

The Rank of \mathcal{U}_V -Generated Modules

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Abstract

Let \mathcal{U} be a nonempty set of R -modules and V be a submodule of $\oplus_{\Lambda} U_{\lambda}$, $U_{\lambda} \in \mathcal{U}$ for all $\lambda \in \Lambda$. A \mathcal{U}_V generated module is a generalization of \mathcal{U} -generated module by using the concept of V -coexact sequence. We say that an R -module N is generated by \mathcal{U}_V if there is an epimorphism from V to N . In this paper, we introduce the definition of rank of \mathcal{U}_V -generated modules. Furthermore, we investigate some properties of rank of \mathcal{U}_V -generated modules.

Keywords: \mathcal{U} -generated module, \mathcal{U}_V -generated module, rank.

1. Introduction

Let R be a ring and let $A \xrightarrow{f} B \xrightarrow{g} C$ be an exact sequence of R -modules, i.e. $\text{Im } f = \text{Ker } g (= g^{-1}(0))$. This exact sequence can be generalized to a quasi-exact sequence by replacing the submodule 0 with any submodule $U \subseteq C$ (Davvaz and Parnian-Garamaleky, 1999). In this case, the sequence is called U -exact (in B). As a dual of a U -exact sequence, V -coexact sequence (V a submodule of A) is defined as follows. A sequence $A \xrightarrow{f} B \xrightarrow{g} C$ is V -coexact if $f(V) = \text{Ker } g$ (Davvaz and Parnian-Garamaleky, 1999). The quasi-exact sequences can be used to generalize the Schanuel Lemma (Anvariye dan Davvaz, 2005). Furthermore, this sequence is used to generalize some notions in homological algebra (Davvaz and Shabani-Solt, 2002). In 2002, the U -split sequence is introduced, and the connection between this sequence and projective modules (Anvariye dan Davvaz, 2002).

Motivated by the generalization of the exact sequence to U -exact sequence and V -coexact sequence, a sub exact sequence is introduced (Fitriani et al., 2016). Furthermore, a sub-exact sequence is used to establish the X -sub-linearly independent module as a generalization of linearly independent module relative to a family of R -modules (Fitriani et al., 2017).

Let \mathcal{U} be a family of R -modules. Then, as a dual of an X -sub-linearly independent module, a \mathcal{U}_V -generated module is introduced as the generalization of a \mathcal{U} -generated module (Fitriani et al., 2018a). Furthermore, a basis and free module relative to a family of R -module is established by using the concept of X -sub-linearly independent module, and a \mathcal{U}_V -generated module (Fitriani et al., 2018b). The motivation of the definition of \mathcal{U}_V -generated module is from a generator class of modules (Anderson and Fuller, 1992).

The rank of a finitely generated module M is defined as the number of minimal generators of M (Adkins and Weintraub, 1992). In this paper, we introduce the definition of the rank of \mathcal{U}_V -generated modules, and we investigate some properties of the rank of \mathcal{U}_V -generated modules.

2. Results

Let \mathcal{U} be a non-empty set of R -modules and R -module M be a finitely \mathcal{U}_V -generated module. Hence, there is an epimorphism from V to M , where V is a submodule of $\{U_{\lambda}\}_{\Lambda}$. The set $\{U_{\lambda}\}_{\Lambda}$ is \mathcal{U}_V -generator for N . Furthermore, the set $\{U_{\lambda}\}_{\Lambda}$ is minimal \mathcal{U}_V -generator for M if $\Lambda = \min\{\Lambda_V | M \text{ is } \mathcal{U}_V\text{-generated, } V \subseteq \oplus_{\Lambda_V} U_{\lambda}\}$. We define rank of \mathcal{U}_V -generated modules as follows:

Definition 1 Let \mathcal{U} be a non-empty set of R -modules and M be a finitely \mathcal{U}_V -generated R -module, for some submodule V of $\oplus_{\Lambda} U_{\lambda}$ with modules $U_{\lambda} \in \mathcal{U}$, for all $\lambda \in \Lambda$. The rank of M relative to \mathcal{U} , we denote it by $\text{rank}(M)_{\mathcal{U}}$, is the minimal cardinality of Λ , where Λ is the index of minimal \mathcal{U}_V -generators of M .

