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The Locating-Chromatic Number of Origami Graphs

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Abstract: The locating-chromatic number of a graph combines two graph concepts, namely coloring vertices and partition dimension of a graph. The locating-chromatic number is the smallest k such that G has a locating k-coloring, denoted by $\chi_L(G)$. This article proposes a procedure for obtaining a locating-chromatic number for an origami graph and its subdivision (one vertex on an outer edge) through two theorems with proofs.

Keywords: locating-chromatic number; origami graphs; subdivision

MSC: 05C12; 05C15

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1. Introduction

The study of the partition dimension of connected graphs was introduced by Chartrand et al. [1,2] with the aim of finding a new method for attacking the problem of determining the metric dimension in graphs. The application of these metric dimensions can be seen in the navigation of a robot modeled by a graph [3,4], solving the problem of chemical data classification, and determining how to represent a set of chemical compounds in such a way that different compounds have different representations [5,6]. The concept of the locating-chromatic number was first introduced by Chartrand et al. in 2002, with two obtained graph concepts, namely coloring vertices and partition dimensions of a graph [7]. Finding the locating-chromatic number of a graph is one of the interesting (and un-completely solved) problems of graph theory. Let $G = (V, E)$ be a connected graph; the distance $d(x, y)$ between two of its vertices x and y is the length of the shortest path between them. Let c be a proper k-coloring of G with color $\{1, 2, ..., k\}$, and $\Pi = \{C_1, C_2, ..., C_k\}$ be a partition of $V(G)$ that is induced by the coloring c. The color code $c_\Pi(v)$ of v is the ordered k-tuple $(d(v, C_1), d(v, C_2), ..., d(v, C_k)))$, where $d(v, C_i) = \min\ \{d(v, x) : x \in C_i\}$ for any $i \in \{1, 2, 3, ..., k\}$. If all distinct vertices of G have distinct color codes, then c is called a k-locating coloring of G. The locating-chromatic number denoted by $\chi_L(G)$ is the smallest k such that G has a locating k-coloring. Let c be a locating k-coloring on graph $G(V, E)$. Furthermore, the locating-chromatic number has been determined for a few graph classes; for example, if P_n is a path of order $n \geq 3$ then the locating-chromatic number is 3; for a cycle C_n if $n \geq 3$ is odd, $\chi_L(C_n) = 3$ was obtained, and if n is even, $\chi_L(C_n) = 4$ was obtained; for a double star graph $(S_{a,b})$, $1 \leq a \leq b$ and $b \geq 2$, $\chi_L(S_{a,b}) = b + 1$ was obtained. Let $\Pi = \{S_1, S_2, ..., S_k\}$ be the partition of $V(G)$ induced by c. A vertex $v \in G$ is called a dominant vertex if $d(v, S_i) = 1$, where $v \notin S_i$. Chartrand et al. characterized all graphs of order n with the locating-chromatic number $n - 1$ [8].

The problem of determining the locating-chromatic number of any general graph is an NP-hard problem [9]. This means that to determine the locating-chromatic number of any given graph, we need a specific algorithm. In 2012, Baskoro and Purwasih proposed a procedure to obtain the locating-chromatic number of corona products of two graphs [9]. In

2014, Asmiati obtained the locating-chromatic number of a non-homogeneous amalgamation of stars [10]. Moreover, to determine the locating-chromatic number of disconnected graphs, graphs with dominant vertices and graphs of two components have been discussed in [11–13]. In 2019, the characterization of the locating chromatic number of powers of paths and a condition (sharp upper and lower bounds) for the locating chromatic number of powers of cycles were discussed [14] (see [15] for a discussion of the necessary and sufficient conditions for a pair of two specific start graphs to be an odd mean graph). Asmiati et al. determined the locating-chromatic number of some Petersen graphs; $P(n,1)$ four for odd $n \geq 3$ or five for even $n \geq 4$ were obtained [16], and in [17] results were obtained for certain barbell graphs. Syofyan et al. have succeed in determined the locating-chromatic number of homogeneous lobsters [18]. In [19], Asmiati obtained the locating-chromatic number for non-homogeneous caterpillar graphs and non-homogeneous firecracker graphs. Next, Irawan and Asmiati in 2018 determined the locating-chromatic number of subdivision firecrackers graphs [20] and in [21] obtained the certain operation of generalized Petersen graphs $sP(n,1)$. In 2014, Behtoei and Anbarloei determined the locating-chromatic number of the joining of two arbitrary graphs [22]. In addition to that, in this article we propose a procedure for obtaining the locating-chromatic number for an origami graph and its subdivision (one vertex on an outer edge). The following definition of an origami graph is taken from [23]. Let $n \in \mathbb{N}$ with $n \geq 3$. An origami graph O_n is a graph with $V(O_n) = \{u_i, v_i, w_i : i \in \{1, ..., n\}\}$ and $E(O_n) = \{u_i w_i, u_i v_i, v_i w_i : i \in \{1, ..., n\}\} \cup \{u_i u_{i+1}, w_i u_{i+1} : i \in \{1, ..., n-1\}\} \cup \{u_n u_1, w_n u_1\}$ (see Figure 1 for an example). Meanwhile, a subdivision of an origami graph O_n^* is a graph with $V(O_n^*) = \{u_i, v_i, x_i, w_i : i \in \{1, ..., n\}\}$ and $E(O_n^*) = \{u_i w_i, u_i v_i, v_i x_i, x_i w_i : i \in \{1, ..., n\}\} \cup \{u_i u_{i+1}, w_i u_{i+1} : i \in \{1, ..., n-1\}\} \cup \{u_n u_1, w_n u_1\}\}$ (see Figure 2 for an example).

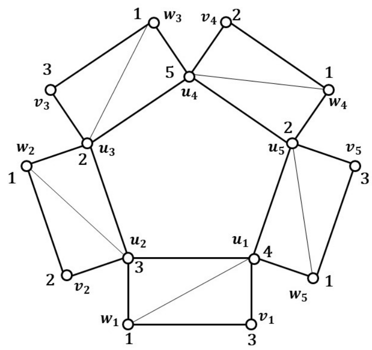

Figure 1. An origami graph O_5.

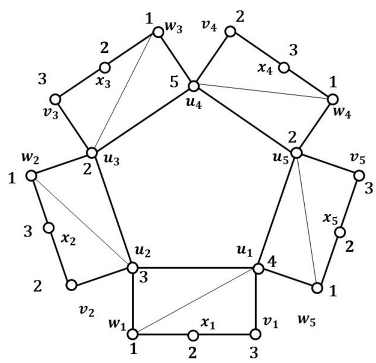

Figure 2. A subdivision of an origami graph O_5^*.

2. Results and Discussions

Let c be a locating coloring in a connected graph G and $N(q)$ denote the set of neighbor of a vertex q in G. If p and q are distinct vertices of G such that $d(p,w) = d(q,w)$ for all $w \in V(G) - \{p,q\}$, then $c(p) \neq c(q)$. In particular, if p and q are non-adjacent vertices such that $N(p) = N(q)$, then $c(p) \neq c(q)$ [7].

