 1 \leq i \leq n\} \cup \{u_{n+1}\};$$

$$C_2 = \{u_i | \text{for odd } i, 3 \leq i \leq n\} \cup \{v_i | \text{for even } i, 2 \leq i \leq n-1\} \cup \{u_{n+i} | \text{for even } i, 2 \leq i \leq \lfloor \frac{n}{2} \rfloor - 2\} \cup \{u_{n+i}$$

for even  $i, \lfloor \frac{n}{2} \rfloor + 2 \leq i \leq n - 1 \} \cup \{v_{n+i} | \text{for odd } i, 1 \leq i \leq n\}$ ;

$$C_3 = \{u_i | \text{for even } i, 2 \leq i \leq \lfloor \frac{n}{2} \rfloor - 2\} \cup \{u_i | \text{for even } i, \lfloor \frac{n}{2} \rfloor + 2 \leq i \leq n - 1\} \cup \{v_i | \text{for odd } i, 1 \leq i \leq n\} \cup \{u_{n+i} | \text{for odd } i, 3 \leq i \leq n\} \cup \{v_{n+i} | \text{for even } i, 2 \leq i \leq n - 1\} \cup \{x_i | \text{for even } i, i \geq 2\};$$

$$C_4 = \{u_1\} \cup \{w_{n+i} | 1 \leq i \leq n\} \cup \{x_i | \text{for odd } i, i \geq 1\};$$

$$C_5 = \{u_{\lfloor \frac{n}{2} \rfloor + 1}\} \cup \{u_{n + \lfloor \frac{n}{2} \rfloor}\}.$$

For  $\lfloor \frac{n}{2} \rfloor$  even  $n \geq 7$ , the color codes of all the vertices of  $V(B_{0n}^S)$  are :

$$c_{\Pi}(u_i) = \begin{cases} 0, & \text{for } 2^{nd} \text{ ordinate, odd } i, 3 \leq i \leq n, n \geq 7 \\ & \text{for } 3^{rd} \text{ ordinate, even } i, 2 \leq i \leq \lfloor \frac{n}{2} \rfloor - 2, n \geq 7 \\ & \text{for } 3^{rd} \text{ ordinate, even } i, \lfloor \frac{n}{2} \rfloor + 2 \leq i \leq n - 1, n \geq 7 \\ & \text{for } 4^{th} \text{ ordinate, } i = 1 \\ & \text{for } 5^{th} \text{ ordinate, } i = \lfloor \frac{n}{2} \rfloor \\ 2, & \text{for } 3^{rd} \text{ ordinate, } i = \lfloor \frac{n}{2} \rfloor \\ i - 1, & \text{for } 4^{th} \text{ ordinate, } 2 \leq i \leq \lfloor \frac{n}{2} \rfloor, n \geq 7 \\ n - i + 1, & \text{for } 4^{th} \text{ ordinate, } \lfloor \frac{n}{2} \rfloor + 1 \leq i \leq n, n \geq 7 \\ i - \lfloor \frac{n}{2} \rfloor, & \text{for } 5^{th} \text{ ordinate, } \lfloor \frac{n}{2} \rfloor + 1 \leq i \leq n, n \geq 7 \\ \lfloor \frac{n}{2} \rfloor - i, & \text{for } 5^{th} \text{ ordinate, } 1 \leq i \leq \lfloor \frac{n}{2} \rfloor - 1, n \geq 7 \\ 1, & \text{otherwise.} \end{cases}$$