Let $\mathcal{U} = \{U_{\lambda}\}_{\Lambda}$ be a non-empty set of R -modules and R -module M is finitely \mathcal{U}_V -generated. Then there exists a finite index set $E \subseteq \Lambda$ such that M is \mathcal{U}_V -generated and $V \subseteq \oplus_E U_e$, for all $e \in E$. We have $\{U_e\}_{e \in E}$ is \mathcal{U}_V -generator for M . If E is a minimal \mathcal{U}_V -generator for M which has the minimal cardinality, i.e. $E = \min\{\Lambda_V | M \text{ is } \mathcal{U}_V\text{-generated, } V \subseteq \oplus_{\Lambda_V} U_{\lambda}\}$,

then

$$\text{rank}(M)_{\mathcal{U}} = |\{U_e\}_E|.$$

Then, we give some examples of the rank of module generated by \mathcal{U}_V , where \mathcal{U} is a family of R -modules.

Example 1 Let $\mathcal{U} = \{\mathbb{Z}_p | p \text{ prime}\}$ be a family of \mathbb{Z} -modules and M be an abelian group of order q^2 , where q prime. We assume that M is a \mathcal{U}_V generated module. If q prime and M is group of order q^2 , then $M \cong \mathbb{Z}_{q^2}$ or $M \cong \mathbb{Z}_q \times \mathbb{Z}_q$.

If $M \cong \mathbb{Z}_{q^2}$, then M is not \mathcal{U}_V generated module. So, we have $M \cong \mathbb{Z}_q \times \mathbb{Z}_q$ and hence the number of minimal \mathcal{U}_V -generators ($V = \mathbb{Z}_q \times \mathbb{Z}_q$) of M is 2. Therefore, $\text{rank}(M)_{\mathcal{U}} = 2$.

Example 2 Let $\mathcal{U} = \{\mathbb{Z}_{p^n} | p \text{ prime}, n \in \mathbb{N}\}$ be a family of \mathbb{Z} -modules and M be an abelian group of order q^2 , where q prime. If $M \cong \mathbb{Z}_{q^2}$, then $\text{rank}(M)_{\mathcal{U}} = 1$ (where $V = \mathbb{Z}_{q^2}$). If $M \cong \mathbb{Z}_q \times \mathbb{Z}_q$, then $\text{rank}(M)_{\mathcal{U}} = 2$ (where $V = \mathbb{Z}_q \times \mathbb{Z}_q$).

Example 3 Let $\mathcal{U} = \{\mathbb{Z}_{p^n} | p \text{ prime}, n \in \mathbb{N}\}$ be a family of \mathbb{Z} -modules and M be an abelian group of order 8. If M is an abelian group of order 8, then M is isomorphic to exactly one of the following groups: \mathbb{Z}_8 , $\mathbb{Z}_4 \times \mathbb{Z}_2$, or $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. We have the following conditions:

1. If $M \cong \mathbb{Z}_8$, then $\text{rank}(M)_{\mathcal{U}} = 1$ (where $V = \mathbb{Z}_{2^3}$).
2. If $M \cong \mathbb{Z}_4 \times \mathbb{Z}_2$, then $\text{rank}(M)_{\mathcal{U}} = 2$ (where $V = \mathbb{Z}_{2^2} \times \mathbb{Z}_2$).
3. If $M \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, then $\text{rank}(M)_{\mathcal{U}} = 3$ (where $V = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$).

We recall that $\mu(\{0\}) = 0$ and if R is PID, then any R -submodule M of R is an ideal, so $\mu(M) = 1$ (Adkins and Weintraub, 1992). For \mathcal{U}_V -generated modules, we have the following properties:

Remark 1 Let \mathcal{U} be a family of R -modules.

1. $\text{rank}(0)_{\mathcal{U}} = 1$;
2. $\text{rank}(W)_{\mathcal{U}} = 1$, for any direct summand W of $U \in \mathcal{U}$.
3. If \mathcal{U} a family of all complemented R -modules, then $\text{rank}(W)_{\mathcal{U}} = 1$, for any submodule W of V , $V \in \mathcal{U}$.
4. If \mathcal{U} a family of all free R -modules, then $\text{rank}(P)_{\mathcal{U}} = 1$, for any projective R -module P .

If R -module N is \mathcal{U}_V -generated, then N' is \mathcal{U}_V -generated, for every homomorphic image N' of N (Fitriani et al., 2018a). Therefore, we have the following proposition.

