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Characteristics of Bayes Estimator in the Geometric Distribution with Prior Beta

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Characteristics of Bayes Estimator in the Geometric Distribution with Prior Beta

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Abstract. This study aims to examine the unbiased, minimum variance (efficient), and consistent characteristics of Bayes estimator in the Geometric distribution with prior Beta. Based on the results of simulation studies it is found that the Bayes estimator in the Geometric distribution with prior Beta are symptotically unbiased estimator for values $\theta < 0{,}5$ and is biased for others, are efficient for the number of samples sizes large and values $\theta \leq 0{,}6$ and not efficient for others and consistent when value $\theta \leq 0{,}5$ and inconsistent for other.

Keyword: Bayes estimator, unbiased, efficient, consistent, geometric distribution.

1. Introduction

Bayes method is a method that combines the likelihood function and prior distribution of parameters to get the posterior distribution which is the basis in the estimation parameters [1]. Estimation of parameters using the Bayes method has been widely applied in various fields of research, namely spatial analysis, small area estimation, survival analysis and others such as those carried out by Widiarti et al., Rizki et al., Hartono et al., Yanuar et al., Fulop and Li, Kinyanjui and Korir, Pusponegoro and Rachmawati, Aw and Cabral [2-10]. In the Bayes method, parameter estimation is done by looking at unknown parameters as random variables that have an initial distribution of these parameters, this distribution is called a prior distribution[11].

Prior distributions are usually obtained based on the subjective beliefs of the researchers themselves. However, for the distribution of samples from the Exponential distribution family, one way is to use the prior conjugate [11]. Geometric distribution is a probability distribution with a success event symbolized by θ and a failure event is denoted by$(1-\theta)$ and is a distribution that comes from the family of Exponential distribution. The random variable of this distribution is the amount of effort needed to get the first success. One of the prior conjugates for the Exponential distribution family with parameters stating probability is the Beta distribution. The Beta distribution is a continuous distribution at interval $(0{,}1)$ and has two parameters namely α and β [12].

Characteristics of estimators need to be examined in order to see the quality of an estimator. Estimator is called a good estimator if it meets certain characteristics. Some characteristics of estimators are unbiased, efficient, consistent, sufficient statistics and complete statistics. An estimator is consistent for its parameters if the sample size gets larger then the estimator will approach the parameter and if the sample size becomes infinite then the estimator will be equal to the parameter or the Mean Square Error (MSE) of the estimator is equal to zero [13]. Thus, this study will examine the quality of Bayes estimators in the Geometric distribution with prior Beta. The characteristics studied are unbiased, efficient and consistent using MSE.

2. Method

2.1. Research Data

The data used in this are simulation data generated using R-Studio Software version 1.2.1.335. The data generated by scenario $\theta \sim$ Beta (α, β) and $y \sim$ Geometrik (θ) as well as with a varying number of sample sizes is 20, 50, 100, 500 and 1000. Paired values (α,β) are set as many as nine pairs is (2,18), (2,8), (2,5), (2,3), (2,2), (3,2), (5,2), (8,2) and (18,2) which results θ value is 0.1, 0,2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8 and 0.9.

2.2. Bayes Method

Bayes method is a method that combines the likelihood function and prior distribution of parameters to get the posterior distribution which is the basis in the estimation parameters [1]. If $B_1, B_2, \dots, B_k$ is an exclusive and complete event, then for each occurrence of B_i conditional A is:

$$P(B_i|A) = \frac{P(B_i \cap A)}{\sum_{i=1}^{k} P(B_i \cap A)} = \frac{P(B_i)P(A|B_i)}{\sum_{i=1}^{k} P(B_i)P(A|B_i)} \tag{1}$$

Equation 1 is Bayes rule [14].

