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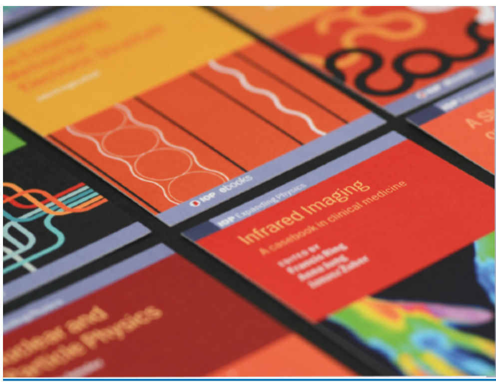

Determining the Noetherian Property of Generalized Power Series Modules by Using *X*-Sub-Exact Sequence

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Abstract. The Noetherian property of the generalized power series module can determine in several ways. This paper uses the sub-exact sequence of modules over a ring R to determine this property. This investigation not only determines the Noetherian property of the generalized power series module but also the Noetherian property of its submodule. Furthermore, we give a construction of $R[[S]]$-homomorphism between the generalized power series modules.

Keyword: noetherian, strictly ordered monoid, generalized power series modules, exact sequence, sub-exact sequence.

1. Introduction

The exact sequence of modules is one of the essential concepts in module theory [1], [2]. In [3], Fitriani et al. introduced a sub-exact sequence of modules. This concept is motivated by the quasi exact sequence established by Davvaz and Parnian-Garamaleky [4]. Furthermore, they use this concept to generalize the generator of modules related to a family of modules over a ring R [5]. Moreover, using a generalization of a linearly independent set of modules [6], they obtained a basis and free modules related to a family of modules [7].

Given ring R, monoid $(S, \leq)$ with a strictly ordered, and a monoid homomorphism ω from S to End(R). In 2019, Faisol and Fitriani gave some conditions for skew GPSM to be a T[[S, ω]]-Noetherian module over a ring R[[S, ω]] [8]. This sufficient condition is a generalization of the previous results [9], which were obtained by applying the properties in [10], generalizing the sufficient conditions in [11], and using the relations specified in [12].

Varadarajan [13] introduce the generalized power series module (GPSM). This module is a module over the generalized power series ring (we call it by GPSR), introduced by Ribenboim [14]. Moreover, the results of Ribenboim construction were generalized by Mazurek and Ziembowski [15] by utilizing the monoid homomorphism used in the convolution multiplication operation. In addition to constructing GPSM, Varadarajan [16] also provides necessary and sufficient conditions that GPSM is a Noetherian module. In this paper, we give a method to determine the Noetherian property of the generalized power series module. We use the concept of the sub-exact sequence to determine this property. In this way, we also can determine the Noetherian property of its submodules. Moreover, we give a construction of $R[[S]]$-homomrphism between the generalized power series modules.

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2. The Main Results

Let R be a commutative ring with $1_R \in R$ and S be a monoid with strictly ordered. Let N_1, N_2, and N_3 be three modules over ring R. The set $N_i[[S]]$ consists of all function μ from S to N_i such that the support of f is Artinian and narrow (we denote support of f by supp(f), that is the set of $s \in S$, where $f(s)$ is not equal to 0), for $i = 1, 2, 3$. We can write the set as follow:

$$N_i[[S]] = \{\mu : S \to N_i \mid \text{supp}(\mu) \text{ is Artinian and narrow}\},$$

$i = 1, 2, 3$.

Before we give a condition when a submodule $L[[S]]$ of $N_2[[S]]$ is Noetherian $R[[S]]$-module, we recall that if L is a submodule of N, then $L[[S]]$ is a submodule of $N_2[[S]]$ as a module over $R[[S]]$. Let

$$L[[S]] = \{\mu \in N_2[[S]] \mid \mu(s) \in L, \text{ for all } s \in S\}.$$

The set $L[[S]]$ is a submodule of $N_2[[S]]$.

Let K, L, M be R-modules and X be R-submodules of L. Recall that the triple (K, L, M) is said to be *X-sub-exact at L* if there exist f and g such that the sequence $K \xrightarrow{f} X \xrightarrow{g} M$ is exact. In the following proposition, we give a condition when a submodule $L[[S]]$ of $N_2[[S]]$ is Noetherian.

Proposition 1. Let R be a commutative ring with $1 \in R$ and $(S, \leq)$ be a monoid with a strictly ordered. Let N_1, N_2, and N_3 are R-modules, and L is a submodule of N_2 over R.
If the triple $(N_1[[S]], N_2[[S]], N_3[[S]])$ is $L[[S]]$-sub-exact as an $R[[S]]$-module, $N_1[[S]]$ and $N_3[[S]]$ are Noetherian $R[[S]]$-modules, then $L[[S]]$ is a Noetherian $R[[S]]$-module.

