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The Formula to Count The Number of Vertices Labeled Order Six Connected Graphs with Maximum Thirty Edges without Loops

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Abstract. If for every pair of vertices in a graph G(V,E) there exist minimum one path joining them, then G is called connected, otherwise the graph is called disconnected. If n vertices and m edges are given then numerous graphs are able to be created. The graphs created might be disconnected or connected, and also maybe simple or not. A simple graph is a graph whose no paralled edges nor loops. A loop is an edges that connects the same vertex while paralled edges are edges that connecting the same pair of vertices. In this research we will discuss the formula to count the number of connected vertex labeled order six graph containing at most thirty edges and may contain fifteen parallel edges without loops.

Keywords: connected graph, vertices labeled, order six, loopless, parallel edges

1. Introduction

Nowadays, graph theory emerges as one of the active branches in mathematics. That condition cannot be separated from the role of graph theory in daily-life. The flexibility representation of a graph can be adopted to represent many real-life problems fuels graph theory to lead this branch. By using graph theory, we can visualize real-problem easily. A point $v_i \in V$ in graph can represent an object such as a city, an airport, a depot, a train station, a computer, and so on , while an edge $e_{ij} \in E$ of a graph which connects a pair v_i, v_j of vertices in V can represent a road, a train track, and so on. There fore, given a graph G(V,E) where $V \neq \emptyset$, $V=\{v_1, v_2, \dots v_n\}$, and $E = \{e_{ij} | \; v_i, v_j \in V\}$, a graph V can, for example, represent computer networks where the points in V represent the computers, and the edges in E represent the cable or transmission line in the system. The flexibility of the graph occurs because of the way of drawing the graph itself. No one can claim that his way of drawing a graph is the only single correct way [1]. The application of graph theory appear in many fields cuch as in biology, chemistry, agriculture, medicine, engineering, and so on. In biology, for example, the used of graph theory in biological network is explored [2] , in representing DNA and phylogeny [3-4], in agriculture [5], in psychology [6], etc. The rest of this article will contain literature review in Section 2, graphs observation and construction is given in Section 3, results and discussion will be provided in Section 4, and the conclusion is given in Section 5.

2. Literature Review

The history of graph enumeration was laid back in 1874 when Cayley found relationship with graph theory when he calculated the number of hydrocarbons C_nH_{2n+2} isomers. Cayley found that counting the number of isomers is related to rooted tree enumeration [7]. [7]. In 1964, Slomenski used the graph concepts to observed the hydrocarbon's additive structural properties [8]. Given n number of vertices and m number of edges, numerous graphs can be created. A simple graph is a graph whose no loops nor parallel edges. In 2007 Bóna gave the method to enumerate forests and trees [9], and Stanley provides the use of generating function for enumeration [10-11]. How to count the number of disconnected vertices labeled of order maximum of four graphs is observed in [12]. The number of disconnected vertices labeled order five graphs containing no parallel edges also observed in [13], and in 2019 the formula to count the number of connected vertices labeled order five graphs with maximum five parallel edges is proposed in [14].

3. Graphs Observation and Construction

Given n = 6 and 5 ≤ m ≤ 30, numerous graphs can be created. Among those graphs that can be formed, we restrict and only take into account connected graphs with 6 ≤ t ≤ 15, where t is the number of edges that connect different pair of vertices. Moreover, after doing that sorting, we count isomorphic graphs as one graph. The following figures show some patterns of the graphs that can be formed, and due to limited space, we only display some patterns here.