In the following subsection, the locating-chromatic number of origami graphs O_n and their subdivisions called O_n^* is described.

2.1. Locating-Chromatic Number of Origami Graphs

Theorem 1. *Let O_n be an origami graph for $n \geq 3$. Then, the locating-chromatic number of O_n,*

$$\chi_L(O_n) = \begin{cases} 4, & \text{for } n = 3 \\ 5, & \text{otherwise}. \end{cases}$$

Proof. Let $n \in \mathbb{N}$ with $n \geq 3$. An origami graph O_n is a graph with $V(O_n) = \{u_i, v_i, w_i : i \in \{1,...,n\}\}$ and $E(O_n) = \{u_iw_i, u_iv_i, v_iw_i : i \in \{1,...,n\}\} \cup \{u_iu_{i+1}, w_iu_{i+1} : i \in \{1,...,n-1\}\} \cup \{u_nu_1, w_nu_1\}$. Next, to prove the theorem, we consider the following two cases:

Case 1. $\chi_L(O_3) = 4$

First, we determine the lower bound of $\chi_L(O_3)$. In the origami graphs O_n for $n \geq 3$, there are three adjacent vertices (complete graph with three vertices, denoted by K_3); we then need at least 3-locating coloring. Without loss of generality, we assign three colors for any K_3 in O_n for $n \geq 3$, and then the three vertices are dominant vertices. As a result, if we use three colors it is not enough because there are more than one K_3 in O_n for $n \geq 3$. Therefore, $\chi_L(O_3) \geq 4$.

Next, we determine the upper bound of $\chi_L(O_3) \leq 4$. To show that 4 is an upper bound for the locating-chromatic number for the origami graph O_3 we describe a locating coloring c using four colors as follows:

$c(u_i) = i, i = 1,2,3.$

$$c(v_i) = \begin{cases} 2, & \text{for } i = 1,3 \\ 3, & \text{for } i = 2. \end{cases}$$

$c(w_i) = 4, i = 1,2,3.$

The coloring c will create the partition Π on $V(O_3)$. We shall show that the color codes of all vertices in O_3 are different. We have: $c_\Pi(u_1) = (0,1,1,1)$; $c_\Pi(u_2) = (1,0,1,1)$; $c_\Pi(u_3) = (1,1,0,1)$; $c_\Pi(v_1) = (1,0,2,1)$; $c_\Pi(v_2) = (2,1,0,1)$; $c_\Pi(v_3) = (2,0,1,1)$; $c_\Pi(w_1) = (1,1,2,0)$; $c_\Pi(w_2) = (2,1,1,0)$; $c_\Pi(w_3) = (1,1,1,0)$. Since the color codes of all vertices O_3 are different, c is a locating-chromatic coloring. Thus, $\chi_L(O_3) \leq 4$.

Case 2. $\chi_L(O_n) = 5$, for $n \geq 4$

To determine the lower bound, we will show that four colors are not enough. For a contradiction, assume that there exists a 4-locating coloring c on O_n for $n \geq 4$. We assign $\{c(u_i), c(v_i), c(w_i), c(u_{i+1})\} = \{1,2,3,4\}$, where $c(v_i) \neq c(u_{i+1})$ because $d(v_i, x) = d(u_{i+1}, x)$, $x \in \{u_i, v_i\}$. Observe that, on O_n for $n \geq 4$, there are n vertices u_i whose degree is 5. As a result, at least two vertices w_k, w_l, $k \neq l$ have the same color codes, which is a contradiction. Therefore, $\chi_L(O_n) \geq 5$, for $n \geq 4$.

To show the upper bound for the locating-chromatic number of origami graphs O_n for $n \geq 4$, let us differentiate some subcases.

Subcase 1. (Odd n), for $\lceil \frac{n}{2} \rceil$ odd, $n \geq 5$
Let c be a coloring of origami graph O_n, $\lceil \frac{n}{2} \rceil$ odd, and $n \geq 5$; we make the partition Π of $V(O_n)$:
$C_1 = \{w_i | 1 \leq i \leq n\}$;
$C_2 = \{u_i | \text{ for odd } i, 3 \leq i \leq n\} \cup \{v_i | \text{ for even } i, 2 \leq i \leq n-1\}$;

$C_3 = \{u_i|$ for even $i, 2 \leq i \leq \lceil \frac{n}{2} \rceil - 1\} \cup \{u_i|$ for even $i, \lceil \frac{n}{2} \rceil + 3 \leq i \leq n-1\} \cup \{v_i|$ for odd $i, 1 \leq i \leq n\}$;
$C_4 = \{u_1\}$;
$C_5 = \{u_{\lceil \frac{n}{2} \rceil + 1}\}$.

For $\lceil \frac{n}{2} \rceil$ odd, the color codes of all the vertices of $V(O_n)$ are:
$C_1 = \{w_i|1 \leq i \leq n\}$.
For $i = 1$, we have:

$$c_\Pi(w_i) = (0, 2, 1, i, \lceil \tfrac{n}{2} \rceil - i + 1).$$

For $2 \leq i \leq \lceil \frac{n}{2} \rceil, n \geq 5$ we have:

$$c_\Pi(w_i) = (0, 1, 1, i, \lceil \tfrac{n}{2} \rceil - i + 1).$$

For $i = \lceil \frac{n}{2} \rceil + 1$ we have:

$$c_\Pi(w_i) = (0, 1, 2, n - i + 1, i - \lceil \tfrac{n}{2} \rceil).$$

For $\lceil \frac{n}{2} \rceil + 2 \leq i \leq n, n \geq 5$ we have:

$$c_\Pi(w_i) = (0, 1, 1, n - i + 1, i - \lceil \tfrac{n}{2} \rceil).$$

$C_2 = \{u_i|$ for odd $i, 3 \leq i \leq n\} \cup \{v_i|$ for even $i, 2 \leq i \leq n-1\}$.
For i odd, $3 \leq i \leq \lceil \frac{n}{2} \rceil, n \geq 5$ we have:

$$c_\Pi(u_i) = (1, 0, 1, i - 1, \lceil \tfrac{n}{2} \rceil - i + 1).$$

For i odd, $\lceil \frac{n}{2} \rceil + 2 \leq i \leq n, n \geq 5$ we have:

$$c_\Pi(u_i) = (1, 0, 1, n - i + 1, i - \lceil \tfrac{n}{2} \rceil - 1).$$

For i even, $2 \leq i \leq \lceil \frac{n}{2} \rceil - 1, n \geq 5$ we have:

$$c_\Pi(v_i) = (1, 0, 1, i, \lceil \tfrac{n}{2} \rceil - i + 2).$$

For $i = \lceil \frac{n}{2} \rceil + 1$, we have:

$$c_\Pi(v_i) = (1, 0, 3, n - i + 2, 1).$$

For i even, $\lceil \frac{n}{2} \rceil + 3 \leq i \leq n - 1, n \geq 9$ we have:

$$c_\Pi(v_i) = (1, 0, 1, n - i + 2, i - \lceil \tfrac{n}{2} \rceil).$$

$C_3 = \{u_i|$ for even $i, 2 \leq i \leq \lceil \frac{n}{2} \rceil - 1\} \cup \{u_i|$ for even $i, \lceil \frac{n}{2} \rceil + 3 \leq i \leq n-1\} \cup \{v_i|$ for odd $i, 1 \leq i \leq n\}$.
For $i = 1$, we have:

$$c_\Pi(v_i) = (1, 2, 0, i, \lceil \tfrac{n}{2} \rceil).$$

For i odd, $3 \leq i \leq \lceil \frac{n}{2} \rceil, n \geq 5$ we have:

$$c_\Pi(v_i) = (1, 1, 0, i, \lceil \tfrac{n}{2} \rceil - i + 2).$$

For i odd, $\lceil \frac{n}{2} \rceil + 2 \leq i \leq n, n \geq 9$ we have:

$$c_\Pi(v_i) = (1, 1, 0, n - i + 2, i - \lceil \tfrac{n}{2} \rceil).$$

For i even, $2 \leq i \leq \lceil \frac{n}{2} \rceil - 1, n \geq 5$ we have:

$$c_\Pi(u_i) = (1, 1, 0, i - 1, \lceil \tfrac{n}{2} \rceil - i + 1).$$

For i even, $\lceil \frac{n}{2} \rceil + 3 \leq i \leq n - 1, n \geq 9$ we have:

$$c_\Pi(u_i) = (1, 1, 0, n - i + 1, i - \lceil \tfrac{n}{2} \rceil - 1).$$

For $C_4 = \{u_1\}$, we have:

$$c_\Pi(u_1) = (1, 1, 1, 0, \lceil \tfrac{n}{2} \rceil - 1).$$

For $C_5 = \{u_{\lceil \frac{n}{2} \rceil + 1}\}$, we have:

$$c_\Pi(u_{\lceil \frac{n}{2} \rceil + 1}) = (1, 1, 2, \lceil \tfrac{n}{2} \rceil - 1, 0).$$

Since for n odd all vertices have different color codes, c is a locating coloring of origami graphs O_n, so that $\chi_L(O_n) \leq 5$, for $\lceil \frac{n}{2} \rceil$ odd, $n \geq 5$.

Subcase 2. (Odd n), for $\lceil \frac{n}{2} \rceil$ even, $n \geq 7$.
Let c be a coloring of origami graph O_n, $\lceil \frac{n}{2} \rceil$ even, and $n \geq 7$; we make the partition Π of $V(O_n)$ as follows:
$C_1 = \{w_i | 1 \leq i \leq n\}$;
$C_2 = \{u_i |$ for odd $i, 3 \leq i \leq n\} \cup \{v_i |$ for even $i, 2 \leq i \leq n-1\}$;
$C_3 = \{u_i |$ for even $i, 2 \leq i \leq \lceil \frac{n}{2} \rceil - 2\} \cup \{u_i |$ for even $i, \lceil \frac{n}{2} \rceil + 2 \leq i \leq n-1\} \cup \{v_i |$ for odd $i, 1 \leq i \leq n\}$;
$C_4 = \{u_1\}$;
$C_5 = \{u_{\lceil \frac{n}{2} \rceil}\}$.

For $\lceil \frac{n}{2} \rceil$ even, the color codes of all the vertices of $V(O_n)$ are:
$C_1 = \{w_i | 1 \leq i \leq n\}$.

For $i = 1$, we have:

$$c_\Pi(w_i) = (0, 2, 1, i, \lceil \tfrac{n}{2} \rceil - i).$$

For $2 \leq i \leq \lceil \frac{n}{2} \rceil - 1, n \geq 7$ we have:

$$c_\Pi(w_i) = (0, 1, 1, i, \lceil \tfrac{n}{2} \rceil - i).$$

For $i = \lceil \frac{n}{2} \rceil$, we have:

$$c_\Pi(w_i) = (0, 1, 2, n - i + 1, i - \lceil \tfrac{n}{2} \rceil + 1).$$

For $\lceil \frac{n}{2} \rceil + 1 \leq i \leq n, n \geq 7$ we have:

$$c_\Pi(w_i) = (0, 1, 1, n - i + 1, i - \lceil \tfrac{n}{2} \rceil + 1).$$

$C_2 = \{u_i |$ for odd $i, 3 \leq i \leq n\} \cup \{v_i |$ for even $i, 2 \leq i \leq n-1\}$.

For i odd, $3 \leq i \leq \lceil \frac{n}{2} \rceil - 1, n \geq 7$ we have:

$$c_\Pi(u_i) = (1, 0, 1, i - 1, \lceil \tfrac{n}{2} \rceil - i).$$

For i odd, $\lceil \frac{n}{2} \rceil + 1 \leq i \leq n, n \geq 7$ we have:

$$c_\Pi(u_i) = (1, 0, 1, n - i + 1, i - \lceil \tfrac{n}{2} \rceil).$$

For i even, $2 \leq i \leq \lceil \frac{n}{2} \rceil - 2, n \geq 7$ we have:

$$c_\Pi(v_i) = (1, 0, 1, i, \lceil \tfrac{n}{2} \rceil - i + 1).$$

For $i = \lceil \frac{n}{2} \rceil$, we have:

$$c_\Pi(v_i) = (1, 0, 3, i, i - \lceil \tfrac{n}{2} \rceil + 1).$$

For i even, $\lceil \frac{n}{2} \rceil + 3 \leq i \leq n - 1, n \geq 7$ we have:

$$c_\Pi(v_i) = (1, 0, 1, n - i + 2, i - \lceil \tfrac{n}{2} \rceil + 1).$$

$C_3 = \{u_i|$ for even $i, 2 \leq i \leq \lceil \frac{n}{2} \rceil - 2\} \cup \{u_i|$ for even $i, \lceil \frac{n}{2} \rceil + 2 \leq i \leq n-1\} \cup \{v_i|$ for odd $i, 1 \leq i \leq n\}$.

For $i = 1$ we have:

$$c_\Pi(v_i) = (1, 2, 0, i, \lceil \tfrac{n}{2} \rceil - i + 1).$$

For i odd, $3 \leq i \leq \lceil \frac{n}{2} \rceil - 1, n \geq 7$ we have:

$$c_\Pi(v_i) = (1, 1, 0, i, \lceil \tfrac{n}{2} \rceil - i + 1).$$

For i odd, $\lceil \frac{n}{2} \rceil + 1 \leq i \leq n, n \geq 7$ we have:

$$c_\Pi(v_i) = (1, 1, 0, n - i + 2, i - \lceil \tfrac{n}{2} \rceil + 1).$$

For i even, $2 \leq i \leq \lceil \frac{n}{2} \rceil - 2, n \geq 7$ we have:

$$c_\Pi(u_i) = (1, 1, 0, i - 1, \lceil \tfrac{n}{2} \rceil - i).$$

For i even, $\lceil \frac{n}{2} \rceil + 2 \leq i \leq n, n \geq 7$ we have:

$$c_\Pi(u_i) = (1, 1, 0, n - i + 1, i - \lceil \tfrac{n}{2} \rceil).$$

$C_4 = \{u_1\}$, we have:

$$c_\Pi(u_1) = (1, 1, 1, 0, \lceil \tfrac{n}{2} \rceil - 1).$$

$C_5 = \{u_{\lceil \frac{n}{2} \rceil}\}$, we have:

$$c_\Pi(u_{\lceil \frac{n}{2} \rceil}) = (1, 1, 2, \lceil \tfrac{n}{2} \rceil - 1, 0).$$

Since for n odd all vertices have different color codes, c is a locating coloring of origami graphs O_n, so that $\chi_L(O_n) \leq 5$, for $\lceil \frac{n}{2} \rceil$ even, $n \geq 7$.