2.3. Unbiased

The estimator $\hat{\theta}$ is said to be unbiased for θ if the expected value is the same as the parameter of the probability distribution parameter.

$$E(\hat{\theta}) = \theta \tag{2}$$

and it is said to be asymptotically biased if the estimator is biased but when the sample size is greater the expected value is the same as the parameter

$$\lim_{n\to\infty} E(\hat{\theta}) = \theta \tag{3}$$

or

$$\lim_{n\to\infty} Biased(\hat{\theta}) = 0 \tag{4}$$

with

$$Biased(\hat{\theta}) = E(\hat{\theta}) - \theta \tag{5}$$

and n is the number of sizes sample [13].

2.4. Efficient

The estimator $\hat{\theta}$ is said to be efficient for θ parameter if the estimator $\hat{\theta}$ has the smallest variance. If $\hat{\theta}_1$ and $\hat{\theta}_2$ are estimator for θ, then estimator $\hat{\theta}_1$ is said to be more efficiant than estimator $\hat{\theta}_2$ if:

$$Var(\hat{\theta}_1) \le Var\,(\hat{\theta}_2) \tag{6}$$

for $\hat{\theta}_1$ and $\hat{\theta}_2$ is unbiased estimator for θ [14]. If $\hat{\theta}$ is an unbiased estimator for θ, then the estimator $\hat{\theta}$ is said to be efficiant if and only if the variance of the estimator $\hat{\theta}$ reaches the Rao-Cramer lower bound.

$$Var(\hat{\theta}) \ge \frac{1}{n\,I(\theta)} \tag{7}$$

with $\frac{1}{n\,I(\theta)}$ is the Rao-Cramer lower bound and $I(\theta)$ Fisher's information [15].

$$I(\theta) = E\left\{\left[\frac{\partial \ln f(x;\theta)}{\partial\theta}\right]^2\right\}$$

or (8)

$$I(\theta) = -E\left\{\left[\frac{\partial^2 \ln f(x;\theta)}{\partial\theta^2}\right]\right\}$$

2.5. Consistent
The estimator $\hat{\theta}$ is said to be consistent for θ parameter if the estimator is convergent in probability θ or if the sample size gets larger then the estimator will approach the parameter and if the sample size becomes infinite then the estimator will be equal to the parameter.

$$\lim_{n\to\infty} p(\hat{\theta}_n - \theta| \geq \varepsilon) = 0$$

atau (9)

$$\lim_{n\to\infty} E((\hat{\theta} - \theta)^2) = 0$$

for every $\varepsilon > 0$ [16].

The consistency of an estimator also can be evaluated based on the value of MSE. If $\hat{\theta}$ is an estimator of parameter θ, then the MSE for $\hat{\theta}$ is defined as follows:

$$MSE(\hat{\theta}) = E[\hat{\theta} - E(\hat{\theta})]^2 + [E(\hat{\theta}) - \theta]^2$$

$$= Var(\hat{\theta}) + [Bias(\hat{\theta})]^2 \quad (11)$$

If $\hat{\theta}$ is the an unbiased estimator then MSE will be equal to the variance $\hat{\theta}$ [13].

3. Result and Discussion

3.1 Bayes Estimator
In the Bayes Estimator, the probability function $f(y_i|\theta)$ is expressed by the conditional probability function $f(y_i|\theta)$, so that for $Y_1, \dots, Y_n$ a random sample of a population with a Geometric distribution with parameter θ can be written as follows:

$$Y_i \sim Geo(\theta) \Leftrightarrow f(y_i|p) = \begin{cases} \theta(1-\theta)^{y-1} & ;\ y_i\ = 1,2,3,\dots; i = 1,2\dots,n\,; 0 \leq \theta \leq 1 \\ 0 & ;\ y_i\ \ lainnya \end{cases} \quad (12)$$

The prior distribution for $Y_i \sim Geo(\theta)$, is $\theta \sim Beta(\alpha, \beta)$. So the probability function of the prior distribution is

$$f(\theta) = \begin{cases} \frac{1}{B(\alpha,\beta)}\ \theta^{\alpha-1}(1-\theta)^{\beta-1} & ; 0 < \theta < 1 \\ 0 & ; \theta\ lainnya \end{cases} \quad (13)$$

