Journal of Physics: Conference Series

PAPER • OPEN ACCESS

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To cite this article: F A Pertiwi *et al* 2021 *J. Phys.: Conf. Ser.* **1751** 012026

Calculating the Number of vertices Labeled Order Six Disconnected Graphs which Contain Maximum Seven Loops and Even Number of Non-loop Edges Without Parallel Edges

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Abstract. A labeled graph is a graph where each vertex or edge is given a value or label. A line/edge with the same starting and ending vertex is called a loop. If given n vertices and m edges, then there are lot disconnected vertices labeled graphs that can be formed. Among those graphs, there are graphs with maximum seven loops and whose non-loop edges are even. In this research, we will discuss the formula for finding that kinds of graphs.

Keywords: disconnected graphs, vertices labeled, order six, loops, parallel edges.

1. Introduction

The concept of graph theory was first introduced in 1736 by a Swiss mathematician named Leonhard Euler on solving the problem of the Konigsberg bridge, Kaliningrad, Russia. Graph theory is a branch of mathematics that presents discrete objects and the relationships between them, with vertices/points representing objects and edges/lines representing relationships between objects. The application of graph theory is widely used in the fields of biology, chemistry, agriculture, medicine, engineering, and so on. Those application include: the used of graph theory in biological network [1] , in representing DNA and phylogeny [2-5], in cryptography [6-7], in agriculture [8], etc, and graph enumerative had led by Cayley in 1874 while he found there was a relationship between isomer structure of hydrocarbon C_nH_{2n+2} and rooted tree [9]. Some theoretical background related with graph enumeration can be found in [10-12].

In this research, we will discuss the formula to count the number of disconnected vertices labeled graphs of order six containing a maximum loop of seven without parallel lines. Literature review is given in the next section where we provide a brief information about some investigations done relating with graph enumeration. In Section 3 we provide graphs constructions and pattern obtained, In Section 4 we give results and discussion, follows by conclusion.

2. Literature Review

Graph G (V, E) is said to be connected if for every two different vertices or points in G, there is a path connecting these points. Else, the graph is not connected. Parallel lines or parallel edges are two or more lines connecting a pair of points in a graph and a loop is a line that has the same starting and end point. A simple graph is graph that containing no parallel lines nor loops. A labeled graph is a graph where each point or line has a label. If only the points are labeled then the labeling is called point labeling, whereas if only lines are labeled it is called line labeling and if the points and lines are labeled it is called total labeling.

If given graph G(V,E) with n number of vertices and m edges, there are a lot of graphs that can be formed. Those graphs obtained are possible connected or disconnected graphs. The formula for counting the number of disconnected verteices labelled graphs of order maximum our is investigated [13], in 2016 the number of disonnected vertex labelled graphs of order five without parallel edges is investigated in [14], and how to count the number of connected vertices labelled graph of order five with maximum 5 parallel edges is investigated in [15].

3. Graphs construction and the patterns obtained

Assume m as the number of lines, g as the number of lines that are no loops, ℓ_i as the number of *i* loops and j_i as the number of vertices which are incident to *i* loop, $i = 0, 1, 2, \ldots n$, thus $m = g + \sum_{i=0}^{m-g} j_i \cdot l_i$

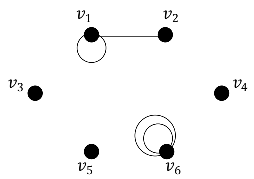

Figure 1. An example of disconnected vertices labeled graph of order six

In figure 1 we can see an example of disconnected vertices labeled graph of order six which contains three loops (one loop incidence to vertex v_1 and two loops are incidence to vertex v_6). In that example g =1, $\ell_1 = 1$ and $\ell_2 = 1$. From this example, $m = g + j_2 \cdot \ell_2 + j_1 \cdot \ell_1 = 1 + (1 \times 2) + (1 \times 1) = 4$

To determine the number of disconnected vertices labeled graphs of order six without parallel edges, we did the following: first, we construct all possible pattern for given m, m≥1; second,we calculate all possible number of graphs that can be obtained using the patterns. Note that during this process we group isomorphism graphs and counted them as 1; next we grouping them by m and g and then observed for possible pattern of numbers obtained. Then, we try to predict the formula of the sequence, and finally prove the formula obtained. We give some patterns and the number of the graphs obtained in the following:

(i) For $n = 6,\ 1 \le m \le 10,\ 1 \le g \le 10,$ and $\ell_i = 0$ with $i = 1,2,3, \ldots ,7$ (no loops).