$$c_{\Pi}(v_i) = \begin{cases} 0, & \text{for } 2^{nd} \text{ ordinate, even } i, 2 \leq i \leq n - 1, n \geq 7 \\ & \text{for } 3^{rd} \text{ ordinate odd } i, 1 \leq i \leq n, n \geq 7 \\ 2, & \text{for } 2^{nd} \text{ ordinate, } i = 1 \\ 3, & \text{for } 3^{rd} \text{ ordinate, } i = \lfloor \frac{n}{2} \rfloor \\ i, & \text{for } 4^{th} \text{ ordinate, } 1 \leq i \leq \lfloor \frac{n}{2} \rfloor, n \geq 7 \\ n - i + 2, & \text{for } 4^{th} \text{ ordinate, } \lfloor \frac{n}{2} \rfloor + 1 \leq i \leq n, n \geq 7 \\ \lfloor \frac{n}{2} \rfloor - i + 1, & \text{for } 5^{th} \text{ ordinate, } 1 \leq i \leq \lfloor \frac{n}{2} \rfloor, n \geq 7 \\ i - \lfloor \frac{n}{2} \rfloor + 1, & \text{for } 5^{th} \text{ ordinate, } \lfloor \frac{n}{2} \rfloor + 1 \leq i \leq n, n \geq 7 \\ 1, & \text{otherwise.} \end{cases}$$

$$c_{\Pi}(w_i) = \begin{cases} 0, & \text{for } 1^{st} \text{ ordinate, } 1 \leq i \leq n, n \geq 7 \\ 2, & \text{for } 2^{nd} \text{ ordinate, } i = 1 \\ & \text{for } 3^{rd} \text{ ordinate, } i = \lfloor \frac{n}{2} \rfloor \\ i, & \text{for } 4^{th} \text{ ordinate, } 2 \leq i \leq \lfloor \frac{n}{2} \rfloor, n \geq 7 \\ n - i + 1, & \text{for } 4^{th} \text{ ordinate, } \lfloor \frac{n}{2} \rfloor + 1 \leq i \leq n, n \geq 7 \\ \lfloor \frac{n}{2} \rfloor - i, & \text{for } 5^{th} \text{ ordinate, } 1 \leq i \leq \lfloor \frac{n}{2} \rfloor, n \geq 7 \\ i - \lfloor \frac{n}{2} \rfloor + 1, & \text{for } 5^{th} \text{ ordinate, } \lfloor \frac{n}{2} \rfloor + 1 \leq i \leq n, n \geq 7 \\ 1, & \text{otherwise.} \end{cases}$$

$$c_{\Pi}(u_{n+i}) = \begin{cases} i - 1, & \text{for } 1^{st} \text{ ordinate, } 2 \leq i \leq \lfloor \frac{n}{2} \rfloor, n \geq 7 \\ n - i + 1, & \text{for } 1^{st} \text{ ordinate, } \lfloor \frac{n}{2} \rfloor + 1 \leq i \leq n, n \geq 7 \\ 0, & \text{for } 1^{st} \text{ ordinate, } i = 1 \\ & \text{for } 2^{nd} \text{ ordinate, even } i, 2 \leq i \leq \lfloor \frac{n}{2} \rfloor - 2, n \geq 7 \\ & \text{for } 2^{nd} \text{ ordinate, even } i, \lfloor \frac{n}{2} \rfloor + 2 \leq i \leq n - 1, n \geq 7 \\ & \text{for } 3^{rd} \text{ ordinate, odd } i, 3 \leq i \leq n, n \geq 7 \\ & \text{for } 5^{th} \text{ ordinate, } i = \lfloor \frac{n}{2} \rfloor \\ 2, & \text{for } 2^{nd} \text{ ordinate, } i = \lfloor \frac{n}{2} \rfloor \\ \lfloor \frac{n}{2} \rfloor - i, & \text{for } 5^{th} \text{ ordinate, } 1 \leq i \leq \lfloor \frac{n}{2} \rfloor - 1, n \geq 7 \\ i - \lfloor \frac{n}{2} \rfloor, & \text{for } 5^{th} \text{ ordinate, } \lfloor \frac{n}{2} \rfloor + 1 \leq i \leq n, n \geq 7 \\ 1, & \text{otherwise.} \end{cases}$$