The likelihood function of $Y_i \sim Geo(\theta)$:

$$f(y_1, y_2, \dots, y_n|\theta) = \prod_{i=1}^{n} \theta(1-\theta)^{y_i - 1} = \theta^n (1-\theta)^{\sum_{i=1}^{n} y_i - n} \quad (14)$$

Joint probability density function of $Y_1, \dots, Y_n$ and θ is:

$$f(y_1, y_2, \dots, y_n; \theta) = f(y_1, y_2, \dots, y_n|\theta)\, f(\theta)$$
$$= \theta^n (1-\theta)^{\sum_{i=1}^n y_i - n} \frac{1}{B(\alpha,\beta)} \theta^{\alpha-1}(1-\theta)^{\beta-1}$$
$$= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \theta^{\alpha+n-1}(1-\theta)^{\beta+\sum_{i=1}^n y_i - n - 1} \tag{15}$$

Marginal function of Y_i is:

$$m(y_i y_1, y_2, \dots, y_n) = \int_{-\infty}^{\infty} f(y_1, y_2, \dots, y_n; \theta) d\theta$$
$$= \int_0^1 \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \theta^{\alpha+n-1}(1-\theta)^{\beta+\sum_{i=1}^n y_i - n - 1}$$
$$= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \frac{\Gamma(\alpha+n)\Gamma(\beta+\sum_{i=1}^n y_i - n)}{\Gamma(\alpha+\beta+\sum_{i=1}^n y_i)} \tag{16}$$

The posterior distribution is:

$$f(\theta|y_1, y_2, \dots, y_n) = \frac{f(y_1, y_2, \dots, y_n; \theta)}{m(y_1, y_2, \dots, y_n)}$$
$$= \frac{\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \theta^{\alpha+n-1}(1-\theta)^{\beta+\sum_{i=1}^n y_i - n - 1}}{\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \frac{\Gamma(\alpha+n)\Gamma(\beta+\sum_{i=1}^n y_i - n)}{\Gamma(\alpha+\beta+\sum_{i=1}^n y_i)}}$$
$$= \frac{\Gamma(\alpha+\beta+\sum_{i=1}^n y_i)}{\Gamma(\alpha+n)\Gamma(\beta+\sum_{i=1}^n y_i - n)} \theta^{\alpha+n-1}(1-\theta)^{\beta+\sum_{i=1}^n y_i - n - 1} \tag{17}$$

So, the Bayes estimator for θ is:

$$\hat{\theta}^B = E\big(f(\theta|y_1, y_2, \dots, y_n)\big) = \int_0^1 \theta\, f(\theta|y_1, y_2, \dots, y_n)\; d\theta$$
$$= \int_0^1 \theta\, \frac{\Gamma(\alpha+\beta+\sum_{i=1}^n y_i)}{\Gamma(\alpha+n)\Gamma(\beta+\sum_{i=1}^n y_i - n)} \theta^{\alpha+n-1}(1-\theta)^{\beta+\sum_{i=1}^n y_i - n - 1} d\theta$$
$$= \frac{\alpha+n}{\alpha+\beta+\sum_{i=1}^n y_i} \tag{18}$$

3.2 Umbiased

The estimator $\hat{\theta}^B$ is said to be unbiased for θ if $E(\hat{\theta}^B) = \theta$.