These patterns obtained when we constructed graphs whose no loops (only g exist). Not all patterns obtained are given here due to limitation of space. We give the three patterns below which are obtained under the given condition above.

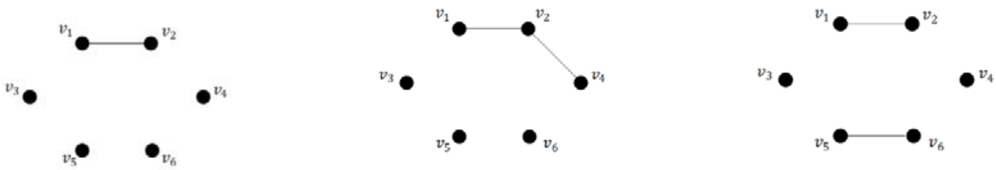

Figure 2. Some patterns obtained for $= 6,\ 1 \le m \le 10,\ 1 \le g \le 10$, and $\ell_i = 0$ with $i = 1,2,3,\dots,7$

For these patterns, note that maximum of g is 10 because the graphs constructed must be disconnected graphs.

Table 1. The number of disconnected vertices labeled graphs where $n = 6, 1 \le m \le 5,\ 1 \le g \le 10$, and $\ell_i = 0$ with $i = 1,2,3,\dots,7$.

m	1	2	3	4	5	6	7	8	9	10
Number of graphs obtained	15	105	365	975	957	715	345	210	60	6

(ii) For $n = 6$, $1 \le m \le 7, g = 0$, and $1 \le \ell_i \le 7$ with $i = 1,2,3,\dots,7$ (no line exists, except loops)

There patterns obtained when we constructed disconnected graphs without line except loops. From the observed patterns we only give three patterns to put here due to same reason as above.

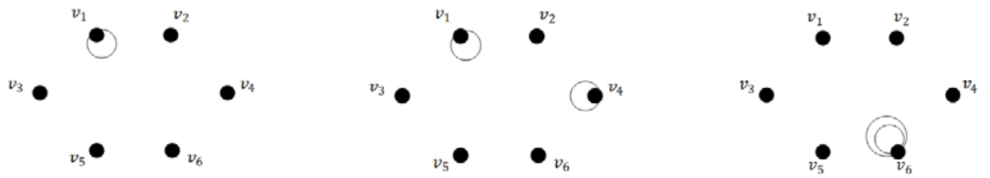

Figure 3. Some patterns obtained for $n = 6$, $g = 0$, dan $1 \le \ell_i \le 7$ with $i = 1,2,3,\dots,7$

The following table show the number of disconnected graphs obtained under the given condition above.

Table 2. The number of disconnected vertices labeled graphs where $n = 6, 1 \le m \le 7$, $g = 0$, and $1 \le \ell_i \le 7$ with $i = 1,2,3,\dots,7$

m	1	2	3	4	5	6	7
Number of graphs obtained	6	21	56	126	252	462	792

(iii) For $n = 6,\ m \ge 3,\ g = 2,4,6,8,10$ and $1 \le \ell_i \le 7$ with $i = 1,2,3,\dots,7$

Note that the patterns obtained under the above condition are the patterns when the number of nonloop lines are even (g even, and g>0). Also, notice that the highest value of g is 10 because of the disconnectivity property. The following are three patterns that obtained. Again, not all patterns are displays because of limitation of space.

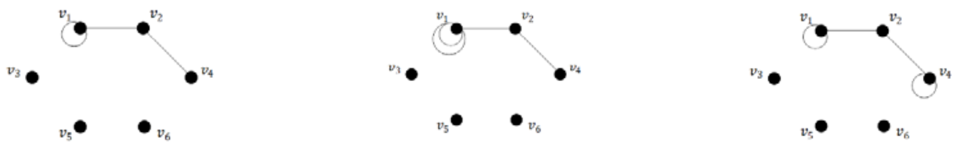

Figure 4. Some patterns obtained for $n = 6$, $g > 0, g$ even, dan $1 \le \ell_i \le 7$ with $i = 1,2,3,\dots,7$

The following table show the graphs obtained under the given condition above. We want to mention here that we are observed the pattern for $3 \le m \le 17$ and not continue further because when the observation reach m = 17 we find the pattern for the grouping graphs.