$$c_{\Pi}(v_{n+i}) = \begin{cases} i, & \text{for } 1^{st} \text{ ordinate, } 2 \leq i \leq \lfloor \frac{n}{2} \rfloor - 1, n \geq 7 \\ n - i + 2, & \text{for } 1^{st} \text{ ordinate, } \lfloor \frac{n}{2} \rfloor + 1 \leq i \leq n, n \geq 7 \\ 0, & \text{for } 2^{nd} \text{ ordinate, odd } i, 1 \leq i \leq n, n \geq 7 \\ & \text{for } 3^{rd} \text{ ordinate, even } i, 2 \leq i \leq n - 1, n \geq 7 \\ 2, & \text{for } 3^{rd} \text{ ordinate, } i = 1 \\ 3, & \text{for } 2^{nd} \text{ ordinate, } i = \lfloor \frac{n}{2} \rfloor \\ \lfloor \frac{n}{2} \rfloor - i + 1, & \text{for } 5^{th} \text{ ordinate, } 1 \leq i \leq \lfloor \frac{n}{2} \rfloor, n \geq 7 \\ i - \lfloor \frac{n}{2} \rfloor + 1, & \text{for } 5^{th} \text{ ordinate, } \lfloor \frac{n}{2} \rfloor + 1 \leq i \leq n, n \geq 7 \\ 1, & \text{otherwise.} \end{cases}$$

$$c_{\Pi}(w_{n+i}) = \begin{cases} i, & \text{for } 1^{st} \text{ ordinate, } 1 \leq i \leq \lfloor \frac{n}{2} \rfloor, n \geq 7 \\ n - i + 1, & \text{for } 1^{st} \text{ ordinate, } \lfloor \frac{n}{2} \rfloor + 1 \leq i \leq n, n \geq 7 \\ 2, & \text{for } 2^{nd} \text{ ordinate, } i = \lfloor \frac{n}{2} \rfloor \\ & \text{for } 3^{rd} \text{ ordinate, } i = 1 \\ 0, & \text{for } 4^{th} \text{ ordinate, } 1 \leq i \leq n, n \geq 7 \\ \lfloor \frac{n}{2} \rfloor - i, & \text{for } 5^{th} \text{ ordinate, } 1 \leq i \leq \lfloor \frac{n}{2} \rfloor - 1, n \geq 7 \\ i - \lfloor \frac{n}{2} \rfloor + 1, & \text{for } 5^{th} \text{ ordinate, } \lfloor \frac{n}{2} \rfloor \leq i \leq n, n \geq 7 \\ 1, & \text{otherwise.} \end{cases}$$

$$c_{\Pi}(x_i) = \begin{cases} s - i + 1, & \text{for } 1^{st} \text{ ordinate, } i > \lfloor \frac{s}{2} \rfloor, s \geq 2 \\ i + 1, & \text{for } 1^{st} \text{ ordinate, } i \leq \lfloor \frac{s}{2} \rfloor, s \geq 2 \\ 1, & \text{for } 1^{st} \text{ ordinate, } i = s \\ & \text{for } 3^{rd} \text{ ordinate, odd } i, i \geq 1 \\ & \text{for } 4^{th} \text{ ordinate, even } i, i \geq 2 \\ i, & \text{for } 2^{nd} \text{ ordinate, } i \leq \lfloor \frac{s}{2} \rfloor \\ s - i + 2, & \text{for } 2^{nd} \text{ ordinate, } i > \lfloor \frac{s}{2} \rfloor \\ 2, & \text{for } 3^{rd} \text{ ordinate, } s = 1 \\ i + \lfloor \frac{n}{2} \rfloor - 2, & \text{for } 5^{th} \text{ ordinate, } i \leq \lfloor \frac{s}{2} \rfloor \\ s - i + \lfloor \frac{n}{2} \rfloor, & \text{for } 5^{th} \text{ ordinate, } i > \lfloor \frac{s}{2} \rfloor \\ 0, & \text{otherwise.} \end{cases}$$

Since for odd  $n$  all vertices have different color codes,  $c$  is a locating coloring for subdivision of certain barbell operation of origami graphs  $B_{O_n}^s$ , so that  $\chi_L(B_{O_n}^s) \leq 5$ , for  $\lfloor \frac{n}{2} \rfloor$  even,  $n \geq 7$ .