$$E(\hat{\theta}^B) = E\left(\frac{\alpha+n}{\alpha+\beta+\sum_{i=1}^n Y_i}\right)$$

$$= \frac{(\alpha + n)\theta}{\alpha\theta + \beta\theta + n} \tag{19}$$

Because $E(\hat{\theta}^B) \neq \theta$, then $\hat{\theta}^B$ is biased estimator for θ. But, $\hat{\theta}^B$ is asymptotically unbiased because:

$$\lim_{n\to\infty} E(\hat{\theta}^B) = \lim_{n\to\infty} \frac{\alpha\theta + n\theta}{\alpha\theta + \beta\theta + n} \tag{20}$$

$$= \lim_{n\to\infty} \frac{\frac{\alpha\theta}{n} + \frac{n\theta}{n}}{\frac{\alpha\theta}{n} + \frac{\beta\theta}{n} + \frac{n}{n}}$$

$$= \frac{0 + \theta}{0 + 0 + 1} = \theta$$

3.3 *Efficient*

The estimator $\hat{\theta}^B$ is efficient for θ parameter if the estimator $\hat{\theta}^B$ has the smallest variance.

$$Var(\hat{\theta}^B) = Var\left(\frac{\alpha + n}{\alpha + \beta + \sum_{i=1}^{n} Y_i}\right)$$

$$= (\alpha + n)^2\, Var\left(\frac{1}{\alpha + \beta + \sum_{i=1}^{n} Y_i}\right) \tag{21}$$

Because $Var\left(\frac{1}{\alpha+\beta+\sum_{i=1}^{n} Y_i}\right)$ cannot be solved analytically, the value of variance will be shown by simulation using R-Studio *software* version 1.2.1335.

3.4 Consistent

The estimator $\hat{\theta}^B$ is consistent for θ if $\lim_{n\to\infty} E((\hat{\theta} - \theta)^2) = \lim_{n\to\infty} MSE(\hat{\theta}^B) = 0$.

$$\lim_{n\to\infty} E((\hat{\theta}^B - \theta)^2) = \lim_{n\to\infty} MSE(\hat{\theta}^B)$$

$$= \lim_{n\to\infty} \{Var(\hat{\theta}^B) + [bias(\hat{\theta}^B)]^2\} \tag{22}$$

Because $Var(\hat{\theta}^B)$ cannot be solved analytically, the value of MSE will be shown by simulation using R-Studio *software* version 1.2.1335

3.5 Simulation

Data simulation using R-Studio Software version 1.2.1.335 with several pairs of (α, β) and varying number of sample sizes yields the bias, variance and MSE values of the Bayes estimator for the Geometric distribution. Tables 1 to 3 display the simulation results for these values.

Table 1. The bias value of the Bayes estimator

α	β	θ	n	Bias $\hat{\theta}^B$	α	β	θ	n	Bias $\hat{\theta}^B$	α	β	θ	n	Bias $\hat{\theta}^B$
			20	0,015606				20	0,270064				20	1,056814
2	18	0,1	50	0,013049	2	3	0,4	50	0,269691	5	2	0,7	50	1,460895
			100	0,012367				100	0,265345				100	1,620098

			500	0,011222				500	0,263759				500	1,789975
			1000	0,011024				1000	0,261983				1000	1,805650
			20	0,055389				20	0,487611				20	1,121733
			50	0,054096				50	0,504101				50	1,887139
2	8	0,2	100	0,051252	2	2	0,5	100	0,500914	8	2	0,8	100	2,404464
			500	0,049785				500	0,503232				500	3,044202
			1000	0,049275				1000	0,500697				1000	3,140313
			20	0,119766				20	0,748747				20	0,816880
			50	0,118141				50	0,855411				50	1,793407
2	5	0,3	100	0,113565	3	2	0.6	100	0,881539	18	2	0,9	100	2,959372
			500	0,110819				500	0,911317				500	6,086961
			1000	0,110019				1000	0,911573				1000	6,975545