Subcase 3. (even n), for $\frac{n}{2}$ odd, $n \geq 6$.
Let c be a coloring of origami graph O_n, $\frac{n}{2}$ odd, and $n \geq 6$; we make the partition Π of $V(O_n)$:
$C_1 = \{w_i | 1 \leq i \leq \frac{n}{2} - 1\} \cup \{w_i | \frac{n}{2} + 1 \leq i \leq n\}$;
$C_2 = \{u_i|$ for odd $i, 3 \leq i \leq n-1\} \cup \{v_i|$ for even $i, 2 \leq i \leq n\}$;
$C_3 = \{u_i|$ for even $i, 2 \leq i \leq n\} \cup \{v_i|$ for odd $i, 1 \leq i \leq n-1\}$;
$C_4 = \{u_1\}$;
$C_5 = \{w_{\frac{n}{2}}\}$.

For $\frac{n}{2}$ odd, the color codes of all the vertices of $V(O_n)$ are:
$C_1 = \{w_i | 1 \leq i \leq \frac{n}{2} - 1\} \cup \{w_i | \frac{n}{2} + 1 \leq i \leq n\}$.
For $i = 1$, we have:

$$c_\Pi(w_i) = (0, 2, 1, i, \tfrac{n}{2} - i + 1).$$

For $2 \leq i \leq \frac{n}{2} - 1, n \geq 6$ we have:

$$c_\Pi(w_i) = (0, 1, 1, i, \tfrac{n}{2} - i + 1).$$

For $\frac{n}{2} + 1 \leq i \leq n, n \geq 6$ we have:

$$c_\Pi(w_i) = (0, 1, 1, n - i + 1, i - \tfrac{n}{2} + 1).$$

$C_2 = \{u_i|$ for odd $i, 3 \leq i \leq n-1\} \cup \{v_i|$ for even $i, 2 \leq i \leq n\}$.
For i odd, $3 \leq i \leq \frac{n}{2}, n \geq 6$ we have:

$$c_\Pi(u_i) = (1, 0, 1, i - 1, \tfrac{n}{2} - i + 1).$$

For i odd, $\frac{n}{2}+2 \leq i \leq n-1, n \geq 6$ we have:

$$c_\Pi(u_i) = (1,0,1,n-i+1,i-\tfrac{n}{2}).$$

For i even, $2 \leq i \leq \frac{n}{2}-1, n \geq 6$ we have:

$$c_\Pi(v_i) = (1,0,1,i,\tfrac{n}{2}-i+2).$$

For i even, $\frac{n}{2}+1 \leq i \leq n-1, n \geq 6$ we have:

$$c_\Pi(v_i) = (1,0,1,n-i+2,i-\tfrac{n}{2}+1).$$

$C_3 = \{u_i |$ for even $i, 2 \leq i \leq n\} \cup \{v_i |$ for odd $i, 1 \leq i \leq n-1\}$.
For $i = 1$, we have:

$$c_\Pi(v_i) = (1,3,0,i,\tfrac{n}{2}-i+2).$$

For i odd, $3 \leq i \leq \frac{n}{2}-2, n \geq 10$ we have:

$$c_\Pi(v_i) = (1,1,0,i,\tfrac{n}{2}-i+2)$$

For $i = \frac{n}{2}$, we have:

$$c_\Pi(v_i) = (2,1,0,i,1).$$

For i odd, $\frac{n}{2}+2 \leq i \leq n-1, n \geq 6$ we have:

$$c_\Pi(v_i) = (1,1,0,n-i+2,i-\tfrac{n}{2}+1).$$

For i even, $2 \leq i \leq \frac{n}{2}-1, n \geq 6$ we have:

$$c_\Pi(u_i) = (1,1,0,i-1,\tfrac{n}{2}-i+1).$$

For i even, $\frac{n}{2}+1 \leq i \leq n, n \geq 6$ we have:

$$c_\Pi(u_i) = (1,1,0,n-i+1,i-\tfrac{n}{2}).$$

For $C_4 = \{u_1\}$, we have:

$$c_\Pi(u_1) = (1,2,1,0,\tfrac{n}{2}-i+1).$$

For $C_5 = \{w_{\frac{n}{2}}\}$, we have:

$$c_\Pi(w_{\frac{n}{2}}) = (2,1,1,\tfrac{n}{2},0).$$

Since for n even all vertices have different color codes, c is a locating coloring of origami graphs O_n, so that $\chi_L(O_n) \leq 5$, for $\frac{n}{2}$ odd, $n \geq 6$.

Subcase 4. (even n), for $\frac{n}{2}$ even, $n \geq 4$.
Let c be a coloring of origami graph O_n, $\frac{n}{2}$ even, and $n \geq 4$; we make the partition Π of $V(O_n)$ as follows:
$C_1 = \{w_i | 1 \leq i \leq \frac{n}{2}\} \cup \{w_i | \frac{n}{2}+2 \leq i \leq n\}$;
$C_2 = \{u_i |$ for odd $i, 3 \leq i \leq n-1\} \cup \{v_i |$ for even $i, 2 \leq i \leq n\}$;
$C_3 = \{u_i |$ for even $i, 2 \leq i \leq n\} \cup \{v_i |$ for odd $i, 1 \leq i \leq n-1\}$;
$C_4 = \{u_1\}$;
$C_5 = \{w_{\frac{n}{2}+1}\}$.