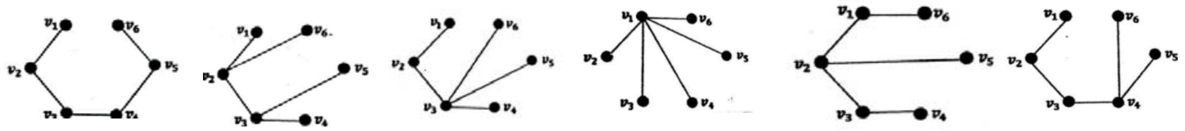

Figure 1. Some patterns obtained for m =5 and t = 5

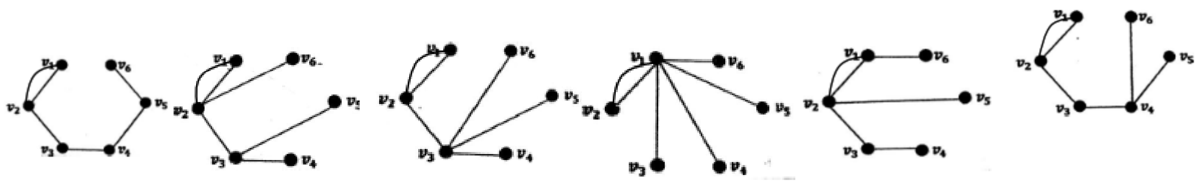

Figure 2. Some patterns obtained for m =6, t=5

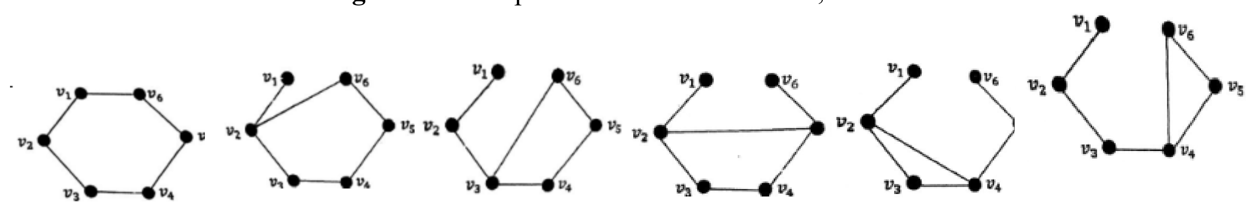

Figure 3. Some patterns obtained for m =6, t=6

4. Results and Discusion

From the process observation and construction, the graphs obtained are grouped into m and t, where t is the number of edges connecting different pairs of vertices in graph. Note that parallel edges are not contributes to t, i.e. parallel edges are conted as one. By that procedure we get Table 1.

Table 1. The number of connected vertices labeled order six graphs with maximum thirty edges and may contain fifteen parallel edges, without loops.

m	The number of connected vertices labeled order six graphs with thirty edges and may contained fifteen parallel edges, without loops										
	t										
	5	6	7	8	9	10	11	12	13	14	15
5	1296										
6	6480	1980									
7	19440	11880	3330								
8	45360	41580	23310	4620							
9	90720	110880	93240	36960	6660						
10	163296	249480	279720	166320	59940	2460					
11	272160	498960	699300	554400	299700	24600	1155				
12	427680	914760	1538460	1524600	1098900	135300	12705	420			
13	641520	1568160	3076920	3659040	3296700	541200	76230	5040	150		
14	926640	2548260	5714280	7927920	8571420	1758900	330330	32760	1950	15	
15	1297296	3963960	9999990	15855840	19999980	4924920	1156155	152880	13650	210	1
16	1769040	5945940	16666650	29729700	42857100	12312300	3468465	573300	68250	1575	15
17	2358720	8648640	26666640	52852800	85714200	28142400	9249240	1834560	273000	8400	120
18	3084480	12252240	41212080	89849760	161904600	59802600	22462440	5197920	928200	35700	680
19	3965760	16964640	61818120	147026880	291428280	119605200	50540490	13366080	2784600	128520	3060
20	5023296	23023440	90349560	232792560	503376120	227249880	106696590	31744440	7558200	406980	11628
21	6279120	30697920	129070800	358142400	838960200	413181600	213393180	70543200	18895500	1162800	38760
22		40291020	180699120	537213600	1355243400	723067800	407386980	148140720	44089500	3052350	116280
23			248461290	787913280	2129668200	1223653200	746876130	296281440	96996900	7461300	319770
24				1132625340	3265491240	2010287400	1321396230	567872760	202811700	17160990	817190
25					4898236860	3216459840	2265250680	1048380480	405623400	37442160	1961256
26						5025718500	3775417800	1872108000	780045000	78004500	4457400
27							6135053925	3244987200	1448655000	156009000	9657700
28								5475915900	2607579000	300874500	20058300
29									4563263250	561632400	40116600
30										1017958725	77558760

By analyzing the numbers in every column, we can derive Table 2 as follows:

Table 2. An alternative form of Table 1

m	The number of connected vertices labeled order six graphs with thirty edges and may contained fifteen parallel edges, without loops										
	t										
	5	6	7	8	9	10	11	12	13	14	15
5	1 x 1296										
6	5 x 1296	1 x 1980									
7	15 x 1296	6 x 1980	1 x 3330								
8	35 x 1296	21 x 1980	7 x 3330	1 x 4620							
9	70 x 1296	56 x 1980	28 x 3330	8 x 4620	1 x 6660						
10	126 x 1296	126 x 1980	84 x 3330	36 x 4620	9 x 6660	1 x 2460					
11	210 x 1296	252 x 1980	210 x 3330	120 x 4620	45 x 6660	10 x 2460	1 x 1155				
12	330 x 1296	462 x 1980	462 x 3330	330 x 4620	165 x 6660	55 x 2460	11 x 1155	1 x 420			
13	495 x 1296	792 x 1980	924 x 3330	792 x 4620	495 x 6660	220 x 2460	66 x 1155	12 x 420	1 x 150		