**Subcase 2.3.** (even  $n$ ), for  $\frac{n}{2}$  odd,  $n \geq 6$

Let  $c$  be a coloring for subdivision of certain barbell operation of origami graph  $B_{O_n}^s$ , for  $\frac{n}{2}$  odd,  $n \geq 6$  we make the partition  $\Pi$  of  $V(B_{O_n}^s)$ :

$$\begin{aligned} C_1 &= \{w_i | 1 \leq i \leq \frac{n}{2} - 1\} \cup \{w_i | \frac{n}{2} + 1 \leq i \leq n\} \cup \{u_{n+1}\}; \\ C_2 &= \{u_i | \text{for odd } i, 3 \leq i \leq n - 1\} \cup \{v_i | \text{for even } i, 2 \leq i \leq n\} \cup \{u_{n+i} | \text{for even } i, 2 \leq i \leq n\} \cup \{v_{n+i} | \text{for odd } i, 1 \leq i \leq n - 1\}; \\ C_3 &= \{u_i | \text{for even } i, 2 \leq i \leq n\} \cup \{v_i | \text{for odd } i, 3 \leq i \leq n - 1\} \cup \{u_{n+i} | \text{for odd } i, 3 \leq i \leq n - 1\} \cup \{v_{n+i} | \text{for even } i, 2 \leq i \leq n\} \cup \{x_i | \text{for odd } i, i \geq 1\}; \\ C_4 &= \{u_1\} \cup \{w_{n+i} | 1 \leq i \leq \frac{n}{2} - 1\} \cup \{w_{n+i} | \frac{n}{2} + 1 \leq i \leq n\} \cup \{x_i | \text{for odd } i, i \geq 2\}; \\ C_5 &= \{w_{\frac{n}{2}}\} \cup \{w_{n+\frac{n}{2}}\}. \end{aligned}$$

For  $\frac{n}{2}$  odd  $n \geq 6$ , the color codes of all the vertices of  $V(B_{O_n}^s)$  are:

$$c_{\Pi}(u_i) = \begin{cases} 0, & \text{for } 2^{nd} \text{ ordinate, odd } i, 3 \leq i \leq n - 1, n \geq 6 \\ & \text{for } 3^{rd} \text{ ordinate, even } i, 2 \leq i \leq n, n \geq 6 \\ & \text{for } 4^{th} \text{ ordinate, } i = 1 \\ 2, & \text{for } 3^{rd} \text{ ordinate, } i = 1 \\ i - 1, & \text{for } 4^{th} \text{ ordinate, } 2 \leq i \leq \frac{n}{2}, n \geq 6 \\ n - i + 1, & \text{for } 4^{th} \text{ ordinate, } \frac{n}{2} + 1 \leq i \leq n, n \geq 6 \\ \frac{n}{2} - i + 1, & \text{for } 5^{th} \text{ ordinate, } 1 \leq i \leq \frac{n}{2}, n \geq 6 \\ i - \frac{n}{2}, & \text{for } 5^{th} \text{ ordinate, } \frac{n}{2} + 1 \leq i \leq n, n \geq 6 \\ 1, & \text{otherwise.} \end{cases}$$

$$c_{\Pi}(v_i) = \begin{cases} 3, & \text{for } 2^{nd} \text{ ordinate, } i = 1 \\ 0, & \text{for } 2^{nd} \text{ ordinate, even } i, 2 \leq i \leq n, n \geq 6 \\ & \text{for } 3^{rd} \text{ ordinate, odd } i, 1 \leq i \leq n - 1, n \geq 6 \\ i, & \text{for } 4^{th} \text{ ordinate, } 1 \leq i \leq \frac{n}{2}, n \geq 6 \\ n - i + 2, & \text{for } 4^{th} \text{ ordinate, } \frac{n}{2} + 1 \leq i \leq n, n \geq 6 \\ \frac{n}{2} - i + 2, & \text{for } 5^{th} \text{ ordinate, } 1 \leq i \leq \frac{n}{2} - 1, n \geq 6 \\ i - \frac{n}{2} + 1, & \text{for } 5^{th} \text{ ordinate, } \frac{n}{2} \leq i \leq n, n \geq 6 \\ 1, & \text{otherwise.} \end{cases}$$