Proposition 1 Let \mathcal{U} be a non-empty set of R -modules, V be a submodule of $\bigoplus_{\Lambda} U_{\lambda}$ with modules $U_{\lambda} \in \mathcal{U}$, for all $\lambda \in \Lambda$ and R -module M is a finitely \mathcal{U}_V -generated module. Then, $\text{rank}(N)_{\mathcal{U}} \leq \text{rank}(M)_{\mathcal{U}}$, for every homomorphic image N of M .

Proof. Let M be a finitely \mathcal{U}_V -generated module, and N be a homomorphic image of M . Hence, N is an \mathcal{U}_V -generated module (Fitriani et al., 2018a). In other words, every U_V -generator of M is U_V -generator of N and hence $\text{rank}(N)_{\mathcal{U}} \leq \text{rank}(M)_{\mathcal{U}}$.

In general, a submodule of an \mathcal{U}_V -generated module need not be an \mathcal{U}_V -generated. For example, if we take $\mathcal{U} = \{\mathbb{Q}\}$, then \mathbb{Z} -module \mathbb{Q} is an $\mathcal{U}_{\mathbb{Q}}$ -generated module. However, since we can not define an epimorphism from \mathbb{Q} to \mathbb{Z} , \mathbb{Z} -module \mathbb{Z} is not an $\mathcal{U}_{\mathbb{Q}}$ -generated module. Nevertheless, in case M is semisimple, we have the following corollary is a consequence of Proposition 1.

Corollary 1 Let \mathcal{U} be a non-empty set of R -modules and R -module M be a finitely \mathcal{U}_V -generated module. If M is semisimple, then $\text{rank}(N)_{\mathcal{U}} \leq \text{rank}(M)_{\mathcal{U}}$, for every submodule N of M .

Proof. Since every submodule of semisimple module is a direct summand, submodule N of M is a homomorphic image of M . By Proposition , we have $\text{rank}(N)_{\mathcal{U}} \leq \text{rank}(M)_{\mathcal{U}}$.

Proposition 2 Let \mathcal{U} be a non-empty set of R -modules, V_1, V_2 be submodules of $\bigoplus_{\Lambda} U_{\lambda}$ with modules $U_{\lambda} \in \mathcal{U}$, for all $\lambda \in \Lambda$. If R -module M_1 and M_2 are finitely \mathcal{U}_{V_1} -generated and \mathcal{U}_{V_2} -generated, respectively. Then,

$$\text{rank}(M_1 \oplus M_2)_{\mathcal{U}} \leq \text{rank}(M_1)_{\mathcal{U}} + \text{rank}(M_2)_{\mathcal{U}}.$$

Proof. Let $\{U_a\}_A$ and $\{U_b\}_B$ be minimal \mathcal{U}_V -generators for N_1 and N_2 , respectively. If M_1 is \mathcal{U}_{V_1} -generated and M_2 is \mathcal{U}_{V_2} -generated, then $M_1 \oplus M_2$ is $\mathcal{U}_{V_1 \oplus V_2}$ -generated. Therefore, we have $\{U_a\}_A \cup \{U_b\}_B$ is $\mathcal{U}_{V_1 \oplus V_2}$ -generator of $M_1 \oplus M_2$. Hence, $\text{rank}(M_1 \oplus M_2)_{\mathcal{U}} \leq \text{rank}(M_1)_{\mathcal{U}} + \text{rank}(M_2)_{\mathcal{U}}$.

Now, we give the properties of pullback and pushout of \mathcal{U}_V -generated modules.

Proposition 3 Let \mathcal{U} be a non-empty set of R -modules, V_1, V_2 be submodules of $\bigoplus_{\Lambda} U_{\lambda}$, $U_{\lambda} \in \mathcal{U}$, for every $\lambda \in \Lambda$. If R -modules N_1 and N_2 are \mathcal{U}_{V_1} -generated and \mathcal{U}_{V_2} -generated, respectively, $g_1 : X \rightarrow N_1$ and $g_2 : X \rightarrow N_2$ be morphisms of $\mathcal{U}_{V_1 \oplus V_2}$ -generated modules, and Q be a pushout of a pair of morphisms (g_1, g_2) . Then

$$\text{rank}(Q)_{\mathcal{U}} \leq \text{rank}(N_1)_{\mathcal{U}} + \text{rank}(N_2)_{\mathcal{U}}.$$

Proof. Pushout Q of a pair of morphisms (g_1, g_2) is a factor module of $N_1 \oplus N_2$ (Wisbauer, 1991). Therefore, Q is an $\mathcal{U}_{V_1 \oplus V_2}$ -generated module. By Proposition 1 and Proposition 2, we have $\text{rank}(Q)_{\mathcal{U}} \leq \text{rank}(N_1)_{\mathcal{U}} + \text{rank}(N_2)_{\mathcal{U}}$.

