ISSN 1439-8516
CN 11–2039/01

Volume 36 · Number 11 · November 2020

ACTA MATHEMATICA SINICA

Chinese Mathematical Society
Institute of Mathematics
Chinese Academy of Sciences

数学学报，英文版

English Series

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Zentralblatt MATH and
Mathematical Reviews
Acta Math Sin,
Engl Series
36(11) 1203–1314 (2020)
Number 11 · 2020
Printed on acid-free paper

Acta Mathematica Sinica, English Series
数学学报（英文版）
ISSN:1439-8516
CN:11-2039/01

Acta Mathematica Sinica, English Series

2020 Vol. 36 No. 11
Published 15 November 2020

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Acta Mathematica Sinica, English Series
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ISSN:1439-8516
CN:11-2039/O1

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Acta Mathematica Sinica, English Series
Nov., 2020, Vol. 36, No. 11, pp. 1232–1244
Published online: December 15, 2020
https://doi.org/10.1007/s10114-020-8377-6
http://www.ActaMath.com

Acta Mathematica Sinica, English Series

A Complete Characterization of Multivariate Normal Stable Tweedie Models through a Monge–Ampère Property

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Abstract Extending normal gamma and normal inverse Gaussian models, multivariate normal stable Tweedie (NST) models are composed by a fixed univariate stable Tweedie variable having a positive value domain, and the remaining random variables given the fixed one are real independent Gaussian variables with the same variance equal to the fixed component. Within the framework of multivariate exponential families, the NST models are recently classified by their covariance matrices $\mathbf{V}(\mathbf{m})$ depending on the mean vector $\mathbf{m}$. In this paper, we prove the characterization of all the NST models through their determinants of $\mathbf{V}(\mathbf{m})$, also called generalized variance functions, which are power of only one component of $\mathbf{m}$. This result is established under the NST assumptions of Monge–Ampère property and steepness. It completes the two special cases of NST, namely normal Poisson and normal gamma models. As a matter of fact, it provides explicit solutions of particular Monge–Ampère equations in differential geometry.

Keywords Covariance matrix, generalized variance function, Monge–Ampère equation, multivariate exponential family, steepness

MR(2010) Subject Classification 62H05, 62E10, 60E07, 53A99, 62H99

1 Introduction and Motivations

The classical form of the Monge–Ampère equations can be written as

$$\det \mathbf{K}'' = f \quad \text{in } \Theta,$$

where $\Theta \subseteq \mathbb{R}^k$ is some open set, $\mathbf{K} : \Theta \to \mathbb{R}$ is an unknown smooth function, $\mathbf{K}'' = (D^2_{ij}\mathbf{K})_{i,j=1,\ldots,k}$ denotes the Hessian matrix of $\mathbf{K}$ with D the partial differential operator, and

Received September 2, 2018, revised March 16, 2020, accepted June 5, 2020
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f is a given positive function (see, e.g., [9]). It arises in several mathematical problems from analysis and geometry. However, explicit solutions remain generally challenging problems. In particular for $f = 1$, the proofs of the basic Monge–Ampère equation such as "*any strictly convex smooth function* $\mathbf{K}$ *in* $\mathbb{R}^k$ *such that* $\det \mathbf{K}'' = 1$ *must be a quadratic form*" were progressive, different, and according to the dimension: Jörgens [13] for dimension $k = 2$ (see also [16] for an easy proof), Calabi [5] for $k = 3, 4, 5$, and Pogorelov [28] for $k \geq 6$. See Cheng and Yau [6] for a more analytic proof, along the lines of affine geometry, and, for example, Xiong and Bao [30] for another use.

Here, we are interested in the following form of the Monge–Ampère equations:

$$\det \mathbf{K}'' = \begin{cases} a(\mathbf{K}_{\boldsymbol{\nu}_p})^b, & \text{for } 1 \leq p < 2, \\ \exp\{(k+1)\mathbf{K}_{\boldsymbol{\nu}_p}\}, & \text{for } p = 2, \\ a(-\mathbf{K}_{\boldsymbol{\nu}_p})^{-b}, & \text{for } p > 2, \end{cases} \tag{1.1}$$

where $\mathbf{K}$ is an unknown *cumulant function* or *log-Laplace transform*, to be determined, $p \geq 1$ is a given constant such that there exist $a = a(p)$ and $b = b(p, k) > 0$, and $\mathbf{K}_{\boldsymbol{\nu}_p}$ is a given cumulant function on the same domain $\Theta_{\boldsymbol{\nu}_p} = \Theta$ for any σ-finite positive (or probability) measure $\boldsymbol{\nu}_p$ on $\mathbb{R}^k$. Let us recall that in the framework of the probability model $\mathbf{F} = \mathbf{F}(\boldsymbol{\mu}) = \{\mathbf{P}_{\boldsymbol{\theta},\boldsymbol{\mu}}(d\mathbf{x}) := \exp[\mathbf{x}^\top\boldsymbol{\theta} - \mathbf{K}_{\boldsymbol{\mu}}(\boldsymbol{\theta})]\boldsymbol{\mu}(d\mathbf{x}); \boldsymbol{\theta} \in \Theta_{\boldsymbol{\mu}}\}$, named *natural exponential family* (NEF) generated by $\boldsymbol{\mu} \in \mathcal{M}(\mathbb{R}^k)$ not concentrated on an affine subset of $\mathbb{R}^k$, if $\mathbf{X}$ is a random vector distributed as $\mathbf{P}_{\boldsymbol{\theta},\boldsymbol{\mu}}$ then $\mathbb{E}_{\boldsymbol{\theta}}(\mathbf{X}) = \mathbf{K}'_{\boldsymbol{\mu}}(\boldsymbol{\theta}) = (D_i\mathbf{K}_{\boldsymbol{\mu}}(\boldsymbol{\theta}))_{i=1,\ldots,k}$ and $\mathrm{Var}_{\boldsymbol{\theta}}(\mathbf{X}) = \mathbf{K}''_{\boldsymbol{\mu}}(\boldsymbol{\theta})$. Since $\mathbf{K}_{\boldsymbol{\mu}} : \Theta \to \mathbb{R}$ is known to be strictly convex, the function $\mathbf{m}(\boldsymbol{\theta}) = \mathbf{K}'_{\boldsymbol{\mu}}(\boldsymbol{\theta})$ is a one-to-one transformation from $\Theta_{\boldsymbol{\mu}}$ onto $\mathbf{M}_{\mathbf{F}} := \mathbf{K}'_{\boldsymbol{\mu}}(\Theta_{\boldsymbol{\mu}})$ and thus $\mathbf{m} = \mathbf{m}(\boldsymbol{\theta})$ provides an alternative parametrization of the family $\mathbf{F} = \{\mathbf{P}(\mathbf{m}, \mathbf{F}); \mathbf{m} \in \mathbf{M}_{\mathbf{F}}\}$, called the *mean parametrization*. Note that $\mathbf{M}_{\mathbf{F}}$ depends only on $\mathbf{F}$, and not on the choice of the generating measure $\boldsymbol{\mu}$ of $\mathbf{F}$. The covariance matrix of $\mathbf{P}(\mathbf{m}, \mathbf{F})$ can be written as a function of the mean parameter $\mathbf{m}$, $\mathbf{V}_{\mathbf{F}}(\mathbf{m}) = \mathbf{K}''_{\boldsymbol{\mu}}(\boldsymbol{\theta})$, called the variance function of $\mathbf{F}$. Also, the differential of the inverse function of $\mathbf{K}'_{\boldsymbol{\mu}}$ defined by $\mathbf{m} \mapsto \boldsymbol{\theta}(\mathbf{m}) := (\mathbf{K}'_{\boldsymbol{\mu}})^{-1}(\mathbf{m})$ is $\boldsymbol{\theta}'(\mathbf{m}) := [D_i(\mathbf{K}'_{\boldsymbol{\mu}})^{-1}(\mathbf{m})]_{i=1,\ldots,k} = [\mathbf{V}_{\mathbf{F}}(\mathbf{m})]^{-1}$. Together with the mean domain $\mathbf{M}_{\mathbf{F}}$, $\mathbf{V}_{\mathbf{F}}$ characterizes $\mathbf{F}$ within the class of all NEFs. See [24, Chap. 54]. However, the so-called *generalized variance function* $\det \mathbf{V}_{\mathbf{F}}(\mathbf{m}) = \det \mathbf{K}''_{\boldsymbol{\mu}}(\boldsymbol{\theta}(\mathbf{m}))$ does not characterize the NEF $\mathbf{F} = \mathbf{F}(\boldsymbol{\mu})$; and, it is necessary to solve individually the corresponding Monge–Ampère equation.