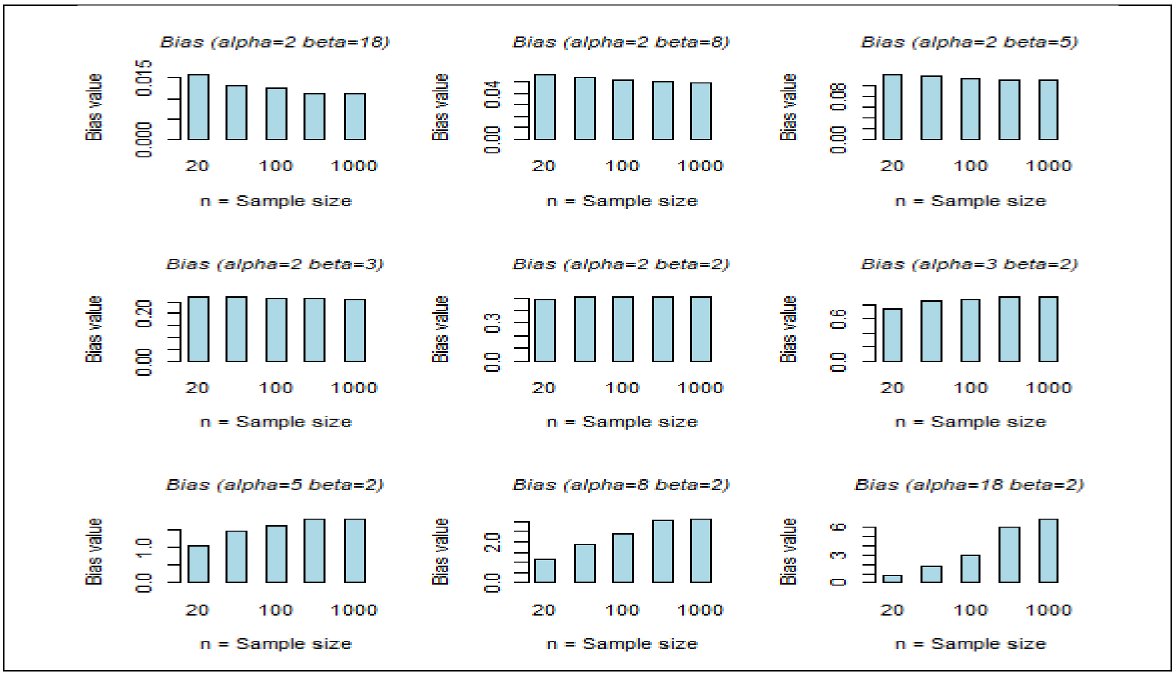

Figure 1. Histogram of the bias value.

Based on Table 1 and Figure 1, it can be seen that the bias value of Bayes estimator for $\theta < 0{,}5$ gets smaller when the number of sample sizes (n) gets bigger and vice versa for $\theta > 0{,}5$ and fluctuative for $\theta = 0.5$. So it can be concluded that the Bayes estimator is asymptotically unbiased estimator for $\theta < 0{,}5$ and biased for the others.

Table 2. The variance value of the Bayes estimator

α	β	θ	n	Var $\hat{\theta}^B$	α	β	θ	n	Var $\hat{\theta}^B$	α	β	θ	n	Var $\hat{\theta}^B$
			20	0,000676				20	0,035284				20	0,171253
2	18	0,1	50	0,000281	2	3	0,4	50	0,015158	5	2	0,7	50	0,234915
			100	0,000142				100	0,006887				100	0,144501

			500	0,000027				500	0,001305				500	0,038901
			1000	0,000013				1000	0,000708				1000	0,021432
			20	0,003512				20	0,087257				20	0,100130
			50	0,001535				50	0,039709				50	0,235316
2	8	0,2	100	0,000749	2	2	0,5	100	0,017910	8	2	0,8	100	0,294061
			500	0,000145				500	0,003702				500	0,121428
			1000	0,000074				1000	0,001906				1000	0,072042
			20	0,010244				20	0,152678				20	0,012953
			50	0,004663				50	0,090424				50	0,074497
2	5	0,3	100	0,002203	3	2	0.6	100	0,048039	18	2	0,9	100	0,210522
			500	0,000442				500	0,010725				500	0,571526
			1000	0,000215				1000	0,005551				1000	0,447854