$$c_{\Pi}(w_i) = \begin{cases} 0, & \text{for } 1^{st} \text{ ordinate, } 1 \leq i \leq \frac{n}{2} - 1, n \geq 6 \\ & \text{for } 1^{st} \text{ ordinate, } \frac{n}{2} + 1 \leq i \leq n, n \geq 6 \\ & \text{for } 5^{th} \text{ ordinate, } i = \frac{n}{2} \\ 2, & \text{for } 2^{nd} \text{ ordinate, } i = 1 \\ i, & \text{for } 4^{th} \text{ ordinate, } 1 \leq i \leq \frac{n}{2}, n \geq 6 \\ n - i + 1, & \text{for } 4^{th} \text{ ordinate, } \frac{n}{2} + 1 \leq i \leq n, n \geq 6 \\ \frac{n}{2} - i + 1, & \text{for } 5^{th} \text{ ordinate, } 1 \leq i \leq \frac{n}{2} - 1, n \geq 6 \\ i - \frac{n}{2} + 1, & \text{for } 5^{th} \text{ ordinate, } \frac{n}{2} + 1 \leq i \leq n, n \geq 6 \\ 1, & \text{otherwise.} \end{cases}$$

$$c_{\Pi}(u_{n+i}) = \begin{cases} i - 1, & \text{for } 1^{st} \text{ ordinate, } 2 \leq i \leq \frac{n}{2}, n \geq 6 \\ n - i + 1, & \text{for } 1^{st} \text{ ordinate, } \frac{n}{2} \leq i \leq n, n \geq 6 \\ 0, & \text{for } 2^{nd} \text{ ordinate, even } i, 2 \leq i \leq n, n \geq 6 \\ & \text{for } 3^{rd} \text{ ordinate, odd } i, 3 \leq i \leq n - 1, n \geq 6 \\ & \text{for } 1^{st} \text{ ordinate, } i = 1 \\ 2, & \text{for } 3^{rd} \text{ ordinate, } i = 1 \\ \frac{n}{2} - i + 1, & \text{for } 5^{th} \text{ ordinate, } 1 \leq i \leq \frac{n}{2}, n \geq 6 \\ i - \frac{n}{2}, & \text{for } 5^{th} \text{ ordinate, } \frac{n}{2} + 1 \leq i \leq n, n \geq 6 \\ 1, & \text{otherwise.} \end{cases}$$

$$c_{\Pi}(v_{n+i}) = \begin{cases} i, & \text{for } 1^{st} \text{ ordinate, } 2 \leq i \leq \frac{n}{2}, n \geq 6 \\ n - i + 2, & \text{for } 1^{st} \text{ ordinate, } \frac{n}{2} + 1 \leq i \leq n, n \geq 6 \\ 0, & \text{for } 2^{nd} \text{ ordinate, odd } i, 1 \leq i \leq n - 1, n \geq 6 \\ & \text{for } 3^{rd} \text{ ordinate, even } i, 2 \leq i \leq n, n \geq 6 \\ 3, & \text{for } 2^{nd} \text{ ordinate, } i = 1 \\ \frac{n}{2} - i + 1, & \text{for } 5^{th} \text{ ordinate, } 1 \leq i \leq \frac{n}{2} - 1, n \geq 6 \\ i - \frac{n}{2} + 1, & \text{for } 5^{th} \text{ ordinate, } \frac{n}{2} \leq i \leq n, n \geq 6 \\ 1, & \text{otherwise.} \end{cases}$$

$$c_{\Pi}(w_{n+i}) = \begin{cases} i, & \text{for } 1^{st} \text{ ordinate, } 1 \leq i \leq \frac{n}{2}, n \geq 6 \\ n - i + 1, & \text{for } 1^{st} \text{ ordinate, } \frac{n}{2} + 1 \leq i \leq n, n \geq 6 \\ 2, & \text{for } 2^{nd} \text{ ordinate, } i = 1 \\ 0, & \text{for } 4^{th} \text{ ordinate, } 1 \leq i \leq \frac{n}{2} - 1, n \geq 6 \\ & \text{for } 4^{th} \text{ ordinate, } \frac{n}{2} + 1 \leq i \leq n, n \geq 6 \\ & \text{for } 5^{th} \text{ ordinate, } i = \frac{n}{2} \\ \frac{n}{2} - i + 1, & \text{for } 5^{th} \text{ ordinate, } 1 \leq i \leq \frac{n}{2} - 1, n \geq 6 \\ i - \frac{n}{2} + 1, & \text{for } 5^{th} \text{ ordinate, } \frac{n}{2} + 1 \leq i \leq n, n \geq 6 \\ 1, & \text{otherwise.} \end{cases}$$