A normal stable Tweedie (NST) family with power parameter $p \geq 1$ is a k-variate NEF on $[0, \infty) \times \mathbb{R}^{k-1}$ generated by the distribution of

$$(X_1, Z_2\sqrt{X_1}, \ldots, Z_k\sqrt{X_1}), \tag{1.2}$$

where $Z_2, \ldots, Z_k$ are iid random variables with standard normal distribution $\mathcal{N}(0, 1)$ independent of X_1 whose distribution belongs to the Tweedie [29] NEF on the positive real line with variance function γm^p, $\gamma > 0$; see, e.g., [1] and the references cited therein. The representation (1.2) is extended to the so-called multiple stable Tweedie (MST) models, where X_1 and the Z_j's are independent and each of them follows a given Tweedie NEF with $p_1, p_j \geq 1$. See, e.g., [4, 8, 19, 26]. The main aim of this paper is the following NST characterization. Consider a steep NEF (i.e., the mean domain is equal to the interior of the closed convex hull of the distribution support) $\mathbf{F}$ on $[0, \infty) \times \mathbb{R}^{k-1}$ governed by a cumulant function or Log-Laplace transform $\mathbf{K}(\boldsymbol{\theta}) = \log \int \exp(\boldsymbol{\theta}^\top \mathbf{x})\boldsymbol{\nu}(d\mathbf{x})$ such that (1.1) holds for $\boldsymbol{\nu}_p$, $p \geq 1$, generator the NST NEF. Following [4] its variance function $\mathbf{V_F}$ also satisfies

$$\det \mathbf{V_F}(\mathbf{m}) = m_1^{p+k-1}, \tag{1.3}$$

for some $p \geq 1$ with $\mathbf{m} = (m_1, \dots, m_k)^\top$. Then $\mathbf{F}$ is an affinity of a NST family (Theorem 3.2).

It is noteworthy that this NST characterization (Theorem 3.2) appears to be a complete one for all $p \geq 1$. Indeed, it includes two cases: normal gamma with $p = 2$ ([18]) and normal Poisson with $p = 1$ ([27]) using two different approaches of proof which are analytical and measure theory, respectively. Also, this type of characterization through Property (1.1) and the generalized variance function (1.3) is connected to some particular cases of the Monge–Ampère equation ([9]) which are referred and widely discussed in the previous papers. See [23] for the Gaussian model with $\det \mathbf{V_F}(\mathbf{m}) = 1$, [17] for the Poisson–Gaussian models and, finally, [10] for the multinomial model. In fact, according to the mean parametrization of NEF, one can re-write both equations (1.1) and (1.3) as

$$\det \mathbf{K}''(\boldsymbol{\theta}(\mathbf{m})) = m_1^{p+k-1}. \tag{1.4}$$

The left member of (1.4) depends on the unknown (cumulant) function $\mathbf{K}$ and the right one is fixed for the model parameter $p \geq 1$ and the dimension $k \in \{1, 2, \dots\}$. Another complete characterization of the NST models has been done through their variance functions in [20] with some associated polynomial functions. Analog works can be reproduced for [26] and, also, in the multivariate geometric ([14]) and discrete ([15]) dispersion models, respectively.

Concerning some potentials of applied statistical aspects of NST, we first have the generalized variance property (1.3) which led to interesting estimators of $\det \mathbf{V_F}(\mathbf{m})$ through explicit but biased maximum likelihood estimators in general and, also, uniformly minimum variance and unbiased estimates. For that, we can refer to [2, 4, 21, 22]. Its special use under Gaussianity (i.e., $\det \mathbf{V_F}(\mathbf{m}) = 1$) has been made via different approaches; see, e.g., Iliopoulos [11] and Jafari [12] with some references cited therein. Note that the likelihood from observations on $(X_1, X_2, \dots, X_k)$ of the NST models shows clearly by (1.2) that X_1 is a sufficient statistic for a variability measure such as the generalized variance. The second statistical aspect of the NST models may be derived from its flexibility compared to the classical normal model, as already indicated in [4] and improved here. Indeed, consider that only the normal terms $(Z_2, \dots, Z_k)$ in (1.2) are observed, then X_1 is an unobserved random effect. The case where $(X_1, X_2, \dots, X_k)$ is normal leads to an analysis of variance with a repeated measures factor (within-subjects independent variable), for which the concept of sphericity (i.e., the equality of variances of the differences between treatment levels) is assumed; that is known as compound symmetry repeated measurements. Using the proposed NST models provides an extension that could be useful as an alternative; see, e.g., Lee and Thompson [25] for a description of this issue from a completely different angle (of generalized hierarchical models with random effects) from ours. Additionally, the proposed NST models can be extended to regression models with covariates such that one can handle a multivariate response vector; see, e.g., Bonat and Jørgensen [3]. The application potential of the NST models holds also for MST models [19]. The disturbance that the NST models introduced in the standard multivariate normal model will eventually propagate to all other components. Finally, some Bayesian approaches can be considered in the sense of Consonni et al. [7], also Kokonendji and Nisa [21].

The rest of the paper is organized in two parts according to the announced NST characterization (1.4). Since the NST property of generalized variance functions (1.3) is already given in [4, Theorem 3.3], Section 2 establishes Equations (1.1) that we shall call the Monge–Ampère property of the NST models (Theorem 2.4). We also point out the continuity property of $\det \mathbf{K}''_{\boldsymbol{\nu}_p}$ with respect to $p > 1$. Its univariate case (Proposition 2.2) appears to be new for the stable Tweedie models on the positive real line. Section 3 states and finally proves Theorem 3.2 of the NST characterization (1.4) via (1.1) and (1.3) with the assumption of steepness.

2 Monge–Ampère Property of NST Models

We first establish the Monge–Ampère property (1.1) for the univariate stable Tweedie models, before showing the same property for all the multivariate NST models.

2.1 Univariate Stable Tweedie Models

Let us briefly review the complete univariate stable Tweedie models through their cumulant functions, which also characterize the generating measures of the families (e.g., [1]).

Definition 2.1 *Let $t > 0$ and $p \in (-\infty, 0] \cup [1, \infty)$. Denote by $\mu_{p,t} = \mu_p^{*t}$ the t-th convolution power of a σ-finite positive measure μ_p. The cumulant function of any univariate stable Tweedie* NEF $F_{p,t} = F(\mu_{p,t}) = \{P(\theta; \mu_{p,t})(dx); \theta \in \Theta(\mu_{p,t})\}$, *generated by $\mu_{p,t}$, is given by $K_{\mu_{p,t}}(\theta) = tK_{\mu_p}(\theta)$ with*

$$K_{\mu_p}(\theta) = \begin{cases} \exp(\theta), & \text{for } p = 1, \\ -\log(-\theta), & \text{for } p = 2, \\ \{1/(2-p)\}\{-(p-1)\theta\}^{(2-p)/(1-p)}, & \text{for } p \neq 1, 2 \end{cases} \tag{2.1}$$

for all θ in their respective canonical domains $\Theta(\mu_p)$ with support S_p of distributions:

$$\Theta(\mu_p) = \begin{cases} \mathbb{R}, & \text{for } p = 0, 1, \\ [0, \infty), & \text{for } p < 0, \\ (-\infty, 0), & \text{for } 1 < p \leq 2, \\ (-\infty, 0], & \text{for } p > 2 \end{cases} \quad \text{and} \quad S_p = \begin{cases} \mathbb{R}, & \text{for } p \leq 0, \\ \mathbb{N}, & \text{for } p = 1, \\ [0, \infty), & \text{for } 1 < p < 2, \\ (0, \infty), & \text{for } p \geq 2. \end{cases}$$