Proof. Since the triple $(N_1[[S]], N_2[[S]], N_3[[S]])$ is $L[[S]]$-sub-exact, based on [3], we have the following sequence of a module over $R[[S]]$ is exact.

$$N_1[[S]] \longrightarrow L[[S]] \longrightarrow N_3[[S]] \tag{1}$$

Since (1) is exact, there are $R[[S]]$-homomorphism f and g, where f is an $R[[S]]$-homomorphism from $N_1[[S]]$ to $L[[S]]$, g is an $R[[S]]$-homomorphism from $L[[S]]$ to $N_3[[S]]$, and Im(f) = Ker(g). By hypothesis, $N_1[[S]]$ and $N_3[[S]]$ are Noetherian modules over $R[[S]]$. Hence based on [17], we have $N[[S]]$ is a Noetherian as a module over $R[[S]]$.

Given three R-modules N_1, N_2, and N_3. Fitriani et al. [3] construct a set $\sigma(N_1, N_2, N_3)$ that consists of all submodules X of N_2 such that the triple (N_1, N_2, N_3) is an X-sub-exact at N_2, i.e.:

$$\sigma(N_1, N_2, N_3) = \{X \text{ submodule of } N_2 \mid (N_1, N_2, N_3) \text{ is an } X\text{-sub exact at } N_2\}.$$

In this case, we construct the set $\sigma(N_1[[S]], N_2[[S]], N_3[[S]])$ that consist of all submodules X of $N_2[[S]]$ such that the triple of generalized power series modules $(N_1[[S]], N_2[[S]], N_3[[S]])$ is an X-sub exact at $N_2[[S]]$, i.e.: $\sigma(N_1[[S]], N_2[[S]], N_3[[S]]) = \{X \leq N_2[[S]] \mid \text{the triple } (N_1[[S]], N_2[[S]], N_3[[S]]) \text{ is an } X\text{-sub exact at } N_2[[S]]\}$.

As a direct consequence of Proposition 1, we have the following result.

Corollary 1. Let R be a commutative ring with $1 \in R$ and $(S, \leq)$ be a monoid with strictly ordered. Let M_1, M_2, and M_3 are modules over ring R. If $N_1[[S]]$ and $N_3[[S]]$ are Noetherian modules over $R[[S]]$, then a submodule X of N_2 is Noetherian, for every $X \in \sigma(N_1[[S]], N_2[[S]], N_3[[S]])$.

Proof. Let $X \in \sigma(N_1[[S]], N_2[[S]], N_3[[S]])$. We have the following exact sequence of $R[[S]]$-modules:

$$N_1[[S]] \longrightarrow X \longrightarrow N_3[[S]]$$

From Proposition 1, we have X is Noether.

In [18], Ziembowski gives a construction of a homomorphism of skew GPSR. Based on his construction, we construct a homomorphism of generalized power series modules in the following proposition.

Proposition 2. Given a commutative ring R with identity element 1. Given a monoid $(S, \leq)$ with a strictly ordered, an endomorphism ω of S such that for every subset Artinian and narrow $T \subseteq S$, $\omega(T)$

is Artinian, narrow, and $h(\omega^{-1}(x)) = h(x)$, for every x is in S, and h is in $R[[S]]$. Let φ be an R-homomorphism from N_2 to N_3, where N_2, N_3 be R-modules. For $\mu \in N_2[[S]]$, we define:

$$\phi: N_2[[S]] \to N_3[[S]]$$
$$\mu \mapsto \bar{\mu},$$

where

$$\bar{\mu}(x) = \begin{cases} \varphi \circ \mu \circ \omega^{-1}(x) & ; \text{ if } x \in \omega(S), \\ 0 & ; \text{ otherwise.} \end{cases} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \ (1)$$

Then ϕ is an $R[[S]]$-homomorphism from $N_2[[S]]$ to $N_3[[S]]$.

Proof. Since $\mathrm{supp}(\bar{\mu}) \subseteq \omega(\mathrm{supp}\ (\mu))$, we have $\bar{\mu} \in N_3[[S]]$. Now, we will show that ϕ is a $R[[S]]$-homomorphism from $N_2[[S]]$ to $N_3[[S]]$.

a. Let $\mu, \beta \in N_2[[S]]$, and $x \in S$. By (1), we have:

$$\begin{aligned} \overline{\mu + \beta}(x) &= \varphi \circ (\mu + \beta) \circ \omega^{-1}(x) \\ &= \varphi((\mu + \beta) \circ \omega^{-1}(x)) \\ &= \varphi(\mu(\omega^{-1}(x)) + \beta(\omega^{-1}(x))) \\ &= \varphi(\mu(\omega^{-1}(x))) + \varphi(\beta(\omega^{-1}(x))) \\ &= \varphi \circ \mu \circ \omega^{-1}(x) + \varphi \circ \beta \circ \omega^{-1}(x) \\ &= \bar{\mu}(x) + \bar{\beta}(x). \end{aligned}$$

This equation implies that $\overline{\mu + \beta} = \bar{\mu} + \bar{\beta}$, and hence $\phi(\mu + \beta) = \phi(\mu) + \phi(\beta)$, for every $\mu, \beta \in N_2[[S]]$.