$$c_{\Pi}(x_i) = \begin{cases} s - i + 1, & \text{for } 1^{st} \text{ ordinate, } i > \lfloor \frac{s}{2} \rfloor, s \geq 2 \\ i + 1, & \text{for } 1^{st} \text{ ordinate, } i \leq \lfloor \frac{s}{2} \rfloor, s \geq 2 \\ 0, & \text{for } 3^{rd} \text{ ordinate, even } i, i \geq 2 \\ & \text{for } 4^{th} \text{ ordinate, odd } i, i \geq 1 \\ i, & \text{for } 2^{nd} \text{ ordinate, } i \leq \lfloor \frac{s}{2} \rfloor, s \geq 2 \\ s - i + 2 & \text{for } 2^{nd} \text{ ordinate, } i > \lfloor \frac{s}{2} \rfloor, s \geq 2 \\ i + \frac{n}{2} - 1 & \text{for } 5^{th} \text{ ordinate, } i \leq \lfloor \frac{s}{2} \rfloor, s \geq 2 \\ s - i + \frac{n}{2} & \text{for } 5^{th} \text{ ordinate, } i > \lfloor \frac{s}{2} \rfloor, s \geq 2 \\ 1, & \text{otherwise.} \end{cases}$$

Since for odd  $n$  all vertices have different color codes,  $c$  is a locating coloring for subdivision of certain barbell operation of origami graphs  $B_{O_n}^s$ , so that  $\chi_L(B_{O_n}^s) \leq 5$ , for  $\frac{n}{2}$  odd,  $n \geq 6$ .

**Subcase 2.4.** (even  $n$ ), for  $\frac{n}{2}$  even,  $n \geq 4$

Let  $c$  be a coloring for subdivision of certain barbell operation of origami graph  $B_{O_n}^s$ ,  $\frac{n}{2}$  even,  $n \geq 4$  we make the partition  $\Pi$  of  $V(B_{O_n}^s)$ :

$$\begin{aligned} C_1 &= \{w_i | 1 \leq i \leq \frac{n}{2} - 1\} \cup \{w_i | \frac{n}{2} + 1 \leq i \leq n\} \cup \{u_{n+1}\}; \\ C_2 &= \{u_i | \text{for odd } i, 1 \leq i \leq n - 1\} \cup \{v_i | \text{for even } i, 2 \leq i \leq n\} \cup \{u_{n+i} | \text{for odd } i, 3 \leq i \leq n\} \cup \{v_{n+i} | \text{for even } i, 1 \leq i \leq n - 1\}; \\ C_3 &= \{u_i | \text{for even } i, 2 \leq i \leq n - 2\} \cup \{v_i | \text{for odd } i, 1 \leq i \leq n - 1\} \cup \{u_{n+i} | \text{for even } i, 3 \leq i \leq n - 1\} \cup \{v_{n+i} | \text{for odd } i, 2 \leq i \leq n\} \cup \{x_i | \text{for odd } i, i \geq 1\}; \\ C_4 &= \{u_n\} \cup \{w_{n+i} | 1 \leq i \leq \frac{n}{2}\} \cup \{w_{n+i} | \frac{n}{2} + 2 \leq i \leq n\} \cup \{x_i | \text{for even } i, i \geq 2\}; \\ C_5 &= \{w_{\frac{n}{2}}\} \cup \{w_{\frac{n}{2}+1}\}. \end{aligned}$$