For $\frac{n}{2}$ even, the color codes of all the vertices of $V(O_n)$ are:
$C_1 = \{w_i | 1 \leq i \leq \frac{n}{2}\} \cup \{w_i | \frac{n}{2}+2 \leq i \leq n\}$.
For $i = 1$ we have:

$$c_\Pi(w_i) = (0,2,1,i,\tfrac{n}{2}-i+2).$$

For $2 \leq i \leq \frac{n}{2}, n \geq 4$ we have:

$$c_\Pi(w_i) = (0,1,1,i,\tfrac{n}{2}-i+2).$$

For $\frac{n}{2}+2 \leq i \leq n, n \geq 4$ we have:

$$c_{\Pi}(w_i) = (0,1,1,n-i+1,i-\tfrac{n}{2}).$$

$C_2 = \{u_i | \text{ for odd } i, 3 \leq i \leq n-1\} \cup \{v_i | \text{ for even } i, 2 \leq i \leq n\}$.
For i odd, $3 \leq i \leq \frac{n}{2}+1, n \geq 8$ we have:

$$c_{\Pi}(u_i) = (1,0,1,i-1,\tfrac{n}{2}-i+2).$$

For i odd, $\frac{n}{2}+3 \leq i \leq n-1, n \geq 8$ we have:

$$c_{\Pi}(u_i) = (1,0,1,n-i+1,i-\tfrac{n}{2}-1).$$

For i even, $2 \leq i \leq \frac{n}{2}, n \geq 4$ we have:

$$c_{\Pi}(v_i) = (1,0,1,i,\tfrac{n}{2}-i+3).$$

For i even, $\frac{n}{2}+2 \leq i \leq n, n \geq 8$ we have:

$$c_{\Pi}(v_i) = (1,0,1,n-i+2,i-\tfrac{n}{2}).$$

$C_3 = \{u_i | \text{ for even } i, 2 \leq i \leq n\} \cup \{v_i | \text{ for odd } i, 1 \leq i \leq n-1\}$.
For $i = 1$, we have:

$$c_{\Pi}(v_i) = (1,3,0,1,\tfrac{n}{2}+1).$$

For i odd, $3 \leq i \leq \frac{n}{2}-1, n \geq 8$ we have:

$$c_{\Pi}(v_i) = (1,1,0,i,\tfrac{n}{2}-i+3).$$

For $i = \frac{n}{2}+1$, we have:

$$c_{\Pi}(v_i) = (2,1,0,i,1).$$

For i odd, $\frac{n}{2}+3 \leq i \leq n-1, n \geq 8$ we have:

$$c_{\Pi}(v_i) = (1,1,0,n-i+2,i-\tfrac{n}{2}).$$

For i even, $2 \leq i \leq \frac{n}{2}, n \geq 4$ we have:

$$c_{\Pi}(u_i) = (1,1,0,i-1,\tfrac{n}{2}).$$

For i even, $\frac{n}{2}+2 \leq i \leq n, n \geq 8$ we have:

$$c_{\Pi}(u_i) = (1,1,0,n-i+1,i-\tfrac{n}{2}-1).$$

For $C_4 = \{u_1\}$, we have:

$$c_{\Pi}(u_1) = (1,2,1,0,\tfrac{n}{2}).$$

For $C_5 = \{w_{\frac{n}{2}}\}$, we have:

$$c_{\Pi}(w_{\frac{n}{2}}) = (2,1,1,\tfrac{n}{2},0).$$

Since for n even all vertices have different color codes, c is a locating coloring of origami graphs O_n, so that $\chi_L(O_n) \leq 5$, for $\frac{n}{2}$ even, $n \geq 4$. this completes the proof of Theorem 1. □

Note that Figure 1 is an example locating coloring for origami graph O_5.

2.2. Locating-Chromatic Number for Subdivision Outer Edge of Origami Graphs

Theorem 2. *Let O_n^* be a subdivision outer edge of origami graphs for $n \geq 3$. Then the locating-chromatic number of O_n^*, $\chi_L(O_n^*) = \begin{cases} 4, & \text{for } n = 3 \\ 5, & \text{otherwise} . \end{cases}$*

Proof. Let O_n^*, $n \geq 3$ be a subdivision of an origami graph; O_n^* is a graph with $V(O_n^*) = \{u_i, v_i, x_i, w_i \ : \ i \in \{1, ..., n\}\}$ and $E(O_n^*) = \{u_i w_i, u_i v_i, v_i x_i, x_i w_i \ : \ i \in \{1, ..., n\}\} \cup \{u_i u_{i+1}, w_i u_{i+1} \ : \ i \in \{1, ..., n-1\}\} \cup \{u_n u_1, w_n u_1\}\}$. Next, to prove the theorem, we consider the following two cases:

Case A. $\chi_L(O_3^*) = 4$
First, we determine the lower bound of $\chi_L(O_3^*)$.

Without loss of generality, we assign $A = \{c(u_i), c(v_i), c(x_i), c(w_i), c(u_{i+1})\} = \{1,2,3\}$. Then, there are three dominant vertices in A, while we still have vertices on other A that must be colored. As a result, there will be two vertices with the same color codes. Thus, $\chi_L(O_3^*) \geq 4$.

Next, we determine the upper bound of $\chi_L(O_3^*) \leq 4$. To show that 4 is an upper bound for the locating-chromatic number for a subdivision outer edge of origami graph O_3^*, we describe a locating coloring c using four colors as follows:

$c(u_i) = i, i = 1,2,3.$

$c(v_i) = \begin{cases} 2, & \text{for } i = 1,3 \\ 3, & \text{for } i = 2. \end{cases}$

$c(w_i) = 4, i = 1,2,3.$

$c(x_i) = i, i = 1,2,3.$

The coloring c will create the partition Π on $V(O_3^*)$. We shall show that the color codes of all vertices in O_3^* are different. We have: $c_\Pi(u_1) = (0,1,1,1)$; $c_\Pi(u_2) = (1,0,1,1)$; $c_\Pi(u_3) = (1,1,0,1)$; $c_\Pi(v_1) = (1,0,2,2)$; $c_\Pi(v_2) = (2,1,0,2)$; $c_\Pi(v_3) = (2,0,1,2)$; $c_\Pi(w_1) = (1,1,2,0)$; $c_\Pi(w_2) = (2,1,1,0)$; $c_\Pi(w_3) = (1,2,1,0)$. $c_\Pi(x_1) = (0,1,3,1)$; $c_\Pi(x_2) = (3,0,1,1)$; $c_\Pi(x_3) = (2,1,0,1)$. Since the color codes of all vertices O_3^* are different, c is a locating-chromatic coloring. Thus, $\chi_L(O_3^*) \leq 4$.

Case B. $\chi_L(O_n^*) = 5$ for $n \geq 4$
Since a subdivision of origami graphs O_n^* for $n \geq 4$ is obtained by origami graph O_n with one added vertex in edge $v_i w_i$, we have $\chi_L(O_n^*) \geq 5$ for $n \geq 4$. The addition of one vertex on the outside does not reduce the number of colors needed because the number of the sets $B = \{c(u_i), c(v_i), c(w_i), c(u_{i+1})\}$ is the same.

To show the upper bound for the locating-chromatic number for a subdivision outer edge of origami graph O_n^* for $n \geq 4$, let us consider different subcases.

Subcase a. (odd n), for $\lceil \frac{n}{2} \rceil$ odd, $n \geq 5$.
Let c be a coloring for a subdivision outer edge of origami graph O_n^*, for $\lceil \frac{n}{2} \rceil$ odd, and $n \geq 5$; we make the partition Π of $V(O_n^*)$:

$C_1 = \{w_i | 1 \leq i \leq n\}$;

$C_2 = \{u_i |$ for odd $i, 3 \leq i \leq n\} \cup \{v_i |$ for even $i, 2 \leq i \leq n-1\} \cup \{x_i |$ for odd $i, 1 \leq i \leq n\}$;

$C_3 = \{u_i |$ for even $i, 2 \leq i \leq \lceil \frac{n}{2} \rceil - 1\} \cup \{u_i |$ for even i, $\lceil \frac{n}{2} \rceil + 3 \leq i \leq n-1\} \cup \{v_i |$ for odd $i, 1 \leq i \leq n\} \cup \{x_i |$ for even $i, 2 \leq i \leq n-1\}$;

$C_4 = \{u_1\}$;

$C_5 = \{u_{\lceil \frac{n}{2} \rceil + 1}\}$.