Letting X be a Tweedie random variable denoted by $Tw_p(\theta, t)$, its probability density (or mass) function can be indicated in terms of

$$P(x; \theta, t, p) = a_p(x; t) \exp\{x\theta - K_{\mu_{p,t}}(\theta)\} \mathbb{1}_{S_p}(x),$$

with $t > 0$, $p \notin (0, 1)$ the power Tweedie index determining the distribution, $\theta \in \Theta(\mu_{p,t})$, $\mathbb{1}_E$ the indicator function of any given event E, and $a_p(x; t)$ the normalizing function detailed as follows. Denoting by $\Gamma(\cdot)$ the classical gamma function and $\alpha := (p-2)/(p-1) = \alpha(p)$ the stability index, we successively have

$$a_p(x; t) = \frac{1}{\pi x} \sum_{k=1}^{\infty} \frac{(-x)^k (\alpha t)^{k\alpha^{-1}} \Gamma(1 + k\alpha^{-1})}{(\alpha - 1)^{\{(\alpha-1)\alpha^{-1}k\}} \Gamma(k+1)} \sin(-k\pi\alpha^{-1}) \mathbb{1}_{\mathbb{R}}(x) \quad \text{for } p < 0,$$

$$a_p(x; t) = \mathbb{1}_{x=0} + \frac{1}{x} \sum_{k=1}^{\infty} \frac{(p-1)^{\alpha k} x^{-k\alpha}}{(2-p)^k t^{(1-\alpha)k} \Gamma(-k\alpha) \Gamma(k+1)} \mathbb{1}_{x>0} \quad \text{for } 1 < p < 2,$$

$$a_p(x;t) = \frac{1}{\pi x}\sum_{k=1}^{\infty}\frac{(p-1)^{\alpha k}t^{(\alpha-1)k}\Gamma(1+\alpha k)}{(p-2)^k x^{\alpha k}\Gamma(1+k)}(-1)^k\sin(-k\pi\alpha)\mathbb{1}_{x>0} \quad \text{for } p>2,$$

$$a_p(x;t) = \begin{cases} (2\pi t)^{-1/2}\exp(-x^2/2)\mathbb{1}_{\mathbb{R}}(x), & \text{for } p=0,\\ \{\Gamma(x+1)\}^{-1}\mathbb{1}_{\mathbb{N}}(x), & \text{for } p=1,\\ (1/t)^{1/t}x^{-1+1/t}\{\Gamma(1/t)\}^{-1}\mathbb{1}_{x>0}, & \text{for } p=2,\\ (2\pi x^3 t)^{-1/2}\exp\{-1/(2x)\}\mathbb{1}_{x>0}, & \text{for } p=3. \end{cases}$$

For $t=1$ one has $\mu_{p,1} := \mu_p$ and the (so-called unit) variance function is $V_p(m) = m^p$ for all m in the mean domain

$$M_p := K'_{\mu_p}(\Theta(\mu_p)) = \begin{cases} \mathbb{R}, & p=0,\\ (0,\infty), & p\neq 0. \end{cases}$$

Therefore, it is clear that the extreme stable models corresponding to $p<0$ are not steep, and they will not be a member anymore of this work from now.

We can now prove the Monge–Ampère property (1.1) for the univariate stable Tweedie models (Definition 2.1 with $t=1$) which are steep and have positive mean domain.

Proposition 2.2 *For $p \geq 1$ the second derivative of K_{μ_p} on the interior of $\Theta(\mu_p)$ satisfies*

(i) $K''_{\mu_p}(\theta) = K_{\mu_p}(\theta)$ *and* $K_{\mu_p}(\,\cdot\,) > 0$ *for* $p=1$,

(ii) $K''_{\mu_p}(\theta) = \{(2-p)K_{\mu_p}(\theta)\}^{p/(2-p)}$ *and* $K_{\mu_p}(\,\cdot\,) > 0$ *for* $1<p<2$,

(iii) $K''_{\mu_p}(\theta) = \exp\{2K_{\mu_p}(\theta)\}$ *and* $K_{\mu_p}(\,\cdot\,) > 0$ *for* $p=2$,

(iv) $K''_{\mu_p}(\theta) = [1/\{-(p-2)K_{\mu_p}(\theta)\}]^{-p/(p-2)}$ *and* $K_{\mu_p}(\,\cdot\,) < 0$ *for* $p>2$; *and, furthermore,*

(v) *the function* $p \mapsto K''_{\mu_p}$ *is continuous for* $p \in (1,\infty)$.

Remark 2.3 The function $p \mapsto K''_{\mu_p}$ is not continuous to the right of $p=1$, because the corresponding compound Poisson-gamma distributions ($1<p<2$) are not discrete. But they are semicontinuous (i.e., they have a mass at zero and positive continuous otherwise).

Proof of Proposition 2.2 From (2.1), we respectively have the following five parts.

(i) For $p=1$, it is obvious because of $K_{\mu_p}(\theta) = \exp(\theta)$.

(ii) For $1<p<2$, one has $K_{\mu_p}(\theta) = \{1/(2-p)\}\{-(p-1)\theta\}^{(2-p)/(1-p)}$ which is equivalent to $\theta = \{1/(1-p)\}\{(2-p)K_{\mu_p}(\theta)\}^{(1-p)/(2-p)}$ with $\Theta_{\mu_p} = (-\infty,0)$. The second derivative of K_{μ_p} leads to the desired result with $K_{\mu_p}(\theta) > 0$.

(iii) For $p=2$, it is trivial from $K_{\mu_p}(\theta) = -\log(-\theta) \Leftrightarrow \theta = -\exp\{-K_{\mu_p}(\theta)\}$ with $\Theta_{\mu_p} = (-\infty,0)$.

(iv) For $p>2$, from $K_{\mu_p}(\theta) = \{-1/(p-2)\}\{-(p-1)\theta\}^{(2-p)/(1-p)}$ one can write $\theta = \{1/(1-p)\}\{(2-p)K_{\mu_p}(\theta)\}^{(1-p)/(2-p)}$ with $\Theta_{\mu_p} = (-\infty,0]$. Then the direct calculation of K''_{μ_p} gives the result with $K_{\mu_p}(\theta) < 0$.

(v) It is obvious for $p \in (1,2)\cup(2,\infty)$. One also has the continuity at $p=2$ by using (2.1), for instance for the right 2^+ of $p=2$:

$$\begin{aligned}\lim_{p\to 2^+} K''_{\mu_p}(\theta) &= \lim_{p\to 2^+}[1/\{-(2-p)K_{\mu_p}(\theta)\}]^{-p/(p-2)}\\ &= \lim_{p\to 2^+}[\{-(p-1)\theta\}^{-(2-p)/(1-p)}]^{-p/(2-p)}\\ &= \lim_{p\to 2^+}[\{-(p-1)\theta\}^{-1}]^{-p/(1-p)}\end{aligned}$$

$$= (1/\theta)^2$$
$$= K''_{\mu_2}(\theta). \qquad \square$$

2.2 Multivariate NST Models

Let $t > 0$ and $p \geq 1$. From (1.2) and Definition 2.1, consider the σ-finite measure $\boldsymbol{\nu}_{p,t} = \boldsymbol{\nu}_p^{*t}$ on $\mathbb{R}^k$ defined by $\boldsymbol{\nu}_{p,t}(d\mathbf{x}) = \mu_{p,t}(dx_1)\prod_{j=2}^{k}\mu_{0,x_1}(dx_j)$. The NST models are multivariate NEF $\mathbf{F}_{p,t} = \mathbf{F}(\boldsymbol{\nu}_{p,t})$, generated by $\boldsymbol{\nu}_{p,t}$ with cumulant functions

$$\mathbf{K}_{\boldsymbol{\nu}_{p,t}}(\boldsymbol{\theta}) = t\mathbf{K}_{\boldsymbol{\nu}_p}(\boldsymbol{\theta}) = tK_{\mu_p}(g(\boldsymbol{\theta})), \tag{2.2}$$

for $g(\boldsymbol{\theta}) = \theta_1 + (\theta_2^2 + \cdots + \theta_k^2)/2$ and $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_k)^\top$ in the canonical domain

$$\boldsymbol{\Theta}(\boldsymbol{\nu}_{p,t}) = \boldsymbol{\Theta}(\boldsymbol{\nu}_p) = \{\boldsymbol{\theta} \in \mathbb{R}^k; g(\boldsymbol{\theta}) \in \Theta(\mu_p)\}. \tag{2.3}$$