14	715 x 1296	1287 x 1980	1716 x 3330	1716 x 4620	1287 x 6660	715 x 2460	286 x 1155	78 x 420	13 x 150	1 x 15	
15	1001 x 1296	2002 x 1980	3003 x 3330	3432 x 4620	3003 x 6660	2002 x 2460	1001 x 1155	364 x 420	91 x 150	14 x 15	1 x 1
16	1365 x 1296	3003 x 1980	5005 x 3330	6435 x 4620	6435 x 6660	5005 x 2460	3003 x 1155	1365 x 420	455 x 150	105 x 15	15 x 1
17	1820 x 1296	4368 x 1980	8008 x 3330	11440 x 4620	12870 x 6660	11440 x 2460	8008 x 1155	4368 x 420	1820 x 150	560 x 15	120 x 1
18	2380 x 1296	6188 x 1980	12376 x 3330	19448 x 4620	24310 x 6660	24310 x 2460	19448 x 1155	12376 x 420	6188 x 150	2380 x 15	680 x 1
19	3060 x 1296	8568 x 1980	18564 x 3330	31824 x 4620	43758 x 6660	48620 x 2460	43758 x 1155	31824 x 420	18564 x 150	8568 x 15	3060 x 1
20	3876 x 1296	11628 x 1980	27132 x 3330	50388 x 4620	75582 x 6660	92378 x 2460	92378 x 1155	75582 x 420	50388 x 150	27132 x 15	11628 x 1
21	4845 x 1296	15504 x 1980	38760 x 3330	77520 x 4620	125970 x 6660	167960 x 2460	184756 x 1155	167960 x 420	125970 x 150	77520 x 15	38760 x 1
22	5985 x 1296	20349 x 1980	54264 x 3330	116280 x 4620	203490 x 6660	293930 x 2460	352716 x 1155	352716 x 420	293930 x 150	203490 x 15	116280 x 1
23		26334 x 1980	74613 x 3330	170544 x 4620	319770 x 6660	497420 x 2460	646646 x 1155	705432 x 420	646646 x 150	497420 x 15	319770 x 1
24			100947 x 3330	245157 x 4620	490314 x 6660	817190 x 2460	1144066 x 1155	1352078 x 420	1352078 x 150	1144066 x 15	817190 x 1
25				346104 x 4620	735471 x 6660	1307504 x 2460	1961256 x 1155	2496144 x 420	2704156 x 150	2496144 x 15	1961256 x 1
26					1081575 x 6660	2042975 x 2460	3268760 x 1155	4457400 x 420	5200300 x 150	5200300 x 15	4457400 x 1
27							5311735 x 1155	7726160 x 420	9657700 x 150	10400600 x 15	9657700 x 1
28								13037895 x 420	17383860 x 150	20058300 x 15	20058300 x 1
29									30421755 x 150	37442160 x 15	40116600 x 1
30										67863915 x 15	77558760 x 1

By observing Table 2, we find that every column can be written as a product of a sequence of number that multiplied with a constant. The sequence of numbers for example in column 1 is : 1, 5, 15, 35, 70, 126, 210, 330, 495, 715, 1001, 1365, 1820, 2380, 3060, 3876, 4845, 5985 and those numbers are multiple by 1296 to get the same value that displayed in Table 1.

Notate that $G(p)_{n,m,t}$ as a connected loopless graph of order n with m edges which may contained parallel edges and t number of edges that connect different pair of vertices, $N(G(p)_{n,m,t})$ as the number of $G(p)_{n,m,t}$. $N(G(p)_{n,m,t}) = |G(p)_{n,m,t}|$.