For for $\lceil \frac{n}{2} \rceil$ odd the color codes of all the vertices of $V(O_n^*)$ are:

$$c_\Pi(u_i)=\begin{cases}0, & \text{for the second component, odd } i, 3\le i\le n\ , n\ge 5\\ & \text{for the third component, even } i, 2\le i\le \lceil\frac{n}{2}\rceil-1, n\ge 5\\ & \text{for the third component, even } i, \lceil\frac{n}{2}\rceil+3\le i\le n-1, n\ge 9\\ & \text{for the fourth component, } i=1\\ & \text{for the fifth component, } i=\lceil\frac{n}{2}\rceil+1\\ 2, & \text{for the third component, } i=\lceil\frac{n}{2}\rceil+1\\ i-1, & \text{for the fourth component, } 2\le i\le\lceil\frac{n}{2}\rceil, n\ge 5\\ n-i+1 & \text{for the fourth component, } \lceil\frac{n}{2}\rceil+1\le i\le n, n\ge 5\\ \lceil\frac{n}{2}\rceil-i, & \text{for the fifth component, } i=1\\ i-\lceil\frac{n}{2}\rceil-1, & \text{for the fifth component, } \lceil\frac{n}{2}\rceil+1\le i\le n, n\ge 5\\ \lceil\frac{n}{2}\rceil-i+1, & \text{for the fifth component, } 2\le i\le\lceil\frac{n}{2}\rceil, n\ge 5\\ 1, & \text{otherwise .}\end{cases}$$

$$c_\Pi(v_i)=\begin{cases}2, & \text{for the first component, } 1\le i\le n, n\ge 5\\ 0, & \text{for the second component, odd } i, 1\le i\le n, n\ge 5\\ & \text{for the third component, even } i, 2\le i\le n-1, n\ge 5\\ i, & \text{for the fourth component, } 1\le i\le\lceil\frac{n}{2}\rceil, n\ge 5\\ n-i+2, & \text{for the fourth component, } \lceil\frac{n}{2}\rceil+1\le i\le n, n\ge 5\\ \lceil\frac{n}{2}\rceil, & \text{for the fifth component, } i=1\\ \lceil\frac{n}{2}\rceil-i+2, & \text{for the fifth component, } 2\le i\le\lceil\frac{n}{2}\rceil, n\ge 5\\ i-\lceil\frac{n}{2}\rceil, & \text{for the fifth component, } \lceil\frac{n}{2}\rceil+1\le i\le n, n\ge 5\\ 1, & \text{otherwise .}\end{cases}$$

$$c_\Pi(w_i)=\begin{cases}0, & \text{for the first component, } 1\le i\le n, n\ge 5\\ 2, & \text{for the third component, } i=\lceil\frac{n}{2}\rceil \text{ and } i=n\\ \lceil\frac{n}{2}\rceil-i+1, & \text{for the fifth component, } 1\le i\le\lceil\frac{n}{2}\rceil, n\ge 5\\ i-\lceil\frac{n}{2}\rceil, & \text{for the fifth component, } \lceil\frac{n}{2}\rceil+1\le i\le n, n\ge 5\\ i, & \text{for the fourth component, } 1\le i\le\lceil\frac{n}{2}\rceil, n\ge 5\\ n-i+1, & \text{for the fourth component, } \lceil\frac{n}{2}\rceil+1\le i\le n, n\ge 5\\ 1, & \text{otherwise .}\end{cases}$$

$$c_\Pi(x_i)=\begin{cases}0, & \text{for the second component, odd } i, 1\le i\le n, n\ge 5\\ & \text{for the third component, even } i, 2\le i\le n-1, n\ge 5\\ i+1, & \text{for the fourth component, } 1\le i\le\lceil\frac{n}{2}\rceil, n\ge 5\\ n-i+2, & \text{for the fourth component, } \lceil\frac{n}{2}\rceil+1\le i\le n, n\ge 5\\ \lceil\frac{n}{2}\rceil-i+2, & \text{for the fifth component, } 1\le i\le\lceil\frac{n}{2}\rceil, n\ge 5\\ i-\lceil\frac{n}{2}\rceil+1, & \text{for the fifth component, } \lceil\frac{n}{2}\rceil+1\le i\le n, n\ge 5\\ 1, & \text{otherwise .}\end{cases}$$

Since for n odd all vertices have different color codes, c is a locating coloring for subdivision of origami graph O_n^*, so that $\chi_L(O_n^*)\le 5$, for $\lceil\frac{n}{2}\rceil$ odd, $n\ge 5$.

Subcase b. (odd n), for $\lceil\frac{n}{2}\rceil$ even, $n\ge 7$.
Let c be a coloring for a subdivision outer edge of origami graph O_n^*, for $\lceil\frac{n}{2}\rceil$ even, and $n\ge 7$; we make the partition Π of $V(O_n^*)$:
$C_1=\{w_i|1\le i\le n\}$;
$C_2=\{u_i|\text{ for odd } i, 3\le i\le n\}\cup\{v_i|\text{ for even } i, 2\le i\le n-1\}\cup\{x_i|\text{ for odd } i, 1\le i\le n\}$;
$C_3=\{u_i|\text{ for even } i, 2\le i\le\lceil\frac{n}{2}\rceil-2\}\cup\{u_i|\text{ for even } i, \lceil\frac{n}{2}\rceil+2\le i\le n-1\}\cup\{v_i|$ for

odd $i, 1 \le i \le n\} \cup \{x_i|$ for even $i, 2 \le i \le n-1\}$;
$C_4 = \{u_1\}$;
$C_5 = \{u_{\lceil \frac{n}{2} \rceil}\}$.