The set $\mathbf{F}_{p,t} = \{\mathbf{P}(\boldsymbol{\theta}; p, t);\ \boldsymbol{\theta} \in \boldsymbol{\Theta}(\boldsymbol{\nu}_{p,t})\}$ of probability distributions $\mathbf{P}(\boldsymbol{\theta}; p, t)(d\mathbf{x}) = \exp\{\boldsymbol{\theta}^\top\mathbf{x} - \mathbf{K}_{\boldsymbol{\nu}_{p,t}}(\boldsymbol{\theta})\}\boldsymbol{\nu}_{p,t}(d\mathbf{x})$ has the mean domain $\mathbf{M}_{\mathbf{F}_{p,t}} := \mathbf{K}'_{\boldsymbol{\nu}_{p,t}}(\Theta(\boldsymbol{\nu}_{p,t})) = \mathbf{M}_{\mathbf{F}_p}$; its variance function and generalized variance function are, respectively,

$$\mathbf{V}_{\mathbf{F}_{p,t}}(\mathbf{m}) = t^{1-p}m_1^{p-2} \cdot \mathbf{m}\mathbf{m}^\top + \mathbf{Diag}_k(0, m_1, \ldots, m_1)$$

and

$$\det \mathbf{V}_{\mathbf{F}_{p,t}}(\mathbf{m}) = t^{1-p}m_1^{p+k-1}.$$

Table 1 depicts a summary for $t = 1$ with $\boldsymbol{\nu}_{p,1} := \boldsymbol{\nu}_p$ such that $\mathbf{S}_p$ denotes the support of distribution(s). We also include two particular cases in italics ($p = 3/2$ and $p = 3$).

Distribution(s)	p	$\mathbf{S}_p$
Normal Poisson	$p = 1$	$\mathbb{N} \times \mathbb{R}^{k-1}$
Normal compound Poisson-gamma	$1 < p < 2$	$[0, \infty) \times \mathbb{R}^{k-1}$
Normal noncentral gamma	$p = 3/2$	$[0, \infty) \times \mathbb{R}^{k-1}$
Normal gamma	$p = 2$	$(0, \infty) \times \mathbb{R}^{k-1}$
Normal positive stable	$p > 2$	$(0, \infty) \times \mathbb{R}^{k-1}$
Normal inverse Gaussian	$p = 3$	$(0, \infty) \times \mathbb{R}^{k-1}$

Table 1 Summary of the NST models with $p \geq 1$, $\mathbf{M}_{\mathbf{F}_p} = (0, \infty) \times \mathbb{R}^{k-1}$ and $\mathbf{S}_p$ ([4])

Here is the Monge–Ampère property (1.1) for the NST models, with the similar comment of Remark 2.3 that we omit.

Theorem 2.4 *For $p \geq 1$, the Hessian determinant of $\mathbf{K}_{\boldsymbol{\nu}_p}$ on the interior of $\boldsymbol{\Theta}(\boldsymbol{\nu}_p)$ satisfies*

(i) $\det \mathbf{K}''_{\boldsymbol{\nu}_p}(\boldsymbol{\theta}) = \{\mathbf{K}_{\boldsymbol{\nu}_p}(\boldsymbol{\theta})\}^k$ *and* $\mathbf{K}_{\boldsymbol{\nu}_p}(\,\cdot\,) > 0$ *for* $p = 1$,

(ii) $\det \mathbf{K}''_{\boldsymbol{\nu}_p}(\boldsymbol{\theta}) = \{(2-p)\mathbf{K}_{\boldsymbol{\nu}_p}(\boldsymbol{\theta})\}^{(p+k-1)/(2-p)}$ *and* $\mathbf{K}_{\boldsymbol{\nu}_p}(\,\cdot\,) > 0$ *for* $1 < p < 2$,

(iii) $\det \mathbf{K}''_{\boldsymbol{\nu}_p}(\boldsymbol{\theta}) = \exp\{(k+1)\mathbf{K}_{\boldsymbol{\nu}_p}(\boldsymbol{\theta})\}$ *and* $\mathbf{K}_{\boldsymbol{\nu}_p}(\,\cdot\,) > 0$ *for* $p = 2$,

(iv) $\det \mathbf{K}''_{\boldsymbol{\nu}_p}(\boldsymbol{\theta}) = [1/\{-(p-2)\mathbf{K}_{\boldsymbol{\nu}_p}(\boldsymbol{\theta})\}]^{(p+k-1)/(p-2)}$ *and* $\mathbf{K}_{\boldsymbol{\nu}_p}(\,\cdot\,) < 0$ *for* $p > 2$; *and, furthermore,*

(v) *the function* $p \mapsto \det \mathbf{K}''_{\boldsymbol{\nu}_p}$ *is continuous for* $p > 1$.

In order to prove this theorem and also for many calculations of determinants below, we need the following Schur representation of determinant in this simplified form:

Lemma 2.5 *Let $\lambda \neq 0$ be a scalar, $\mathbf{a} \in \mathbb{R}^{k-1}$ a vector and $\mathbf{A}$ a given $(k-1)\times(k-1)$ matrix. Then*

$$\det \begin{pmatrix} \lambda & \mathbf{a}^\top \\ \mathbf{a} & \mathbf{A} \end{pmatrix} = \lambda \det(\mathbf{A} - \lambda^{-1}\mathbf{a}\mathbf{a}^\top).$$

Proof of Theorem 2.4 From (2.2) with $t = 1$ and all $p \geq 1$ one has the Hessian

$$\mathbf{K}''_{\boldsymbol{\nu}_p}(\boldsymbol{\theta}) = K''_{\mu_p}(g(\boldsymbol{\theta})) \cdot g'(\boldsymbol{\theta})g'(\boldsymbol{\theta})^\top + K'_{\mu_p}(g(\boldsymbol{\theta})) \cdot g''(\boldsymbol{\theta}),$$

with $g'(\boldsymbol{\theta}) = (1, \theta_2, \ldots, \theta_k)^\top$ and $g''(\boldsymbol{\theta}) = \mathbf{Diag}_k(0, 1, \ldots, 1)$; that is

$$\mathbf{K}''_{\boldsymbol{\nu}_p}(\boldsymbol{\theta}) = K''_{\mu_p}(g(\boldsymbol{\theta}))\,(1, \theta_2, \ldots, \theta_k)\,(1, \theta_2, \ldots, \theta_k)^\top + K'_{\mu_p}(g(\boldsymbol{\theta}))\,\mathbf{Diag}_k(0, 1, \ldots, 1).$$

Using Lemma 2.5 with $\lambda = K''_{\mu_p}(g(\boldsymbol{\theta}))$, $\mathbf{a} = K''_{\mu_p}(g(\boldsymbol{\theta}))\,(\theta_2, \ldots, \theta_k)^\top$ and

$$\mathbf{A} = K''_{\mu_p}(g(\boldsymbol{\theta})) \cdot (\theta_2, \ldots, \theta_k)(\theta_2, \ldots, \theta_k)^\top + K'_{\mu_p}(g(\boldsymbol{\theta})) \cdot \mathbf{I}_{k-1},$$

one here obtains $\mathbf{A} - \lambda^{-1}\mathbf{a}\mathbf{a}^\top = K'_{\mu_p}(g(\boldsymbol{\theta})) \cdot \mathbf{I}_{k-1}$ and therefore

$$\begin{aligned} \det \mathbf{K}''_{\boldsymbol{\nu}_p}(\boldsymbol{\theta}) &= K''_{\mu_p}(g(\boldsymbol{\theta})) \det[K'_{\mu_p}(g(\boldsymbol{\theta}))\,\mathbf{I}_{k-1}] \\ &= K''_{\mu_p}(g(\boldsymbol{\theta}))\,\{K'_{\mu_p}(g(\boldsymbol{\theta}))\}^{k-1}. \end{aligned} \tag{2.4}$$

Now, we check each part of the announced results by using (2.4) and Proposition 2.2.