For  $\frac{n}{2}$  even  $n \geq 4$ , the color codes of all the vertices of  $V(B_{O_n}^s)$  are :

$$c_{\Pi}(u_i) = \begin{cases} 0, & \text{for } 2^{nd} \text{ ordinate, odd } i, 1 \leq i \leq n - 1, n \geq 4 \\ & \text{for } 3^{rd} \text{ ordinate, even } i, 2 \leq i \leq n, n \geq 4 \\ & \text{for } 4^{th} \text{ ordinate, } i = n \\ i, & \text{for } 4^{th} \text{ ordinate, } 1 \leq i \leq \frac{n}{2}, n \geq 4 \\ n - i, & \text{for } 4^{th} \text{ ordinate, } \frac{n}{2} + 1 \leq i \leq n - 1, n \geq 4 \\ \frac{n}{2} - i + 1, & \text{for } 5^{th} \text{ ordinate, } 1 \leq i \leq \frac{n}{2}, n \geq 4 \\ i - \frac{n}{2}, & \text{for } 5^{th} \text{ ordinate, } \frac{n}{2} + 1 \leq i \leq n, n \geq 4 \\ 1, & \text{otherwise.} \end{cases}$$

$$c_{\Pi}(v_i) = \begin{cases} 0, & \text{for } 2^{nd} \text{ ordinate, even } i, 2 \leq i \leq n, n \geq 4 \\ & \text{for } 3^{rd} \text{ ordinate, odd } i, 1 \leq i \leq n - 1, n \geq 4 \\ i + 1, & \text{for } 4^{th} \text{ ordinate, } 1 \leq i \leq \frac{n}{2}, n \geq 4 \\ n - i + 1, & \text{for } 4^{th} \text{ ordinate, } \frac{n}{2} + 1 \leq i \leq n, n \geq 4 \\ \frac{n}{2} - i + 2, & \text{for } 5^{th} \text{ ordinate, } 1 \leq i \leq \frac{n}{2} - 1, n \geq 4 \\ i - \frac{n}{2} + 1, & \text{for } 5^{th} \text{ ordinate, } \frac{n}{2} \leq i \leq n, n \geq 4 \\ 1, & \text{otherwise.} \end{cases}$$

$$c_{\Pi}(w_i) = \begin{cases} 0, & \text{for } 1^{st} \text{ ordinate, } 1 \leq i \leq \frac{n}{2} - 1, n \geq 4 \\ & \text{for } 1^{st} \text{ ordinate, } \frac{n}{2} + 1 \leq i \leq n, n \geq 4 \\ & \text{for } 5^{th} \text{ ordinate, } i = \frac{n}{2} \\ 2, & \text{for } 1^{st} \text{ ordinate, } i = \frac{n}{2} \\ & \text{for } 3^{rd} \text{ ordinate, } i = n \\ i + 1, & \text{for } 4^{th} \text{ ordinate, } 1 \leq i \leq \frac{n}{2}, n \geq 4 \\ n - i, & \text{for } 4^{th} \text{ ordinate, } \frac{n}{2} \leq i \leq n - 1, n \geq 4 \\ \frac{n}{2} - i + 1, & \text{for } 5^{th} \text{ ordinate, } 1 \leq i \leq \frac{n}{2}, n \geq 4 \\ i - \frac{n}{2} + 1, & \text{for } 5^{th} \text{ ordinate, } \frac{n}{2} + 1 \leq i \leq n, n \geq 4 \\ 1, & \text{otherwise.} \end{cases}$$

$$c_{\Pi}(u_{n+i}) = \begin{cases} i - 1, & \text{for } 1^{st} \text{ ordinate, } 2 \leq i \leq \frac{n}{2}, n \geq 4 \\ n - i + 1, & \text{for } 1^{st} \text{ ordinate, } \frac{n}{2} + 1 \leq i \leq n, n \geq 4 \\ 0, & \text{for } 2^{nd} \text{ ordinate, even } i, 2 \leq i \leq n, n \geq 4 \\ & \text{for } 3^{rd} \text{ ordinate, odd } i, 3 \leq i \leq n - 1, n \geq 4 \\ & \text{for } 1^{st} \text{ ordinate, } i = 1 \\ 2, & \text{for } 3^{rd} \text{ ordinate, } i = 1 \\ \frac{n}{2} - i + 1, & \text{for } 5^{th} \text{ ordinate, } 1 \leq i \leq \frac{n}{2}, n \geq 4 \\ i - \frac{n}{2}, & \text{for } 5^{th} \text{ ordinate, } \frac{n}{2} + 1 \leq i \leq n, n \geq 4 \\ 1, & \text{otherwise.} \end{cases}$$

