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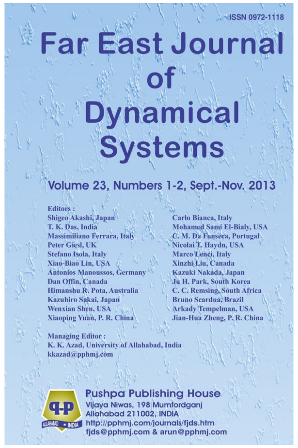

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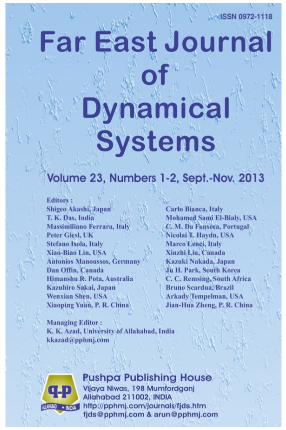

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Articles 1 - 4 of 4 [1]

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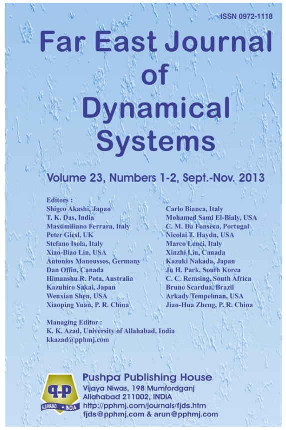

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Abstract

In this paper, we consider a generalization of a double discrete sine-Gordon equation. The generalization is done by introducing a number of parameters in the Lax-pair matrices. By restricting to the traveling wave solution, we derive a three-parameter family of discrete integrable dynamical systems using the so-called staircase methods. Special focus is on the cases where the resulting family of dynamical systems is of low dimension, i.e., two-dimensional. In those cases, the dynamics and bifurcation in the system is described by means of analyzing the level sets of the integral functions. Local bifurcation such as period-doubling bifurcation for map has been detected. Apart from that, we have observed nonlocal bifurcations which involve collision between heteroclinic and homoclinic connection between critical points.

Keywords and phrases:
ΔΔ-sine-Gordon equation, two-dimensional mapping, critical points, integrable systems, bifurcations.

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DYNAMICS AND BIFURCATIONS IN A TWO-DIMENSIONAL MAP DERIVED FROM A GENERALIZED ΔΔ-SINE-GORDON EQUATION

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Abstract

In this paper, we consider a generalization of a double discrete sine-Gordon equation. The generalization is done by introducing a number of parameters in the Lax-pair matrices. By restricting to the traveling wave solution, we derive a three-parameter family of discrete integrable dynamical systems using the so-called staircase methods. Special focus is on the cases where the resulting family of dynamical systems is of low dimension, i.e., two-dimensional. In those cases, the dynamics and bifurcation in the system is described by means of analyzing the level sets of the integral functions. Local bifurcation

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such as period-doubling bifurcation for map has been detected. Apart from that, we have observed nonlocal bifurcations which involve collision between heteroclinic and homoclinic connection between critical points.

1. Introduction

The sine-Gordon equation is a partial differential equation which is known to have soliton solutions, hence it is also called one of the soliton equations. The discretized (both in space and in time) version of the equation could be done in various ways. In this paper, we will follow the version in [4, 9-13], i.e., by describing its Lax-pair. By restriction to traveling wave solution, we derive an ordinary difference equation (see [7]) which is integrable as is the original equation.

In the literature, attention has been devoted to the integrability of the equation, the geometry it generates, symmetry in the system or the classification of integrable system (see [1]). In 2010, Late J. J. Duistermaat wrote a seminal book called Discrete Integrable Systems, QRT Maps, and Elliptic Surfaces [3] which provide us with a novel way of looking at integrable system. This book also originated from a discussion on a generalized discrete sine-Gordon equation between one of the authors of this paper and J. J. Duistermaat as is indicated in the preface of that book.

The mapping which is derived from the sine-Gordon equation is known to be a part of the celebrated Quispel-Roberts-Thompson (QRT) maps [12]. The latter is known as the most general family of Liouville integrable two dimensional maps. In [8], families of integrable mapping on a plane which is not a member of the QRT maps are introduced. Another interesting extension of the study on sine-Gordon equations is found in [14] where non-integrable perturbation is introduced.

Our interest in studying the sine-Gordon discrete dynamical systems is on the dynamics and the bifurcations therein. To do this, we need to have free parameters in the system. For this reason, we introduce a generalization to the sine-Gordon equation (originally this generalization was introduced in [16]). Since integrability is a property to be preserved, we choose to generalize the Lax-pair. By requiring the compatibility of the horizontal and vertical switches, we derive a mapping which we call: *generalized sine-Gordon equation*.

We begin with formulating a generalized sine-Gordon equation, by introducing eight parameters into the Lax-pair matrices. By analyzing the so-called compatibility condition (or commutativity of the multiplication of the matrices), we derive a system of two algebraic homogeneous equations. We have two possibilities: the space of solutions of the system of homogeneous equations is one dimensional or two dimensional. In this paper, we restrict ourselves to consider only the latter. By doing this, we can reduce the number of parameters in the system to three.