(i) For $p = 1$, the result is trivial and $\mathbf{K}_{\boldsymbol{\nu}_p}(\boldsymbol{\theta}) > 0$.

(ii) For $1 < p < 2$, the desired result is obtained as

$$\begin{aligned} \det \mathbf{K}''_{\boldsymbol{\nu}_p}(\boldsymbol{\theta}) &= \{(2-p)K_{\mu_p}(g(\boldsymbol{\theta}))\}^{p/(2-p)}\{(2-p)K_{\mu_p}(g(\boldsymbol{\theta}))\}^{(k-1)/(2-p)} \\ &= \{(2-p)\mathbf{K}_{\boldsymbol{\nu}_p}(\boldsymbol{\theta})\}^{(p+k-1)/(2-p)}, \end{aligned}$$

with $\mathbf{K}_{\boldsymbol{\nu}_p}(\boldsymbol{\theta}) > 0$.

(iii) For $p = 2$, one easily expresses

$$\det \mathbf{K}''_{\boldsymbol{\nu}_p}(\boldsymbol{\theta}) = \exp\{2K_{\mu_p}(g(\boldsymbol{\theta}))\}[\exp\{K_{\mu_p}(g(\boldsymbol{\theta}))\}]^{k-1} = \exp\{(k+1)\mathbf{K}_{\boldsymbol{\nu}_p}(\boldsymbol{\theta})\}.$$

(iv) For $p > 2$, it follows that

$$\begin{aligned} \det \mathbf{K}''_{\boldsymbol{\nu}_p}(\boldsymbol{\theta}) &= [1/\{-(p-2)K_{\mu_p}(g(\boldsymbol{\theta}))\}]^{(p+k-1)/(p-2)} \\ &= [1/\{-(p-2)\mathbf{K}_{\boldsymbol{\nu}_p}(\boldsymbol{\theta})\}]^{(p+k-1)/(p-2)}, \end{aligned}$$

with $\mathbf{K}_{\boldsymbol{\nu}_p}(\boldsymbol{\theta}) < 0$.

(v) For $p \in (1, 2) \cup (2, \infty)$, the function $p \mapsto \det \mathbf{K}''_{\boldsymbol{\nu}_p}$ is obviously continuous. At $p = 2$ one can directly obtain by undergraduate calculations

$$\begin{aligned} \lim_{p\to 2^-} \det \mathbf{K}''_{\boldsymbol{\nu}_p}(\boldsymbol{\theta}) &= \lim_{p\to 2^-} \{(2-p)K_{\mu_p}(g(\boldsymbol{\theta}))\}^{(p+k-1)/(2-p)} \\ &= \lim_{p\to 2^-} \{-(p-1)g(\boldsymbol{\theta})\}^{(p+k-1)/(1-p)} \\ &= \{-g(\boldsymbol{\theta})\}^{-(k+1)} \\ &= \exp\{(k+1)\mathbf{K}_{\boldsymbol{\nu}_2}(\boldsymbol{\theta})\} \\ &= \det \mathbf{K}''_{\boldsymbol{\nu}_2}(\boldsymbol{\theta}) \end{aligned}$$

and also

$$\begin{aligned}\lim_{p\to 2^+}\det \mathbf{K}''_{\boldsymbol{\nu}_p}(\boldsymbol{\theta}) &= \lim_{p\to 2^+}[1/\{-(p-2)K_{\mu_p}(g(\boldsymbol{\theta}))\}]^{(p+k-1)/(p-2)}\\ &= \lim_{p\to 2^+}[1/\{-(p-1)g(\boldsymbol{\theta})\}]^{(p+k-1)/(p-1)}\\ &= \{-g(\boldsymbol{\theta})\}^{-(k+1)}\\ &= \exp\{(k+1)\mathbf{K}_{\boldsymbol{\nu}_2}(\boldsymbol{\theta})\}\\ &= \det\mathbf{K}''_{\boldsymbol{\nu}_2}(\boldsymbol{\theta}).\end{aligned}$$ □

We here thank the anonymous referee for pointing out that, from (2.4), the formula

$$\det\mathbf{K}''(\boldsymbol{\theta}) = K''(g(\boldsymbol{\theta}))\{K'(g(\boldsymbol{\theta}))\}^{k-1}$$

is true for any distribution of X_1, not necessarily the Tweedie ones.

3 Characterization Result of NST Models

3.1 Result and Final Remarks

Before showing the main result with some concluding remarks, we first compile three classical operations of basis, affinity and convolution between two connected NEFs (e.g., [4, 24]). They exhibit the closeness in type of a given NEF through the variance function and generalized variance function.

Proposition 3.1 *Let $\boldsymbol{\mu}$ and $\widetilde{\boldsymbol{\mu}}$ be two σ-finite positive measures on $\mathbb{R}^k$ such that $\mathbf{F}=\mathbf{F}(\boldsymbol{\mu})$, $\widetilde{\mathbf{F}}=\mathbf{F}(\widetilde{\boldsymbol{\mu}})$ and $\mathbf{m}\in\mathbf{M}_{\mathbf{F}}$.*

(i) *If there exists $(\mathbf{a},b)\in\mathbb{R}^k\times\mathbb{R}$ such that $\widetilde{\boldsymbol{\mu}}(d\mathbf{x})=\exp\{\langle\mathbf{a},\mathbf{x}\rangle+b\}\boldsymbol{\mu}(d\mathbf{x})$, then $\mathbf{F}=\widetilde{\mathbf{F}}$: $\Theta(\widetilde{\boldsymbol{\mu}})=\Theta(\boldsymbol{\mu})-\mathbf{a}$ and $\mathbf{K}_{\widetilde{\boldsymbol{\mu}}}(\boldsymbol{\theta})=\mathbf{K}_{\boldsymbol{\mu}}(\boldsymbol{\theta}+\mathbf{a})+b$; for $\widetilde{\mathbf{m}}=\mathbf{m}\in\mathbf{M}_{\mathbf{F}}$, $\mathbf{V}_{\widetilde{\mathbf{F}}}(\widetilde{\mathbf{m}})=\mathbf{V}_{\mathbf{F}}(\mathbf{m})$ and $\det\mathbf{V}_{\widetilde{\mathbf{F}}}(\widetilde{\mathbf{m}})=\det\mathbf{V}_{\mathbf{F}}(\mathbf{m})$.*

(ii)*If $\widetilde{\boldsymbol{\mu}}=\boldsymbol{\varphi}*\boldsymbol{\mu}$ is the image measure of $\boldsymbol{\mu}$ by the affine transformation $\boldsymbol{\varphi}(\mathbf{x})=\mathbf{A}\mathbf{x}+\mathbf{b}$, where $\mathbf{A}$ is a $k\times k$ non-degenerate matrix and $\mathbf{b}\in\mathbb{R}^k$, then: $\Theta(\widetilde{\boldsymbol{\mu}})=\mathbf{A}^\top\Theta(\boldsymbol{\mu})$ and $\mathbf{K}_{\widetilde{\boldsymbol{\mu}}}(\boldsymbol{\theta})=\mathbf{K}_{\boldsymbol{\mu}}(\mathbf{A}^\top\boldsymbol{\theta})+\mathbf{b}^\top\boldsymbol{\theta}$; for $\widetilde{\mathbf{m}}=\mathbf{A}\mathbf{m}+\mathbf{b}\in\boldsymbol{\varphi}(\mathbf{M})$, $\mathbf{V}_{\widetilde{\mathbf{F}}}(\widetilde{\mathbf{m}})=\mathbf{A}\mathbf{V}_{\mathbf{F}}(\boldsymbol{\varphi}^{-1}(\mathbf{m}))\mathbf{A}^\top$ and $\det\mathbf{V}_{\widetilde{\mathbf{F}}}(\widetilde{\mathbf{m}})=(\det\mathbf{A})^2\det\mathbf{V}_{\mathbf{F}}(\mathbf{m})$.*

(iii) *If $\widetilde{\boldsymbol{\mu}}=\boldsymbol{\mu}^{*t}$ is the t-th convolution power of $\boldsymbol{\mu}$ for $t>0$, then: $\Theta(\widetilde{\boldsymbol{\mu}})=\Theta(\boldsymbol{\mu})$ and $\mathbf{K}_{\widetilde{\boldsymbol{\mu}}}(\boldsymbol{\theta})=t\mathbf{K}_{\boldsymbol{\mu}}(\boldsymbol{\theta})$; for $\widetilde{\mathbf{m}}=t\mathbf{m}\in t\mathbf{M}_{\mathbf{F}}$, $\mathbf{V}_{\widetilde{\mathbf{F}}}(\widetilde{\mathbf{m}})=t\mathbf{V}_{\mathbf{F}}(t^{-1}\mathbf{m})$ and $\det\mathbf{V}_{\widetilde{\mathbf{F}}}(\widetilde{\mathbf{m}})=t^k\det\mathbf{V}_{\mathbf{F}}(\mathbf{m})$.*

Part (iii) of Proposition 3.1 allows to consider $t=1$ for the following characterization theorem of the NST family. Here is the characterization result of NST.