Proposition 4. Let \mathcal{U} be a non-empty set of R -modules, V_1, V_2 be submodules of $\bigoplus_{\Lambda} U_{\lambda}$, $U_{\lambda} \in \mathcal{U}$, for every $\lambda \in \Lambda$. If R -modules N_1 and N_2 are \mathcal{U}_{V_1} -generated and \mathcal{U}_{V_2} -generated, respectively, $N_1 \oplus N_2$ be a semisimple module, and P be a pullback of a pair of morphisms (f_1, f_2) , where $f_1 : N_1 \rightarrow N$ and $f_2 : N_2 \rightarrow N$ are morphisms of $\mathcal{U}_{V_1 \oplus V_2}$ -generated modules. Then

$$\text{rank}(P)_{\mathcal{U}} \leq \text{rank}(N_1)_{\mathcal{U}} + \text{rank}(N_2)_{\mathcal{U}}.$$

Proof. Pullback P of a pair of morphisms (g_1, g_2) is a submodule of $N_1 \oplus N_2$ (Wisbauer, 1991). Since $N_1 \oplus N_2$ is a semisimple module, P is a direct summand of $N_1 \oplus N_2$ and hence P is a homomorphic image of $N_1 \oplus N_2$. By Proposition 1 and Proposition 2, we have $\text{rank}(P)_{\mathcal{U}} \leq \text{rank}(N_1)_{\mathcal{U}} + \text{rank}(N_2)_{\mathcal{U}}$.

It is possible that an R -module M is a \mathcal{U}_{V_1} -generated and a \mathcal{U}_{V_2} -generated module. In the following proposition, we will show the connection between V_1 and V_2 by using Five Lemma (Wisbauer, 1991).

Proposition 5 Let \mathcal{U} be a non-empty set of R -modules, V_1, V_2 be submodules of $\bigoplus_{\Lambda} U_{\lambda}$, $U_{\lambda} \in \mathcal{U}$, for every $\lambda \in \Lambda$. If R -modules M is \mathcal{U}_{V_1} -generated and \mathcal{U}_{V_2} -generated, i.e there are epimorphisms $p_1 : V_1 \rightarrow M$ and $p_2 : V_2 \rightarrow M$. Let V_1 be a V_2 -projective module, i.e there is morphism $p : V_1 \rightarrow V_2$ such that $p_2 \circ p = p_1$. If we define $\alpha = p|_{\text{Ker } p_1}$ and we assume that $\alpha(\text{Ker } p_1) \subseteq \text{Ker } p_2$, then we have:

1. If α is monomorphism, then V_1 is isomorphic to a submodule of V_2 ;
2. If α is epimorphism, then V_2 is a \mathcal{U}_{V_1} -generated module.

Proof. If R -modules M is \mathcal{U}_{V_1} -generated and \mathcal{U}_{V_2} -generated, then there are epimorphisms $p_1 : V_1 \rightarrow M$ and $p_2 : V_2 \rightarrow M$. Since V_1 is V_2 -projective, there is morphism $p : V_1 \rightarrow V_2$ such that $p_2 \circ p = p_1$. We define $\alpha = p|_{\text{Ker } p_1}$ and we assume that $\alpha(\text{Ker } p_1) \subseteq \text{Ker } p_2$. Hence, we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \text{Ker } p_1 & \xrightarrow{i_1} & V_1 & \xrightarrow{p_1} & M & \longrightarrow & 0 \\ & & \downarrow \alpha & & \downarrow p & & \parallel & & \\ 0 & \longrightarrow & \text{Ker } p_2 & \xrightarrow{i_2} & V_2 & \xrightarrow{p_2} & M & \longrightarrow & 0 \end{array}$$

By Five Lemma, if α is a monomorphism, then p is a monomorphism. So, V_1 is isomorphic to a submodule of V_2 . If α is an epimorphism, then p is an epimorphism, and hence V_2 is a \mathcal{U}_{V_1} -generated module.