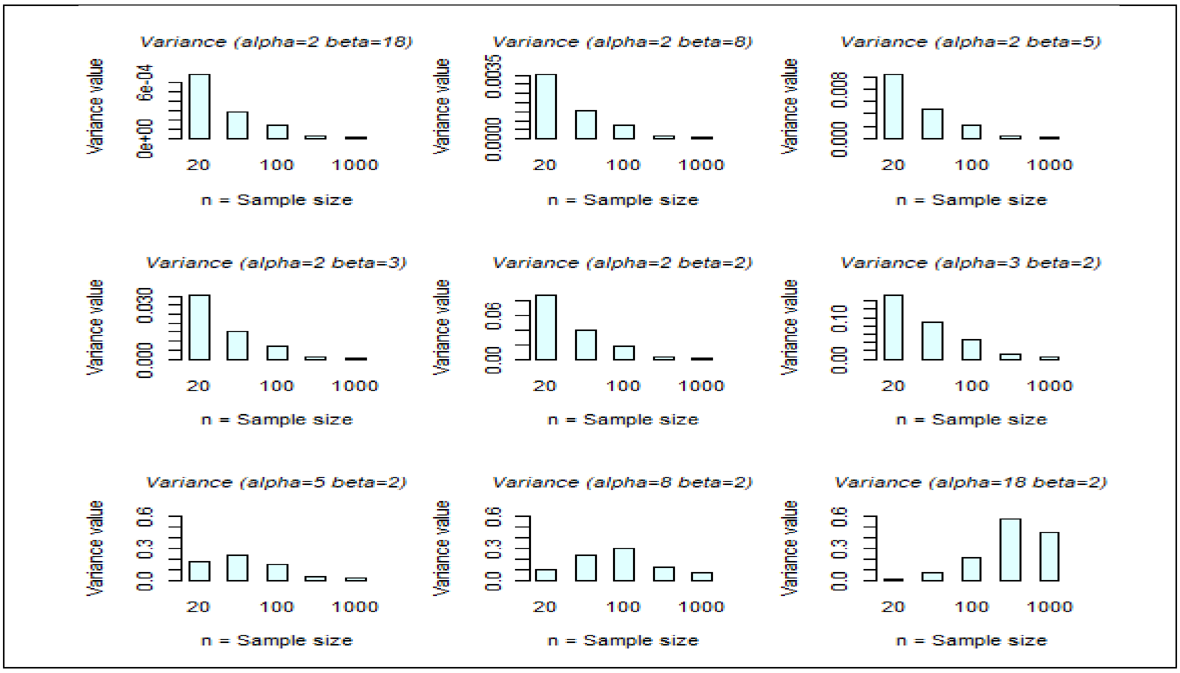

Figure 2. Histogram of the variance value.

Based on Table 2 and Figure 2, it can be seen that the variance value for $\theta \leq 0{,}6$ gets smaller when the number of sample sizes (n) gets bigger and fluctuative for others. The Bayes estimator is efficient when number of samples sizes large and values $\theta \leq 0{,}6$ and inefficient for the others.

Table 3. The MSE value of the Bayes estimator

α	β	θ	n	MSE $\hat{\theta}^B$	α	β	θ	n	MSE $\hat{\theta}^B$	α	β	θ	n	MSE $\hat{\theta}^B$
			20	0,000919				20	0,108219				20	1,288108
2	18	0,1	50	0,000451	2	3	0,4	50	0,087891	5	2	0,7	50	2,369129
			100	0,000295				100	0,077295				100	2,769218

			500	0,000153				500	0,070874				500	3,242913
			1000	0,000135				1000	0,069343				1000	3,281805
			20	0,006580				20	0,325021				20	1,358415
			50	0,004462				50	0,293827				50	3,796610
2	8	0,2	100	0,003376	2	2	0,5	100	0,268825	8	2	0,8	100	6,075506
			500	0,002624				500	0,256944				500	9,388595
			1000	0,002502				1000	0,252603				1000	9,933610
			20	0,024588				20	0,713299				20	0,680247
			50	0,018620				50	0,822152				50	3,290806
2	5	0,3	100	0,015100	3	2	0.6	100	0,825149	18	2	0,9	100	8,968404
			500	0,012723				500	0,841224				500	37,622619
			1000	0,012319				1000	0,836517				1000	49,106082