Theorem 3.2 *Let $p\geq 1$ and $k\in\{1,2,\ldots\}$ be fixed. Consider a steep* NEF $\mathbf{F}$ *on $[0,\infty)\times\mathbb{R}^{k-1}$ governed by its cumulant function $\mathbf{K}$ on $\boldsymbol{\Theta}\subseteq\mathbb{R}^k$ such that (1.1) and (1.3) hold, where (1.1) comes in*

(H1) $\det\mathbf{K}''(\boldsymbol{\theta})=\{\mathbf{K}(\boldsymbol{\theta})\}^k$ *and* $\mathbf{K}(\,\cdot\,)>0$ *for* $p=1$,

(H2) $\det\mathbf{K}''(\boldsymbol{\theta})=\{(2-p)\mathbf{K}(\boldsymbol{\theta})\}^{(p+k-1)/(2-p)}$ *and* $\mathbf{K}(\,\cdot\,)>0$ *for* $1<p<2$,

(H3) $\det\mathbf{K}''(\boldsymbol{\theta})=\exp\{(k+1)\mathbf{K}(\boldsymbol{\theta})\}$ *and* $\mathbf{K}(\,\cdot\,)>0$ *for* $p=2$,

(H4) $\det\mathbf{K}''(\boldsymbol{\theta})=[1/\{-(p-2)\mathbf{K}(\boldsymbol{\theta})\}]^{(p+k-1)/(p-2)}$ *and* $\mathbf{K}(\,\cdot\,)<0$ *for* $p>2$.

Then $\mathbf{F}$ is an NST family governed by $\boldsymbol{\nu}_p$ of Table 1.

For the proof of Theorem 3.2 we will omit the univariate case ($k=1$) because of its triviality, even if it is new as a converse of Theorem 2.1. In general, for $k>1$ solving only

$\det \mathbf{K}'' = f(\mathbf{K})$ for known f is finding the solutions of differential equation, a much harder problem. Hence, adding a given form of generalized variance function must help to the solutions, under supplementary conditions.

For $p = 1$ in Theorem 3.2, we can also refer to [27, Theorems 4.1 and 4.2]) using the method of measure theory as in [17]. Here we shall use an alternative method, says analytical, and already used for $p = 2$ in [18, Theorem 2.1]. A challenging proof will be for the MST models where its generalized variance functions, replacing (1.3) of NST, can be written as follows: $\det \mathbf{V}(\mathbf{m}) = m_1^{q_1} m_2^{q_2} \ldots m_\ell^{q_\ell}$, for given $\ell \in \{0, 1, \ldots, k\}$ and $q_s \in \mathbb{R}$, $s = 1, 2, \ldots, \ell$; see, e.g., [4, 19].

3.2 Proof of Theorem 3.2

We first give in details the proof for $p > 2$. For that, we successively establish the following five lemmas before concluding the proof.

The first one is a reformulation of assumptions leading clearly to $\mathbf{K}(\boldsymbol{\theta}) < 0$ for all $\boldsymbol{\theta} \in \Theta$ such that $\mathbf{M} = \mathbf{K}'(\Theta)$.

Lemma 3.3 *Let $p > 2$ and $k \in \{2, 3, \ldots\}$ be fixed. Consider a steep* NEF $\mathbf{F}$ *on* $[0, \infty) \times \mathbb{R}^{k-1}$ *governed by its cumulant function* $\mathbf{K}$ *on* $\boldsymbol{\Theta} \subseteq \mathbb{R}^k$ *such that* (H4) *of Theorem* 3.2 *and* (1.3) *hold. Then*

$$\mathbf{K}(\boldsymbol{\theta}(\mathbf{m})) = -(p-2)^{-1} m_1^{-(p-2)}, \quad \mathbf{m} \in \mathbf{M} := (0, \infty) \times \mathbb{R}^{k-1}.$$

Proof Since $\det \mathbf{K}''(\boldsymbol{\theta}(\mathbf{m})) = \det \mathbf{V}(\mathbf{m})$, Assumption (H4) of Theorem 3.2 combined to (1.3) gives the desired result from

$$[1/\{-(p-2)\mathbf{K}(\boldsymbol{\theta}(\mathbf{m}))\}]^{(p+k-1)/(p-2)} = m_1^{p+k-1}.$$

The mean domain $\mathbf{M} := (0, \infty) \times \mathbb{R}^{k-1}$ is deduced from the steepness property of $\mathbf{F}$. □

Lemma 3.4 *From Lemma* 3.3 *and denoting* $\mathbf{V}_\mathbf{F}(\mathbf{m}) = (V_{ij}(\mathbf{m}))_{i,j=1,\ldots,k}$ *the variance function of* $\mathbf{F}$*, then*

$$V_{1j}(\mathbf{m}) = m_1^{p-1} m_j, \quad \forall j = 1, \ldots, k.$$

Proof Since one classically has $\mathbf{m} = \mathbf{K}'(\boldsymbol{\theta})$ and $\boldsymbol{\theta}'(\mathbf{m}) = \{\mathbf{V}_\mathbf{F}(\mathbf{m})\}^{-1}$, the differentiation of $\mathbf{K}(\boldsymbol{\theta}(\mathbf{m})) = -(p-2)^{-1} m_1^{-(p-2)}$ of Lemma 3.3 with respect to $\mathbf{m}$ gives

$$\mathbf{m}\{\mathbf{V}_\mathbf{F}(\mathbf{m})\}^{-1}\mathbf{u} = m_1^{1-p}(1, 0, \ldots, 0)\mathbf{u}, \quad \forall \mathbf{u} \in \mathbb{R}^k.$$

In particular for $\mathbf{u} = \mathbf{V}_\mathbf{F}(\mathbf{m})\mathbf{e}_j$ where $(\mathbf{e}_j)_{j=1,\ldots,k}$ is the canonical basis of $\mathbb{R}^k$, the desired result is thus obtained. □

The following lemma introduces the basical cumulant function $\mathbf{K}$ in terms of canonical parameter $\boldsymbol{\theta} = (\theta_1, \theta_2, \ldots, \theta_k)^\top$.

Lemma 3.5 *Under assumptions of Lemma* 3.4*, there is a function* $h : \boldsymbol{\Theta}_1 \subseteq \mathbb{R}^{k-1} \to (-\infty, 0)$ *such that, up to additive constant,*

$$\mathbf{K}(\boldsymbol{\theta}) = -(p-2)^{-1}[-(p-1)\{\theta_1 + h(\theta_2, \ldots, \theta_k)\}]^{(p-2)/(p-1)}$$

on $\boldsymbol{\Theta} = \left\{\boldsymbol{\theta} \in \mathbb{R}^k; \theta_1 + h(\theta_2, \ldots, \theta_k) < 0\right\}$.

Proof Departing from Lemma 3.3 which shows the cumulant function $\mathbf{K}$ in terms of the only first component m_1 of the mean vector $\mathbf{m} = (m_1, m_2, \ldots, m_k)^\top$, we successively investigate $\mathbf{K}$ through integration with respect to the first component θ_1 of the canonical parameter $\boldsymbol{\theta} = (\theta_1, \theta_2, \ldots, \theta_k)^\top$.