Result 1 : Given n vertices (n=6), m, t edges, $5 \leq m \leq 30$, t = 5, t is the number of edges that connect different pair of points, then the number of connected vertices labeled loopless order six graphs with maximum fifteen parallel edges is $N(G(p)_{6,m,5}) = 1296 \times C_4^{(m-1)}$

Proof :

The sequence that appears in the first column is: 1, 5, 15, 35, 70, 126, 210, 330, 495, 715, 1001, 1365, 1820, 2380, 3060, 3876, 4845, 5985

1 5 15 35 70 126 … 1820 2380 3060 3876 4845 5985
4 10 20 35 56 … 560 680 816 969 1140
6 10 15 21 … 120 136 153 171
4 5 6 … 16 17 18
1 1 … 1 1

Notice that the fixed difference appears on the fourth level. The related polynomial that can represent that sequence is order four polynomial: $P_4(m) = A_4m^4 + A_3m^3 + A_2m^2 + A_1m + A_0$

Using the value obtained in the first column we get:

$$1296 = 625a_4 + 125a_3 + 25a_2 + 5a_1 + a_0 \quad (1)$$
$$6480 = 1296a_4 + 216a_3 + 36a_2 + 6a_1 + a_0 \quad (2)$$
$$19440 = 2401a_4 + 343a_3 + 49a_2 + 7a_1 + a_0 \quad (3)$$
$$45360 = 4096a_4 + 512a_3 + 64a_2 + 8a_1 + a_0 \quad (4)$$
$$90720 = 6561a_4 + 729a_3 + 81a_2 + 9a_1 + a_0 \quad (5)$$

Equations (1) – (5) constitute a system of linear equations which can be represented by matrix

$Ax = b$, as follows:

$$\begin{bmatrix} 625 & 125 & 25 & 5 & 1 \\ 1296 & 216 & 36 & 6 & 1 \\ 2401 & 343 & 49 & 7 & 1 \\ 4096 & 512 & 64 & 8 & 1 \\ 6561 & 729 & 81 & 9 & 1 \end{bmatrix} \begin{bmatrix} a_4 \\ a_3 \\ a_2 \\ a_1 \\ a_0 \end{bmatrix} = \begin{bmatrix} 1296 \\ 6480 \\ 19440 \\ 45360 \\ 90720 \end{bmatrix}$$

Solving that matrix we get: $a_4 = \frac{1296}{24},\ a_3 = -\frac{12960}{24}, a_2 = \frac{45360}{24},\ a_1 = -\frac{64800}{24}$, and $a_0 = \frac{31104}{24}$

Therefore : $P_4(\mathrm{m}) = \frac{1296}{24}m^4 - \frac{12960}{24}m^3 + \frac{45360}{24}m^2 - \frac{64800}{24}m + \frac{31104}{24}$

$$= \frac{1296}{24} \times (m^4 - 10m^3 + 35m^2 - 50m + 240)$$
$$= \frac{1296}{24} \times ((m-1)(m-2)(m-3)(m-4))$$
$$= 1296 \times \frac{((m-1)(m-2)(m-3)(m-4))}{4\times3\times2\times1}$$
$$= 1296 \times C_4^{(m-1)}$$

Result 2 : Given n vertices (n = 6), and m, t, edges; 6 ≤ m ≤ 30, t = 6, t is the number of edges connecting different pair of points/vertices (parallel edges are counted as one), then the number of connected verticed labeled loopless graphs of order six with maximum fifteen parallel edges is
$\mathrm{N}(\mathrm{G}(p)_{6,\mathrm{m},6}) = 1980 \times C_5^{(m-1)}$

Proof :
The sequence that appears in the first column is: 1, 6, 21, 56, 126, 252, 462, 792, 1287, 2002, 3003, 4368, 6188, 8568, 11628, 15504, 20349, 26334.

1 6 21 56 126 252 462 … 4368 6188 8568 11628 15504 20349 26334
5 15 35 70 126 210 … 1820 2380 3060 3876 4845 5985
10 20 35 56 84 … 560 680 816 969 1140
10 15 21 28 … 120 136 153 171
5 6 7 … 16 17 18
1 1 … 1 1

Notice that the fixed difference appears on the fifth level. The related polynomial that can represent that sequence is order five polynomial: $Q_5(\mathrm{m}) = \mathrm{A}_5\mathrm{m}^5 + \mathrm{A}_4\mathrm{m}^4 + \mathrm{A}_3\mathrm{m}^3 + \mathrm{A}_2\mathrm{m}^2 + \mathrm{A}_1\mathrm{m} + \mathrm{A}_0$
Using the value obtained in the first column we get:
$1980 = 7776a_5 + 1296a_4 + 216a_3 + 36a_2 + 6a_1 + a_0$ (6)
$11880 = 16807a_5 + 2401a_4 + 343a_3 + 49a_2 + 7a_1 + a_0$ (7)
$41580 = 32768a_5 + 4096a_4 + 512a_3 + 64a_2 + 8a_1 + a_0$ (8)
$110880 = 59049a_5 + 6561a_4 + 729a_3 + 81a_2 + 9a_1 + a_0$ (9)
$249480 = 100000a_5 + 10000a_4 + 1000a_3 + 100a_2 + 10a_1 + a_0$ (10)
$498960 = 161051a_5 + 14641a_4 + 1331a_3 + 121a_2 + 11a_1 + a_0$ (11)