3. Conclusions

Let $\mathcal{U} = \{U_{\lambda}\}_{\Lambda}$ be a non-empty set of R -modules and R -module M is finitely \mathcal{U}_V -generated. Then there exists a finite index set $E \subseteq \Lambda$ such that M is \mathcal{U}_V -generated and $V \subseteq \bigoplus_{e \in E} U_e$, for all $e \in E$. We have $\{U_e\}_{e \in E}$ is \mathcal{U}_V -generator for M . If E is a minimal \mathcal{U}_V -generator for M which has the minimal cardinality, then $\text{rank}(M)_{\mathcal{U}} = |\{U_e\}_E|$. Furthermore, $\text{rank}(N)_{\mathcal{U}} \leq \text{rank}(M)_{\mathcal{U}}$, for every homomorphic image N of M .

If R -module M_1 and M_2 are finitely \mathcal{U}_{V_1} -generated and \mathcal{U}_{V_2} -generated, respectively. Then, $\text{rank}(M_1 \oplus M_2)_{\mathcal{U}} \leq \text{rank}(M_1)_{\mathcal{U}} + \text{rank}(M_2)_{\mathcal{U}}$. This implies if R -modules N_1 and N_2 are \mathcal{U}_{V_1} -generated and \mathcal{U}_{V_2} -generated, respectively, $g_1 : X \rightarrow N_1$ and $g_2 : X \rightarrow N_2$ be morphisms of $\mathcal{U}_{V_1 \oplus V_2}$ -generated modules, and Q be a pushout of a pair of morphisms (g_1, g_2) . Then $\text{rank}(Q)_{\mathcal{U}} \leq \text{rank}(N_1)_{\mathcal{U}} + \text{rank}(N_2)_{\mathcal{U}}$. Besides that, if R -modules N_1 and N_2 are \mathcal{U}_{V_1} -generated and \mathcal{U}_{V_2} -generated, respectively, $N_1 \oplus N_2$ be a semisimple module, and P be a pullback of a pair of morphisms (f_1, f_2) , where $f_1 : N_1 \rightarrow N$ and $f_2 : N_2 \rightarrow N$ are morphisms of $\mathcal{U}_{V_1 \oplus V_2}$ -generated modules. Then $\text{rank}(P)_{\mathcal{U}} \leq \text{rank}(N_1)_{\mathcal{U}} + \text{rank}(N_2)_{\mathcal{U}}$.

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References

- Adkins, W. A., & Weintraub, S. H. (1992). *Algebra "An Approach via Module Theory"*. New York: Springer-Verlag. <https://doi.org/10.1007/978-1-4612-0923-2>
- Anderson, F. W., & Fuller, K. R. (1992). *Rings and categories of Modules*. New York: Springer-Verlag. <https://doi.org/10.1007/978-1-4612-4418-9>
- Anvariye, S. M., & Davvaz, B. (2005). On Quasi-Exact Sequences. *Bull. Korean Math. Soc.*, 42(1), 149-155. <https://doi.org/10.4134/BKMS.2005.42.1.149>
- Anvariye, S. M., & Davvaz, B. (2002). U-Split-Exact Sequences. *Far East J. Math. Sci.(FJMS)*, 4(2), 209-219.
- Davvaz, B., & Parnian-Garamaleky, Y. A. (1999). A Note on Exact Sequences, *Bull.Malays. Math. Sci. Soc.* 22(1), 53-56.
- Davvaz, B., & Shabani-Solt, H. (2002). A generalization of homological algebra. *Journal of the Korean Mathematical Society*, 39(6), 881-898.
- Fitriani, Surodjo, B., & Wijayanti, I. E. (2016). On X-sub-exact Sequences. *Far East J. Math. Sci.(FJMS)*, 100(7), 1055-1065. <http://dx.doi.org/10.17654/MS100071055>
- Fitriani, Surodjo, B., & Wijayanti, I. E. (2017, October). On X-sub-linearly independent modules. In *Journal of Physics: Conference Series* (Vol. 893, No. 1, p. 012008). IOP Publishing. <https://doi.org/10.1088/1742-6596/893/1/012008>
- Fitriani, Wijayanti, I. E., & Surodjo, B. (2018). Generalization of U-Generator and M-Subgenerator Related to Category [M]. *Journal of Mathematics Research*, 10(4), 101-106. <https://doi.org/10.5539/jmr.v10n4p101>
- Fitriani, Wijayanti, I. E., & Surodjo, B. (2018b). A Generalization of Basis and Free Modules Relatives to a Family \mathcal{U} of R -Modules. *J. Phys. Conf. Series*, 1097, 012087, 2018. <https://doi.org/10.1088/1742-6596/1097/1/012087>
- Wisbauer, R. (1991). *Foundation of Module and Ring Theory*. Philadelphia: Gordon and Breach.

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