From Lemma 3.4 one has $V_{11}(\mathbf{m}) = m_1^p$, that involves $D^2_{11}\mathbf{K}(\boldsymbol{\theta}) = \{D_1\mathbf{K}(\boldsymbol{\theta})\}^p$. By integration with respect to θ_1, there exists a function $h : \mathbb{R}^{k-1} \to (-\infty, 0)$ such that

$$D_1\mathbf{K}(\boldsymbol{\theta}) = [-(p-1)\{\theta_1 + h(\theta_2, \ldots, \theta_k)\}]^{-1/(p-1)}. \tag{3.1}$$

One deduces $\theta_1 + h(\theta_2, \ldots, \theta_k) < 0$ because $D_1\mathbf{K}(\boldsymbol{\theta}) = m_1 > 0$ and $p - 1 > 0$. Thus, for particular $\boldsymbol{\theta} = (0, \theta_2, \ldots, \theta_k)^\top$ one has $0 + h(\theta_2, \ldots, \theta_k) < 0$ and, then, $h(\theta_2, \ldots, \theta_k) < 0$.

The derivative of $D_1\mathbf{K}(\boldsymbol{\theta})$ with respect to θ_j gives

$$D^2_{1j}\mathbf{K}(\boldsymbol{\theta}) = \frac{-h'(\theta_2, \ldots, \theta_k)D_1\mathbf{K}(\boldsymbol{\theta})}{(p-1)\{\theta_1 + h(\theta_2, \ldots, \theta_k)\}}, \quad \forall j = 2, \ldots, k, \tag{3.2}$$

and by Lemma 3.4 one obtains besides

$$D^2_{1j}\mathbf{K}(\boldsymbol{\theta}) = \{D_1\mathbf{K}(\boldsymbol{\theta})\}^{p-1}D_j\mathbf{K}(\boldsymbol{\theta}), \quad \forall j = 2, \ldots, k \tag{3.3}$$

with $D_j\mathbf{K}(\boldsymbol{\theta}) = m_j$ and $D^2_{ij}\mathbf{K}(\boldsymbol{\theta}(\mathbf{m})) = V_{ij}(\mathbf{m})$ for $i, j = 1, \ldots, k$. Combining (3.2) and (3.3) one has

$$\{D_1\mathbf{K}(\boldsymbol{\theta})\}^{p-1}D_j\mathbf{K}(\boldsymbol{\theta}) = \frac{-h'(\theta_2, \ldots, \theta_k)D_1\mathbf{K}(\boldsymbol{\theta})}{(p-1)\{\theta_1 + h(\theta_2, \ldots, \theta_k)\}},$$

therefore,

$$D_j\mathbf{K}(\boldsymbol{\theta}) = h'(\theta_2, \ldots, \theta_k)[-(p-1)\{\theta_1 + h(\theta_2, \ldots, \theta_k)\}]^{-1/(p-1)},$$

for all $j = 2, \ldots, k$; and, by integration with respect to θ_j one gets

$$\mathbf{K}(\boldsymbol{\theta}) = -(p-2)^{-1}[-(p-1)\{\theta_1 + h(\theta_2, \ldots, \theta_k)\}]^{(p-2)/(p-1)} + b(\theta_1), \tag{3.4}$$

where b is a real function to be determined.

Hence, the derivative of (3.4) with respect to θ_1 gives

$$D_1\mathbf{K}(\boldsymbol{\theta}) = [-(p-1)\{\theta_1 + h(\theta_2, \ldots, \theta_k)\}]^{-1/(p-1)} + b'(\theta_1),$$

to compare with (3.1) one deduces $b'(\theta_1) = 0$ therefore $b(\theta_1) = b$ a real constant. Then, from both $\mathbf{K}(\boldsymbol{\theta})$ of Lemma 3.3 and of (3.4) one obtains

$$-(p-2)^{-1}[-(p-1)\{\theta_1 + h(\theta_2, \ldots, \theta_k)\}]^{(p-2)/(p-1)} + b = -(p-2)^{-1}m_1^{-(p-2)},$$

with $m_1 = D_1\mathbf{K}(\boldsymbol{\theta})$ above, therefore $b = 0$. Finally, the expression (3.4) with $b = 0$ leads to the announced result. □

Lemma 3.6 *From Lemma* 3.5*, one has the following assertions*:

(i) $D^2_{11}\mathbf{K}(\boldsymbol{\theta}) = m_1^p$.

(ii) $D^2_{1j}\mathbf{K}(\boldsymbol{\theta}) = m_1^{p-1}m_j$, $\forall j = 2, \ldots, k$.

(iii) $D^2_{ij}\mathbf{K}(\boldsymbol{\theta}) = m_1^{p-2}m_im_j + m_1h''_{ij}(\theta_2, \ldots, \theta_k)$, $\forall i, j \in \{2, \ldots, k\}$, *with* $m_j := D_j\mathbf{K}(\boldsymbol{\theta})$ *for all* $j = 1, 2, \ldots, k$.

Proof Parts (i) and (ii) come from Lemma 3.4. Concerning Part (iii) the first partial derivative of $\mathbf{K}(\boldsymbol{\theta})$, from Lemma 3.5, with respect to θ_i gives

$$D_i\mathbf{K}(\boldsymbol{\theta}) = h'_i(\theta_2, \ldots, \theta_k)[-(p-1)\{\theta_1 + h(\theta_2, \ldots, \theta_k)\}]^{-1/(p-1)} =: m_i$$

and, then, the second one can be written successively as

$$\begin{aligned} D_{ij}^2\mathbf{K}(\boldsymbol{\theta}) &= h_i'(\theta_2,\dots,\theta_k)h_j'(\theta_2,\dots,\theta_k)[-(p-1)\{\theta_1+h(\theta_2,\dots,\theta_k)\}]^{-p/(p-1)} \\ &\quad + h_{ij}''(\theta_2,\dots,\theta_k)[-(p-1)\{\theta_1+h(\theta_2,\dots,\theta_k)\}]^{-1/(p-1)} \\ &= m_i m_j[-(p-1)\{\theta_1+h(\theta_2,\dots,\theta_k)\}]^{-(p-2)/(p-1)} \\ &\quad + h_{ij}''(\theta_2,\dots,\theta_k)[-(p-1)\{\theta_1+h(\theta_2,\dots,\theta_k)\}]^{-1/(p-1)} \\ &= m_1^{p-2}m_i m_j + m_1 h_{ij}''(\theta_2,\dots,\theta_k) \end{aligned}$$

for all $i,j=2,\dots,k$. Hence, the lemma is proven. □

Lemma 3.7 *The function* $h:\boldsymbol{\Theta}_1\subseteq\mathbb{R}^{k-1}\to(-\infty,0)$ *of Lemma* 3.5 *satisfies in succession the following properties*:

(i) h *is a convex function.*

(ii) $\det h''(\theta_2,\dots,\theta_k)$ *is a real constant with respect to* $(\theta_2,\dots,\theta_k)$.

(iii) $h''(\theta_2,\dots,\theta_k)=\boldsymbol{\Sigma}_{k-1}$ *is a* $(k-1)\times(k-1)$ *symmetric matrix and not depending on* θ_j, $\forall j\in\{2,\dots,k\}$.

Proof (i) Let $\boldsymbol{\nu}=(\boldsymbol{\nu}_1,\boldsymbol{\nu}_1^c)$ be the generating measure of $\mathbf{F}$. From Lemma 3.5 and since $(\theta_2,\dots,\theta_k)\mapsto\mathbf{K}(0,\theta_2,\dots,\theta_k)=:\mathbf{K}_{\boldsymbol{\nu}_1^c}(\theta_2,\dots,\theta_k)$ is the cumulant function of $\boldsymbol{\nu}_1^c$, the function

$$(\theta_2,\dots,\theta_k)\mapsto\mathbf{K}_{\boldsymbol{\nu}_1^c}(\theta_2,\dots\theta_k)=-(p-2)^{-1}\left[-(p-1)h(\theta_2,\dots,\theta_k)\right]^{(p-2)/(p-1)}$$

is therefore convex. This implies that $(\theta_2,\dots,\theta_k)\mapsto[-(p-2)\mathbf{K}(0,\theta_2,\dots\theta_k)]^{(p-1)/(p-2)}=-(p-1)h(\theta_2,\dots,\theta_k)$ is also convex. Therefore h is convex.