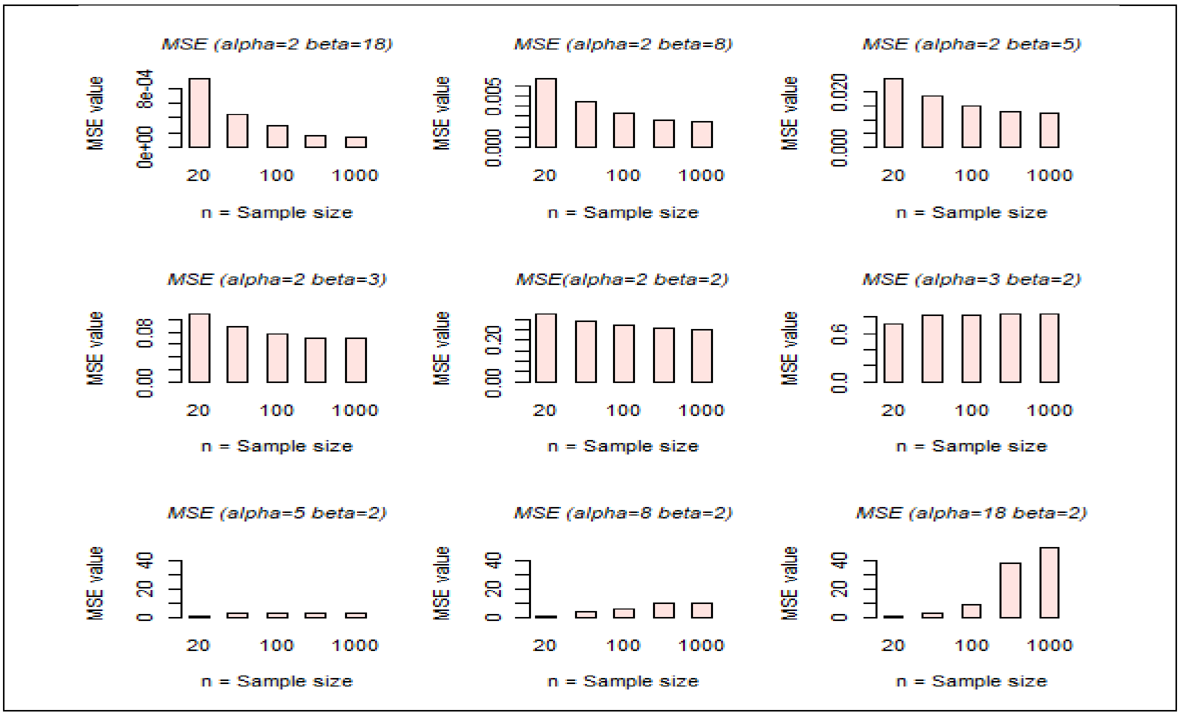

Figure 3. Histogram of the MSE value.

Based on Table 3 and Figure 3, it can be seen that MSE value of the Bayes estimator for $\theta \leq 0,5$ gets smaller when the number of sample sizes (n) gets bigger and vice versa for other. So, the Bayes estimator is consistent when value $\theta \leq 0,5$ and inconsistent for other.

4. Conclusion

Based on the result and discussion it can be concluded that the Bayes estimator on the Geometric distribution with prior Beta are asymptotically unbiased estimator for $\theta < 0,5$ and biased for the others, efficient when number of samples sizes large and values $\theta \leq 0,6$ and inefficient for others and consistent when value $\theta \leq 0,5$ and inconsistent for other.

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