$$c_{\Pi}(v_{n+i}) = \begin{cases} i, & \text{for } 1^{st} \text{ ordinate, } 2 \leq i \leq \frac{n}{2} + 1, n \geq 4 \\ n - i + 2, & \text{for } 1^{st} \text{ ordinate, } \frac{n}{2} + 2 \leq i \leq n, n \geq 4 \\ 0, & \text{for } 2^{nd} \text{ ordinate, odd } i, 1 \leq i \leq n - 1, n \geq 4 \\ & \text{for } 3^{rd} \text{ ordinate, even } i, 2 \leq i \leq n, n \geq 4 \\ 3, & \text{for } 2^{nd} \text{ ordinate, } i = 1 \\ \frac{n}{2} - i + 3, & \text{for } 5^{th} \text{ ordinate, } 1 \leq i \leq \frac{n}{2}, n \geq 4 \\ i - \frac{n}{2}, & \text{for } 5^{th} \text{ ordinate, } \frac{n}{2} + 1 \leq i \leq n, n \geq 4 \\ 1, & \text{otherwise.} \end{cases}$$

$$c_{\Pi}(w_{n+i}) = \begin{cases} i, & \text{for } 1^{st} \text{ ordinate, } 1 \leq i \leq \frac{n}{2}, n \geq 4 \\ n - i + 1, & \text{for } 1^{st} \text{ ordinate, } \frac{n}{2} + 1 \leq i \leq n, n \geq 4 \\ 0, & \text{for } 4^{th} \text{ ordinate, } 1 \leq i \leq \frac{n}{2}, n \geq 4 \\ & \text{for } 4^{th} \text{ ordinate, } \frac{n}{2} + 2 \leq i \leq n, n \geq 4 \\ & \text{for } 5^{th} \text{ ordinate, } i = \frac{n}{2} + 1 \\ 2, & \text{for } 4^{th} \text{ ordinate, } i = \frac{n}{2} + 1 \\ \frac{n}{2} - i + 2, & \text{for } 5^{th} \text{ ordinate, } 1 \leq i \leq \frac{n}{2}, n \geq 4 \\ i - \frac{n}{2}, & \text{for } 5^{th} \text{ ordinate, } \frac{n}{2} + 2 \leq i \leq n, n \geq 4 \\ 1, & \text{otherwise.} \end{cases}$$

$$c_{\Pi}(x_i) = \begin{cases} s - i + 1, & \text{for } 1^{st} \text{ ordinate, } i > \lfloor \frac{s}{2} \rfloor, s \geq 2 \\ i + 1, & \text{for } 1^{st} \text{ ordinate, } i \leq \lfloor \frac{s}{2} \rfloor, s \geq 2 \\ & \text{for } 2^{nd} \text{ ordinate, } i < \lfloor \frac{s}{2} \rfloor, s \geq 2 \\ s - i + 2, & \text{for } 2^{nd} \text{ ordinate, } i \geq \lfloor \frac{s}{2} \rfloor, s \geq 2 \\ 0, & \text{for } 3^{rd} \text{ ordinate, odd } i, i \geq 1 \\ & \text{for } 4^{th} \text{ ordinate, even } i, i \geq 2 \\ i + \frac{n}{2}, & \text{for } 5^{th} \text{ ordinate, } i < \lfloor \frac{s}{2} \rfloor, s \geq 2 \\ s - i + \frac{n}{2} + 1, & \text{for } 5^{th} \text{ ordinate, } i \geq \lfloor \frac{s}{2} \rfloor, s \geq 2 \\ 1, & \text{otherwise.} \end{cases}$$

Since for odd n all vertices have different color codes, c is a locating coloring for subdivison of certain barbell operation of origami graphs  $B_{\delta_n}^s$ , so that  $\chi_L(B_{\delta_n}^s) \leq 5$ , for  $\frac{n}{2}$  even,  $n \geq 4$ . This completes the proof of the theorem. □

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