By solving that system of equations we get:
$\mathrm{a}_5 = \frac{1980}{120}, \mathrm{a}_4 - \frac{29700}{120}, \mathrm{a}_3 = \frac{168300}{120}, \mathrm{a}_2 = -\frac{445500}{120}, \mathrm{a}_1 = \frac{542520}{120}, \mathrm{a}_0 = -\frac{232600}{120}$

Therefore $Q_5(\mathrm{m}) = \frac{1980}{120}m^5 - \frac{29700}{120}m^4 + \frac{168300}{120}m^3 - \frac{445500}{120}m^2 + \frac{542520}{120} - \frac{232600}{120}$

$= \frac{1980}{120}(m^5 - 15m^4 + 85m^3 - 225m^2 + 274m - 120)$

$= \frac{1980}{120}((m-1)(m-2)(m-3)(m-4)(m-5))$

$= 1980 \times \frac{((m-1)(m-2)(m-3)(m-4)(m-5))}{5\times4\times3\times2\times1}$

$= 1980 \times C_5^{(m-1)}$

Observing and continuing by this manner we get:

Result 3: Given n vertices (n = 6), and m, t edges; 7 ≤ m ≤ 30, t = 7 then N(G(p)6,m,7) = $3330 \times C_6^{(m-1)}$
Result 4: Given n vertices (n = 6), and m, t edges ; 8 ≤ m ≤ 30 , t = 8, then N(G(p)6,m,8) = $4620 \times C_7^{(m-1)}$
Result 5 :Given n vertices (n = 6), and m, t edges ;9 ≤ m ≤ 30 , t = 9, then N(G(p)6,m,9) = $6660 \times C_8^{(m-1)}$
Result 6: Given n vertices (n = 6), and m, t edges; 10 ≤ m ≤ 30, t =10, then N(G(p)6,m,10) = $2460 \times C_9^{(m-1)}$
Result 7:Given n vertices (n = 6), and m, t edges,11 ≤ m ≤ 30, t=11, then N(G(p)6,m,11) = $1155 \times C_{10}^{(m-1)}$
Result 8: Given n vertices (n = 6), and m, t edges, 12 ≤ m ≤ 30, t = 12, then N(G(p)6,m,12) = $420 \times C_{11}^{(m-1)}$
Result 9 : Given n vertices (n = 6), and m, t edges, 13 ≤ m ≤ 30, t = 13, then N(G(p)6,m,13) = $150 \times C_{12}^{(m-1)}$
Result 10: Given n vertices (n = 6), and m, t edges, 14 ≤ m ≤ 30, t = 14, then (G(p)6,m,14)=$15 \times C_{13}^{(m-1)}$
Result 11: Given n vertices (n = 6), and m, t edges, 15 ≤ m ≤ 30, t = 15, then N(G(p)6,m,15) =$C_{14}^{(m-1)}$

5. Conclusion

From the above discussion we are able to conclude that if given n vertices (n = 6), and m, t, edges; 5 ≤ m ≤ 30, t≤ m, 5 ≤t ≤15, t is the number of edges connecting different pair vertices (parallel edges are counted as one), then the number of connected vertices labeled loop-lees order six graphs with maximum thirty edges and may contain at most fifteen parallel edges N(G(p)6,m,,t) is:

N(G(p)6,m,,t) = $\sum_{t=5}^{15}$ N(G$(p)_{6,\mathrm{m,t}}$, where N(G(p)6,m,5) = 1296× $C_4^{(m-1)}$, N(G(p)6,m,6) = 1980 × $C_5^{(m-1)}$, N(G(p)6,m,7) =3330 × $C_6^{(m-1)}$, N(G(p)6,m,8) = 4620 × $C_7^{(m-1)}$, N(G(p)6,m,9) = 6660 × $C_8^{(m-1)}$, N(G(p)6,m,10) = 2460 × $C_9^{(m-1)}$, N(G(p)6,m,11) =1155 $C_{10}^{(m-1)}$, N(G(p)6,m,12) = 420 × $C_{11}^{(m-1)}$, N(G(p)6,m,13) = 150 × $C_{12}^{(m-1)}$, N(G(p)6,m,14) = 15 × $C_{13}^{(m-1)}$, N(G(p)6,m,15) = $C_{14}^{(m-1)}$

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