(ii) From Lemma 3.6 and Lemma 2.5 we can write

$$\det\mathbf{K}''(\boldsymbol{\theta})=\det[D_{ij}^2\mathbf{K}(\boldsymbol{\theta})]_{i,j=1,\dots,k}=\lambda\det(\mathbf{A}-\lambda^{-1}\mathbf{a}\mathbf{a}^\top),$$

with $\lambda=m_1^p$, $\mathbf{a}=m_1^{p-1}(m_2,\dots,m_k)^\top$ and $\mathbf{A}=m_1^{-p}\mathbf{a}\mathbf{a}^\top+m_1h_{ij}''(\theta_2,\dots,\theta_k)$. Therefore,

$$\begin{aligned} \det\mathbf{K}''(\boldsymbol{\theta}) &= m_1^p\det\left(\frac{1}{m_1^p}\mathbf{a}\mathbf{a}^\top+m_1h_{ij}''(\theta_2,\dots,\theta_k)-\frac{1}{m_1^p}\mathbf{a}\mathbf{a}^\top\right) \\ &= m_1^{p+k-1}\det h_{ij}''(\theta_2,\dots,\theta_k). \end{aligned}$$

Since $\det\mathbf{K}''(\boldsymbol{\theta})$ must be equal to m_1^{p+k-1}, we deduce that $\det h_{ij}''(\theta_2,\dots,\theta_k)=1$ not depending on θ_j, for all $j\in\{2,\dots,k\}$.

(iii) It is deduced from Parts (i) and (ii) and the result of basic Monge–Ampère equation. □

Theorem 3.8 (Jörgens–Calabi–Pogorelov [5, 13, 28]) *Let* $\mathbf{K}$ *be a convex function on* $\mathbb{R}^k$ *of class* $\mathscr{C}^2$ *such that* $\det\mathbf{K}''(\boldsymbol{\theta})=1$. *Then* $\mathbf{K}''$ *is a constant.*

Theorem 3.8 is used in the sense of Cheng and Yau [6] through the convexity and the analytical property of h on $\boldsymbol{\Theta}_1\subseteq\mathbb{R}^{k-1}$, which is extended on the whole $\mathbb{R}^{k-1}$.

In order to conclude the proof of Theorem 3.2 for $p>2$, it suffices to show that the family $\mathbf{F}$ of Lemmas 3.3–3.7 belongs to the NST family, up to linear transformation (Part (ii) with $\mathbf{b}=\mathbf{0}$ of Proposition 3.1). Indeed, substituting $h''(\theta_2,\dots,\theta_k)=\boldsymbol{\Sigma}_{k-1}$ of Part (iii) of Lemma 3.7 into Part (iii) of Lemma 3.6, the variance function of $\mathbf{F}$ is deduced from Lemma 3.6 as

$$\mathbf{V}_{\mathbf{F}}(\mathbf{m})=[D_{ij}^2\mathbf{K}(\boldsymbol{\theta}(\mathbf{m}))]_{i,j=1,\dots,k}$$

$$= \begin{pmatrix} m_1^p & m_1^{p-1}(m_2,\dots,m_k) \\ m_1^{p-1}(m_2,\dots,m_k)^\top & m_1^{-p}(m_2,\dots,m_k)^\top(m_2,\dots,m_k)+m_1\boldsymbol{\Sigma}_{k-1} \end{pmatrix}$$
$$= m_1^{p-2}\mathbf{m}\mathbf{m}^\top + m_1 \begin{pmatrix} 0 & \mathbf{0}_{k-1}^\top \\ \mathbf{0}_{k-1} & \boldsymbol{\Sigma}_{k-1} \end{pmatrix},$$

where $\mathbf{0}_{k-1}^\top$ is the null vector of $\mathbb{R}^{k-1}$. Applying Cholesky's decomposition, there is a triangular matrix $\mathbf{T}$ such that $\boldsymbol{\Sigma}_{k-1} = \mathbf{T}\mathbf{T}^\top$. Denote $\widetilde{\mathbf{F}}$ the image of the NEF $\mathbf{F}$ by linear transformation $\mathbf{x} \mapsto \mathbf{Bx}$ of $\mathbb{R}^k$ with $\mathbf{B} = \begin{pmatrix} 1 & \mathbf{0}_{k-1}^\top \\ \mathbf{0}_{k-1} & \mathbf{T}^{-1} \end{pmatrix}$, where $\mathbf{T}^{-1}$ is the $(k-1)\times(k-1)$ inverse matrix of $\mathbf{T}$. Then, by Formula (54.14) of Kotz et al. [24] and Part (ii) of Proposition 3.1, one can successively verify

$$\begin{aligned}
\mathbf{V}_{\widetilde{\mathbf{F}}}(\widetilde{\mathbf{m}}) &:= \mathbf{B}\mathbf{V}_{\mathbf{F}}\left(\mathbf{B}^{-1}\mathbf{m}\right)\mathbf{B}^\top \\
&= \mathbf{B}\left\{ m_1^{p-2}(\mathbf{B}^{-1}\mathbf{m})(\mathbf{B}^{-1}\mathbf{m})^\top + m_1 \begin{pmatrix} 0 & \mathbf{0}_{k-1}^\top \\ \mathbf{0}_{k-1} & \mathbf{T}\mathbf{T}^\top \end{pmatrix} \right\}\mathbf{B}^\top \\
&= m_1^{p-2}\mathbf{m}\mathbf{m}^\top + m_1\mathbf{B}\begin{pmatrix} 0 & \mathbf{0}_{k-1}^\top \\ \mathbf{0}_{k-1} & \mathbf{T}\mathbf{T}^\top \end{pmatrix}\mathbf{B}^\top \\
&= m_1^{p-2}\mathbf{m}\mathbf{m}^\top + m_1 \begin{pmatrix} 1 & \mathbf{0}_{k-1}^\top \\ \mathbf{0}_{k-1} & \mathbf{T}^{-1} \end{pmatrix}\begin{pmatrix} 0 & \mathbf{0}_{k-1}^\top \\ \mathbf{0}_{k-1} & \mathbf{T}\mathbf{T}^\top \end{pmatrix}\begin{pmatrix} 1 & \mathbf{0}_{k-1}^\top \\ \mathbf{0}_{k-1} & \left(\mathbf{T}^\top\right)^{-1} \end{pmatrix} \\
&= m_1^{p-2}\mathbf{m}\mathbf{m}^\top + m_1\mathbf{Diag}_k(0,1,\dots,1).
\end{aligned}$$

Finally, the NST characterization through variance functions in [20, Theorem 3.1] allows to conclude the proof of Theorem 3.2 for $p > 2$.

The remainder proof of Theorem 3.2 for $1 \le p < 2$ is similar to the previous case ($p > 2$) through the following two points around of Lemma 3.5. (A) $p = 1$: we have to consider $\mathbf{K}(\boldsymbol{\theta}) = \exp(\theta_1)h(\theta_2,\dots,\theta_k)$, for $\boldsymbol{\Theta} = \left\{\boldsymbol{\theta} \in \mathbb{R}^k; h(\theta_2,\dots,\theta_k) > 0\right\}$ with $h > 0$ and $\mathbf{K}(\boldsymbol{\theta}(\mathbf{m})) > 0$. (B) $p \in (1,2)$: that will be $\mathbf{K}(\boldsymbol{\theta}) = (2-p)^{-1}[-(p-1)\{\theta_1 + h(\theta_2,\dots,\theta_k)\}]^{(p-2)/(p-1)}$, for $\boldsymbol{\Theta} = \left\{\boldsymbol{\theta} \in \mathbb{R}^k; \theta_1 + h(\theta_2,\dots,\theta_k) < 0\right\}$ with $h < 0$ and $\mathbf{K}(\boldsymbol{\theta}(\mathbf{m})) > 0$.

Acknowledgements We sincerely thank the Associate Editor and the anonymous referees for their valuable comments and suggestions. Part of this work was performed whilst the first author was at the University of Lampung - Indonesia.

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