b. Let $\mu \in N_2[[S]]$, $h \in R[[S]]$, and $x \in S$. By (1), we get:

$$\begin{aligned} \overline{h\mu}(x) &= \varphi \circ (h\mu) \circ \omega^{-1}(x) \\ &= \varphi\big((h\mu)(\omega^{-1}(x))\big) \\ &= \varphi\big(\textstyle\sum_{s+t=\omega^{-1}(x)} h(s)\, \mu(t)\big) \\ &= \textstyle\sum_{s+t=\omega^{-1}(x)} \varphi(h(s)\mu(t)) \\ &= \textstyle\sum_{s+t=\omega^{-1}(x)} h(s)\varphi(\mu(t)) \\ &= \textstyle\sum_{\omega^{-1}(u)+\omega^{-1}(v)=\omega^{-1}(x)} h(\omega^{-1}(u))\varphi(\mu(\omega^{-1}(v)))\ ;\ s = \omega^{-1}(u) \text{ dan } t = \omega^{-1}(v) \\ &= \textstyle\sum_{\substack{\omega^{-1}(u)+\omega^{-1}(v)=\omega^{-1}(x) \\ \omega^{-1}(u+v)=\omega^{-1}(x) \\ u+v=x}} h(u)\varphi(\mu(\omega^{-1}(v)))\ ;\ h\big(\omega^{-1}(u)\big) = h(u) \\ &= \textstyle\sum_{u+v=x} h(u)(\varphi \circ \mu \circ \omega^{-1})(v) \\ &= \textstyle\sum_{u+v=x} h(u)\bar{\mu}(v) \\ &= h\bar{\mu}(x). \end{aligned}$$

Hence, for every $\mu \in N_2[[S]]$, $h \in R[[S]]$, we have $\phi(h\mu) = \overline{h\mu} = h\bar{\mu} = h\,\phi(\mu)$.

From a-b, we can conclude that ϕ is an R[[S]]-homomorphism from $N_2[[S]]$ to $N_3[[S]]$.

Given an R-module M, we recall that a submodule N of M is a direct summand of M if there exists $K \leq M$ such that $M = N \oplus K$, i.e., $M = N + K$, and $N \cap K = 0$. In this case, every $m \in M$ can be uniquely written as $m = a + b$, where $a \in N$, and $b \in K$ [17]. Next, we will use the construction of R[[S]]-homomorphism in Proposition 2 to provide the Noetherian property of the GPSM.

Proposition 3. Given a commutative ring R with $1 \in R$ and a monoid $(S, \leq)$ with a strictly ordered. Let N_1, N_2, and N_3 are R-modules, and $L[[S]]$ is a direct summand of $N_2[[S]]$ as an $R[[S]]$-module. If $(N_1[[S]], N_2[[S]], N_3[[S]])$ is $L[[S]]$-sub-exact as an $R[[S]]$-module, $N_1[[S]]$ and $N_3[[S]]$ are Noetherian $R[[S]]$-modules, then $N_2[[S]]$ is Noether.

Proof. By hypothesis ($N_1[[S]]$, $N_2[[S]]$, $N_3[[S]]$) is $L[[S]]$-sub-exact as an $R[[S]]$-module. Since $N_1[[S]]$ and $N_3[[S]]$ are Noetherian $R[[S]]$-modules, based on Proposition 1, we get $L[[S]]$ is Noether.
Since $L[[S]]$ is a direct summand, there exists a submodule K of N_2[[S]] such that $N_2[[S]] = L[[S]] \oplus K$. Then every $\mu \in M_2[[S]]$ can uniquely write as $\mu = \mu' + \mu''_2$, where $\mu' \in L[[S]]$, and $\mu'' \in K$.
Besides that, the triple ($N_1[[S]]$, $N_2[[S]]$, $N_3[[S]]$) is $L[[S]]$-sub-exact implies that there are two $R[[S]]$-homomorphisms f and g such that the following sequence is exact.

$$N_1[[S]] \xrightarrow{f} L[[S]] \xrightarrow{g} N_3[[S]],$$

i.e., $\mathrm{Im}(f) = \mathrm{Ker}(g)$.
Thus, we can define an $R[[S]]$-homomorphism

$$g': N_2\big[[S]\big] \to N_3\big[[S]\big],$$

where $g' = \begin{cases} g(\mu); \text{if } \mu \in L\big[[S]\big]; \\ 0 \quad ; \ \text{otherwise.} \end{cases}$
Hence, we get the following diagram of $R[[S]]$-module:

$$\begin{tikzcd}
N_1[[S]] \arrow[r, "f"] \arrow[dr, "i \circ f"'] & L[[S]] \arrow[r, "g"] \arrow[d, "i"] & N_3[[S]] \\
& N_2[[S]] \arrow[ur, "g'"'] &
\end{tikzcd}$$

Based on [3], the following sequence of $R[[S]]$-module is exact.

$$N_1[[S]] \xrightarrow{i \circ f} N_2[[S]] \xrightarrow{g'} N_3[[S]].$$

Since $N_1[[S]]$ and $N_3[[S]]$ are Noetherian, based on Proposition 1, $N_2[[S]]$ is Noetherian.

Conclusion
Based on the results, we can conclude that we can use the concept of a sub-exact sequence of modules over $R[[S]]$ to determine the Noetherian property of generalized power series modules. Besides that, we also can determine the Noetherian property of its submodule.

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