For $\lceil \frac{n}{2} \rceil$ even, the color codes of all the vertices of $V(O_n^*)$ are:

$$c_\Pi(u_i) = \begin{cases} 0, & \text{for the second component, odd } i, 3 \le i \le n, n \ge 7 \\ & \text{for the third component, even } i, 2 \le i \le \lceil \frac{n}{2} \rceil - 2, n \ge 7 \\ & \text{for the third component, even } i, \lceil \frac{n}{2} \rceil + 2 \le i \le n-1, n \ge 7 \\ & \text{for the fourth component, } i = 1 \\ & \text{for the fifth component, } i = \lceil \frac{n}{2} \rceil \\ 2, & \text{for the third component, } i = \lceil \frac{n}{2} \rceil \\ i-1, & \text{for the fourth component, } 2 \le i \le \lceil \frac{n}{2} \rceil - 1, n \ge 7 \\ n-i+1, & \text{for the fourth component, odd } i, \lceil \frac{n}{2} \rceil + 1 \le i \le n, n \ge 7 \\ \lceil \frac{n}{2} \rceil - 1, & \text{for the fourth component, } i = \lceil \frac{n}{2} \rceil \\ \lceil \frac{n}{2} \rceil - i, & \text{for the fifth component, } 1 \le i \le \lceil \frac{n}{2} \rceil - 1, n \ge 7 \\ i - \lceil \frac{n}{2} \rceil, & \text{for the fifth component, } \lceil \frac{n}{2} \rceil + 1 \le i \le n, n \ge 7 \\ 1, & \text{otherwise .} \end{cases}$$

$$c_\Pi(v_i) = \begin{cases} 0, & \text{for the second component, even } i, 2 \le i \le n-1, n \ge 7 \\ & \text{for the third component, odd } i, 1 \le i \le n, n \ge 7 \\ 2, & \text{for the first component, } 1 \le i \le n, n \ge 7 \\ i, & \text{for the fourth component, } 1 \le i \le \lceil \frac{n}{2} \rceil, n \ge 7 \\ n-i+2 & \text{for the fourth component, } \lceil \frac{n}{2} \rceil + 1 \le i \le n, n \ge 7 \\ \lceil \frac{n}{2} \rceil - i + 1 & \text{for the fifth component, } 1 \le i \le \lceil \frac{n}{2} \rceil, n \ge 7 \\ i - \lceil \frac{n}{2} \rceil + 1 & \text{for the fifth component, } \lceil \frac{n}{2} \rceil + 1 \le i \le n, n \ge 7 \\ 1, & \text{otherwise .} \end{cases}$$

$$c_\Pi(w_i) = \begin{cases} 0, & \text{for the first component, } 1 \le i \le n, n \ge 7 \\ 2, & \text{for the third component, } i = \lceil \frac{n}{2} \rceil - 1 \text{ and } i = n \\ i, & \text{for the fourth component, } 1 \le i \le \lceil \frac{n}{2} \rceil, n \ge 7 \\ n-i+1, & \text{for the fourth component, } \lceil \frac{n}{2} \rceil + 1 \le i \le n, n \ge 7 \\ \lceil \frac{n}{2} \rceil - i, & \text{for the fifth component, } 1 \le i \le \lceil \frac{n}{2} \rceil - 1, n \ge 7 \\ i - \lceil \frac{n}{2} \rceil + 1, & \text{for the fifth component, } \lceil \frac{n}{2} \rceil \le i \le n, n \ge 7 \\ 1, & \text{otherwise .} \end{cases}$$

$$c_\Pi(x_i) = \begin{cases} 0, & \text{for the second component, odd } i, 1 \le i \le n, n \ge 7 \\ & \text{for the third component, even } i, 2 \le i \le n-1, n \ge 7 \\ i+1, & \text{for the fourth component, } 1 \le i \le \lceil \frac{n}{2} \rceil - 1, n \ge 7 \\ n-i+2, & \text{for the fourth component, } \lceil \frac{n}{2} \rceil \le i \le n, n \ge 7 \\ \lceil \frac{n}{2} \rceil - i + 2, & \text{for the fifth component, } 1 \le i \le \lceil \frac{n}{2} \rceil, n \ge 7 \\ i - \lceil \frac{n}{2} \rceil + 2, & \text{for the fifth component, } \lceil \frac{n}{2} \rceil + 1 \le i \le n, n \ge 7 \\ 1, & \text{otherwise .} \end{cases}$$

Since for n odd all vertices have different color codes, c is a locating coloring for a subdivision of the outer edge of origami graph O_n^*, so that $\chi_L(O_n^*) \le 5$, for $\lceil \frac{n}{2} \rceil$ even, $n \ge 7$.

Subcase c. (even n), for $\frac{n}{2}$ odd, $n \geq 6$.
Let c be a coloring for a subdivision outer edge of origami graph O_n^*, for $\frac{n}{2}$ odd, and $n \geq 6$; we make the partition Π of $V(O_n^*)$:
$C_1 = \{w_i | 1 \leq i \leq \frac{n}{2} - 1\} \cup \{w_i | \frac{n}{2} + 1 \leq i \leq n\}$;
$C_2 = \{u_i |$ for odd $i, 3 \leq i \leq n-1\} \cup \{v_i |$ for even $i, 2 \leq i \leq n\} \cup \{x_i |$ for odd $i, 1 \leq i \leq n-1\}$;
$C_3 = \{u_i |$ for even $i, 2 \leq i \leq n\} \cup \{v_i |$ for odd $i, 1 \leq i \leq n-1\} \cup \{x_i |$ for even $i, 2 \leq i \leq n\}$;
$C_4 = \{u_1\}$;
$C_5 = \{w_{\frac{n}{2}}\}$.

For $\frac{n}{2}$ odd, the color codes of all the vertices of $V(O_n^*)$ are:

$$c_\Pi(u_i) = \begin{cases} 0, & \text{for the second component, odd } i, 3 \leq i \leq n-1, n \geq 6 \\ & \text{for the third component, even } i, 2 \leq i \leq n, n \geq 6 \\ & \text{for the fourth component, } i = 1 \\ 2, & \text{for the second component, } i = 1 \\ i-1, & \text{for the fourth component, } 2 \leq i \leq \frac{n}{2}, n \geq 6 \\ n-i+1, & \text{for the fourth component, } \frac{n}{2}+1 \leq i \leq n, n \geq 6 \\ \frac{n}{2}-i+1, & \text{for the fifth component, } 1 \leq i \leq \frac{n}{2}, n \geq 6 \\ i-\frac{n}{2}, & \text{for the fifth component, } \frac{n}{2}+1 \leq i \leq n, n \geq 6 \\ 1, & \text{otherwise .} \end{cases}$$

$$c_\Pi(v_i) = \begin{cases} 2, & \text{for the first component, } 1 \leq i \leq n, n \geq 6 \\ 0, & \text{for the second component, even } i, 2 \leq i \leq n, n \geq 6 \\ & \text{for the third component, odd } i, 1 \leq i \leq n-1, n \geq 6 \\ i, & \text{for the fourth component, } 1 \leq i \leq \frac{n}{2}, n \geq 6 \\ n-i+2, & \text{for the fourth component, } \frac{n}{2}+1 \leq i \leq n, n \geq 6 \\ \frac{n}{2}-i+2, & \text{for the fifth component, } 1 \leq i \leq \frac{n}{2}, n \geq 6 \\ i-\frac{n}{2}+1, & \text{for component, fifth component, } \frac{n}{2}+1 \leq i \leq n, n \geq 6 \\ 1, & \text{otherwise .} \end{cases}$$