Table 3. The number of disconnected vertices labeled graphs where $n = 6, 1 \leq m \leq 7, g = 0,$ even, and $1 \leq \ell_i \leq 7$ with $i = 1,2,3, \ldots ,7$

m	g				
	2	4	6	8	10
3	630				
4	2205				
5	5880	5850			
6	13230	20475			
7	26460	54600	4290		
8	48510	122850	15015		
9	83160	245700	40040	1260	
10		450450	90090	4410	
11		772200	180180	11760	36
12			330330	26460	126
13			566280	52920	336
14				97020	756
15				166320	1512
16					2772
17					4752

4. Results and Discusion

Based on the results obtained from Table 1 , 2 and 3, we get the following table:

Table 4. The number of disconnected vertices labeled graph of order six that containing even nonloops edges and loops maximum seven, without parallel edges.

m	g					
	0	2	4	6	8	10
1	6					
2	21	1 x 105				
3	56	6× 105				
4	126	21× 105	1 x 975			
5	252	56× 105	6× 975			
6	462	126× 105	21× 975	1 x 715		
7	792	252× 105	56× 975	6× 715		
8		462× 105	126× 975	21× 715	1 x210	
9		792× 105	252× 975	56× 715	6× 210	
10			462× 975	126× 715	21× 210	1 x 6
11			792× 975	252× 715	56× 210	6× 6
12				462× 715	126× 210	21× 6
13				792× 715	252× 210	56× 6
14					462×210	126× 6
15					792×210	252× 6
16						462× 6
17						792× 6

Gathering information from Table 4, we can derive another Table as follows:

Table 5. Sequence of numbers appear in every column

m	g					
	0	2	4	6	8	10
1	6					
2	21	1				
3	56	6				
4	126	21	1			
5	252	56	6			
6	462	126	21	1		
7	792	252	56	6		
8		462	126	21	1	
9		792	252	56	6	
10			462	126	21	1
11			792	252	56	6
12				462	126	21
13				792	252	56
14					462	126
15					792	252
16						462
17						792

From the above table, we can see that there are pattern of numbers on every column, and those numbers constitute a sequence. Notate $G^d(\ell)_{n,m,g}$ as disconnected vertices labeled graphs of order n with m edges and may contain loops without parallel edges, N($G^d(\ell)_{n,m,g}$) = the number of $G^d(\ell)_{n,m,g}$ = $|G^d(\ell)_{n,m,g}|$

Result 1. Given n = 6, 1 ≤ m ≤ 7 and g = 2, then $N(G^d(\ell)_{6,m,g=0}) = C_5^{(m+5)}$

Proof:

From Table 5 we get the sequence: 6, 21, 56, 126, 252, 462, 792

6 21 56 126 252 462 792

15 35 70 126 210 330

20 35 56 84 120

15 21 28 36

6 7 8

1 1

Since the fixed difference occurs on the fifth level, then the fifth order polynomial can represent this sequence.

$$P_5(m) = a_5m^5 + a_4m^4 + a_3m^3 + a_2m^2 + a_1m + a_0 \quad (1)$$

By substituting m =1,2,…,6 to (1) we get the following:

$$6 = a_5+a_4 + a_3 + a_2 + a_1 + a_0 \quad (2)$$
$$21 = 32a_5+16a_4 + 8a_3 + 4a_2 + 2a_1 + a_0 \quad (3)$$
$$56 = 243a_5+81a_4 + 27a_3 + 9a_2 + 3a_1 + a_0 \quad (4)$$
$$126 = 1024a_5+256a_4 + 64a_3 + 16a_2 + 4a_1 + a_0 \quad (5)$$
$$252 = 3125a_5+625a_4 + 125a_3 + 25a_2 + 5a_1 + a_0 \quad (6)$$
$$462 = 7776a_5+1296a_4 + 216a_3 + 36a_2 + 6a_1 + a_0 \quad (7)$$

Solving the system of linear equation (2) - (7) we get these values:
$a_5 = \frac{-288}{-34560}$, $a_4 = \frac{-4320}{-34560}$, $a_3 = \frac{-24480}{-34560}$, $a_2 = \frac{-64800}{-34560}$, $a_1 = \frac{-78912}{-34560}$, and $a_0 = \frac{-34560}{-34560}$