$$c_\Pi(w_i) = \begin{cases} 0, & \text{for the first component, } 1 \leq i \leq \frac{n}{2}-1, n \geq 6 \\ & \text{for the first component, } \frac{n}{2}+1 \leq i \leq n, n \geq 6 \\ & \text{for the fifth component, } i = \frac{n}{2} \\ 2, & \text{for the first component, } i = \frac{n}{2} \\ & \text{for the second component, } i = n \\ i, & \text{for the fourth component, } 1 \leq i \leq \frac{n}{2}, n \geq 6 \\ n-i+1, & \text{for the fourth component, } \frac{n}{2}+1 \leq i \leq n, n \geq 6 \\ \frac{n}{2}-i+1, & \text{for the fifth component, } 1 \leq i \leq \frac{n}{2}, n \geq 6 \\ i-\frac{n}{2}+1, & \text{for the fifth component, } \frac{n}{2}+1 \leq i \leq n, n \geq 6 \\ 1, & \text{otherwise .} \end{cases}$$

$$c_\Pi(x_i) = \begin{cases} 0, & \text{for the second component, odd } i, 1 \le i \le n-1, n \ge 6 \\ & \text{for the third component, even } i, 2 \le i \le n, n \ge 6 \\ i+1, & \text{for the fourth component, } 1 \le i \le \frac{n}{2}, n \ge 6 \\ n-i+2, & \text{for the fourth component, } \frac{n}{2}+1 \le i \le n, n \ge 6 \\ \frac{n}{2}-i+2, & \text{for the fifth component, } 1 \le i \le \frac{n}{2}-1, n \ge 6 \\ i-\frac{n}{2}+2, & \text{for the fifth component, } \frac{n}{2}+1 \le i \le n, n \ge 6 \\ 1, & \text{otherwise.} \end{cases}$$

Since for n even all vertices have different color codes, c is a locating coloring for a subdivision of the outer edge of origami graph O_n^*, so that $\chi_L(O_n^*) \le 5$, for $\frac{n}{2}$ odd, $n \ge 6$.

Subcase d. (even n), for $\frac{n}{2}$ even, $n \ge 4$.
Let c be a coloring of subdivision origami graph O_n^*, for $\frac{n}{2}$ even, and $n \ge 4$; we make the partition Π of $V(O_n^*)$:
$C_1 = \{w_i | 1 \le i \le \frac{n}{2}\} \cup \{w_i | \frac{n}{2}+2 \le i \le n\}$;
$C_2 = \{u_i | \text{ for odd } i, 3 \le i \le n-1\} \cup \{v_i | \text{ for even } i, 2 \le i \le n\} \cup \{x_i | \text{ for odd } i, 1 \le i \le n-1\}$;
$C_3 = \{u_i | \text{ for even } i, 2 \le i \le n\} \cup \{v_i | \text{ for odd } i, 1 \le i \le n-1\} \cup \{x_i | \text{ for even } i, 2 \le i \le n\}$;
$C_4 = \{u_1\}$;
$C_5 = \{w_{\frac{n}{2}+1}\}$.

For $\frac{n}{2}$ even the color codes of all the vertices of $V(O_n^*)$ are:

$$c_\Pi(u_i) = \begin{cases} 0, & \text{for the second component, odd } i, 3 \le i \le n-1, n \ge 4 \\ & \text{for the third component, even } i, 2 \le i \le n, n \ge 4 \\ & \text{for the fourth component, } i = 1 \\ 2, & \text{for the second component, } i = 1 \\ i-1, & \text{for the fourth component, } 2 \le i \le \frac{n}{2}+1, n \ge 4 \\ n-i+1, & \text{for the fourth component, } \frac{n}{2}+2 \le i \le n, n \ge 4 \\ \frac{n}{2}, & \text{for the fifth component, } i = 1 \\ \frac{n}{2}-i+2, & \text{for the fifth component, } 2 \le i \le \frac{n}{2}+1, n \ge 4 \\ i-\frac{n}{2}-1, & \text{for the fifth component, } \frac{n}{2}+2 \le i \le n, n \ge 4 \\ 1, & \text{otherwise .} \end{cases}$$

$$c_\Pi(v_i) = \begin{cases} 2, & \text{for the first component, } 1 \le i \le n, n \ge 4 \\ 0, & \text{for the second component, even } i, 2 \le i \le n, n \ge 4 \\ & \text{for the third component, odd } i, 1 \le i \le n-1, n \ge 4 \\ i, & \text{for the fourth component, } 1 \le i \le \frac{n}{2}, n \ge 4 \\ n-i+2, & \text{for the fourth component, } \frac{n}{2}+1 \le i \le n, n \ge 4 \\ \frac{n}{2}+i, & \text{for the fifth component, } i = 1 \\ \frac{n}{2}-i+3, & \text{for the fifth component, } 2 \le i \le \frac{n}{2}+1, n \ge 4 \\ i-\frac{n}{2}, & \text{for the fifth component, } \frac{n}{2}+2 \le i \le n, n \ge 4 \\ 1, & \text{otherwise .} \end{cases}$$

$$c_\Pi(w_i) = \begin{cases} 0, & \text{for the first component, } 1 \le i \le \frac{n}{2}, n \ge 4 \\ & \text{for the first component, } \frac{n}{2}+2 \le i \le n, n \ge 4 \\ & \text{for the fifth component, } i = \frac{n}{2}+1 \\ 2, & \text{for the first component, } i = \frac{n}{2}+1 \\ & \text{for the second component, } i = n \\ i, & \text{for the fourth component, } 1 \le i \le \frac{n}{2}, n \ge 4 \\ n-i+1, & \text{for the fourth component, } \frac{n}{2}+1 \le i \le n, n \ge 4 \\ \frac{n}{2}-i+2, & \text{for the fifth component, } 1 \le i \le \frac{n}{2}, n \ge 4 \\ i-\frac{n}{2}, & \text{for the fifth component, } \frac{n}{2}+2 \le i \le n, n \ge 4 \\ 1, & \text{otherwise .} \end{cases}$$

$$c_\Pi(x_i) = \begin{cases} 0, & \text{for the second component, odd } i, 1 \le i \le n-1, n \ge 4 \\ & \text{for the third component, even } i, 2 \le i \le n, n \ge 4 \\ i+1, & \text{for the fourth component, } 1 \le i \le \frac{n}{2}, n \ge 6 \\ n-i+2, & \text{for the fourth component, } \frac{n}{2}+1 \le i \le n, n \ge 4 \\ \frac{n}{2}-i+3, & \text{for the fifth component, } 1 \le i \le \frac{n}{2}, n \ge 4 \\ i-\frac{n}{2}+1, & \text{for the fifth component, } \frac{n}{2}+2 \le i \le n, n \ge 4 \\ 1, & \text{otherwise.} \end{cases}$$

Since for n even all vertices have different color codes, c is a locating coloring for a subdivision outer edge of origami graph O_n^*, so that $\chi_L(O_n^*) \le 5$, for $\frac{n}{2}$ even, $n \ge 4$. This completes the proof of Theorem 2. □

Note that Figure 2 is an example locating coloring for a subdivision of the outer edge of origami graph O_5^*.

3. Conclusions

The proving steps of the two theorems we gave earlier show that the locating-chromatic number of origami graphs O_n, $\chi_L(O_n)$ is 4 for $n = 3$ and 5 for $n \ge 4$; the same result holds for a subdivision of the outer edge of origami graph O_n^*. This research can be continued so as to determine the locating-chromatic number for some certain operations of origami graphs.

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