Therefore $P_5(m) = \frac{288}{34560}m^5 + \frac{4320}{34560}m^4 + \frac{24480}{34560}m^3 + \frac{64800}{34560}m^2 + \frac{78912}{34560}m + \frac{34560}{34560}$

$$= \frac{1}{120}(m^5 + 15m^4 + 85m^3 + 225m^2 + 274m + 120)$$
$$= \frac{1}{120}(m+5)(m+4)(m+3)(m+2)(m+1)$$
$$= \frac{(m+5)(m+4)(m+3)(m+2)(m+1)}{5\times4\times3\times2\times1}$$
$$= C_5^{(m+5)}$$

Result 1. Given n = 6, 3 ≤ m ≤ 9 and g = 2, then $N(G^d(\ell)_{6,m,g=2}) = C_5^{(m+3)}$
Proof:
Note that every column in Table 5 consitutes similar sequence of numbers, therefore that sequence also relates with polynomial of order five. However, in Table 4 that sequence of numbers is multiply by a constant with different value for every column. For columns where g = 2, the constant is 105. Therefore, by using the information in Table 3 and 4 we get the following:

$$630 = 243a_5 + 81a_4 + 27a_3 + 9a_2 + 3a_1 + a_0 \quad (8)$$
$$2205 = 1024a_5 + 256a_4 + 64a_3 + 16a_2 + 4a_1 + a_0 \quad (9)$$
$$5880 = 3125a_5 + 625a_4 + 125a_3 + 25a_2 + 5a_1 + a_0 \quad (10)$$
$$13230 = 7776a_5 + 1296a_4 + 216a_3 + 36a_2 + 6a_1 + a_0 \quad (11)$$
$$26460 = 16807a_5 + 2401a_4 + 343a_3 + 49a_2 + 7a_1 + a_0 \quad (12)$$
$$48510 = 32768a_5 + 4096a_4 + 512a_3 + 64a_2 + 8a_1 + a_0 \quad (13)$$

Solving the system of linear equation (8) - (13) we get these values:
$a_5 = \frac{-30240}{-34560}$, $a_4 = \frac{-151200}{-34560}$, $a_3 = \frac{-151200}{-34560}$, $a_2 = \frac{151200}{-34560}$, $a_1 = \frac{181440}{-34560}$, and $a_0 = \frac{0{,}0000008}{-34560}$
Therefore :

$$P_5(m) = \frac{30240}{34560}m^5 + \frac{151200}{34560}m^4 + \frac{151200}{34560}m^3 - \frac{151200}{34560}m^2 - \frac{181440}{34560}m - \frac{0{,}0000008}{34560}$$
$$= \frac{105}{120}(m^5 + 5m^4 + 5m^3 - 5m^2 - 6m)$$
$$= \frac{105}{120}(m+3)(m+2)(m+1)(m)(m-1)$$
$$= 105 \times \frac{(m+3)(m+2)(m+1)(m)(m-1)}{5\times4\times3\times2\times1}$$
$$= 105 \times C_5^{(m+3)}$$

By similar manner, we also get the following results:

Result 3 : Given n = 6, 5 ≤ m ≤ 11 and g = 4, then $N(G^d(\ell)_{6,m,g=4}) = 975\, C_5^{(m+1)}$
Result 4 : Given n = 6, 7 ≤ m ≤ 13 and g = 6, then $N(G^d(\ell)_{6,m,g=6}) = 715\, C_5^{(m-1)}$
Result 5 : Given n = 6, 9 ≤ m ≤ 15 and g = 8, then $N(G^d(\ell)_{6,m,g=8}) = 210\, C_5^{(m-3)}$
Result 6 : Given n = 6, 11 ≤ m ≤ 17 and g = 6, then $N(G^d(\ell)_{6,m,g=10}) = 6\, C_5^{(m-5)}$

5. Conclusion

From the results above, we conclude that given *G* (V,E), |V| = 6, |E| = m , 1 ≤ m ≤ 17 , and g, g even, g is the number of nonloop edges, then the number of connected vertices labelled graphs of order six with maximum seven loops containing no loops is:

$$N(G^d(\ell)_{6,m,g\ \text{even}}) = \sum_{g=0,2,4,6,8,10} N(G^d(\ell)_{6,m,g})$$
$$= N(G^d(\ell)_{6,m,g=0}) + N(G^d(\ell)_{6,m,g=2}) + N(G^d(\ell)_{6,m,g=4}) + N(G^d(\ell)_{6,m,g=6}) + N(G^d(\ell)_{6,m,g=8}) + N(G^d(\ell)_{6,m,g=10})$$

Acknowledgement
The authors wish to thank Universitas Lampung for the support and funding of this research in accordance with the Research Contract No: 1499/UN26.21/PN/2020.

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