Using the so-called staircase method (see [10] or [7] for a general setting), we derive an ordinary discrete integrable dynamical system, with three parameters. Further reduction to the number of parameters in the system can be done by analyzing the integrals of the discrete system. For the case studies where the dimension of the phase space of the discrete system is two or three, we derive seven functions which contain the dynamics for all values of parameter. By analyzing the level sets of these functions, we derive some conclusion on the dynamics and bifurcations in the system. This study is related to [5].

We have observed an interesting local bifurcation of critical point in the system, namely: the period doubling bifurcation, where two period-2 points are created from a critical point. We have observed also a nonlocal bifurcation involving collision of homoclinic and heteroclinic connection between saddle type critical points. Furthermore, we have observed a change of stability of a critical point from a saddle type into an elliptic type of which we have not seen before in the literature.

2. Problem Formulations

A $\Delta\Delta$-sine-Gordon equation on a two-dimensional lattice $\mathbb{Z}^2$ is defined as follows:

$$V_{l,m}V_{l+1,m}V_{l,m+1}V_{l+1,m+1} - pq(V_{l,m}V_{l+1,m+1} - V_{l+1,m}V_{l,m+1}) = 1, \quad (2.1)$$

for fields $V_{l,m}$ defined at the site (l, m) of the lattice, while p, q are arbitrary constants. Let us write $\xi_{l,m}(k) = (V_{l,m}(k), U_{l,m}(k))^T$ for the vector consisting of wave functions at location (l, m) on the lattice, depending on a spectral parameter k. The above equation is derived from the consideration of the following maps:

$$\xi_{l+1,m}(k) = \frac{1}{p-k} M_{l,m}^{\text{hor}} \xi_{l,m}(k),$$

$$\xi_{l,m+1}(k) = \frac{1}{q - \frac{1}{k}} M_{l,m}^{\text{vert}} \xi_{l,m}(k),$$

where

$$M_{l,m}^{\text{hor}} = \begin{pmatrix} p & -V_{l+1,m} \\ -\frac{k^2}{V_{l,m}} & p\frac{V_{l+1,m}}{V_{l,m}} \end{pmatrix} \text{ and } M_{l,m}^{\text{vert}} = \begin{pmatrix} q\frac{V_{l,m+1}}{V_{l,m}} & -\frac{1}{k^2 V_{l,m}} \\ -V_{l,m+1} & q \end{pmatrix}.$$

These two matrices are also known as the Lax-pair matrices. This mapping is well-defined if

$$(M_{l+1,m}^{\text{vert}} M_{l,m}^{\text{hor}} - M_{l,m+1}^{\text{hor}} M_{l,m}^{\text{vert}})\xi_{l,m} = 0$$

for all $(l, m) \in \mathbb{Z}^2$. For the relation with the original sine-Gordon partial differential equation, see [11].

A generalization of the mapping (2.1) is done by generalizing the two matrices:

$$P_{l,m}^{\text{hor}} = \begin{pmatrix} \alpha_1 p & -\alpha_2 V_{l+1,m} \\ -\alpha_3 \frac{k^2}{V_{l,m}} & \alpha_4 p \frac{V_{l+1,m}}{V_{l,m}} \end{pmatrix}$$

and

$$P_{l,m}^{\text{vert}} = \begin{pmatrix} \beta_1 q \dfrac{V_{l,m+1}}{V_{l,m}} & -\beta_2 \dfrac{1}{k^2 V_{l,m}} \\ -\beta_3 V_{l,m+1} & \beta_4 q \end{pmatrix}.$$

Then the compatibility condition leads to the following system of four nonlinear equations:

$$(\beta_1 - \beta_4)\alpha_2 q k^2 V_{l+1,m+1} V_{l,m} - (\alpha_1 - \alpha_4)\beta_2 p = 0,$$

$$(\alpha_1 - \alpha_4)\beta_3 p V_{l+1,m+1} V_{l,m} - (\beta_1 - \beta_4)\alpha_3 q k^2 = 0,$$

$$\alpha_1\beta_1 (V_{l,m+1} V_{l+1,m} - V_{l+1,m+1} V_{l,m}) qp$$

$$+ \alpha_2\beta_3 V_{l+1,m+1} V_{l,m+1} V_{l+1,m} V_{l,m} = \beta_2\alpha_3,$$

$$\alpha_4\beta_4 (V_{l,m+1} V_{l+1,m} - V_{l+1,m+1} V_{l,m}) qp$$

$$+ \alpha_2\beta_3 V_{l+1,m+1} V_{l,m+1} V_{l+1,m} V_{l,m} = \beta_2\alpha_3 \tag{2.2}$$

for all $l, m \in \mathbb{Z}$. In order for these four equations to be consistent with each other, we need to impose some conditions on the parameters α_j and β_j, $j = 1, 2, 3, 4$. One could immediately see that one of the conditions is

$$\alpha_1\beta_1 - \alpha_4\beta_4 = 0. \tag{2.3}$$

If this holds, then the last two equations in (2.2) are consistent.

The first two equations can be written as:

$$\begin{pmatrix} (\beta_1 - \beta_4)\alpha_2 q k^2 & (\alpha_1 - \alpha_4)\beta_2 p \\ (\alpha_1 - \alpha_4)\beta_3 p & (\beta_1 - \beta_4)\alpha_3 q k^2 \end{pmatrix} \begin{pmatrix} V_{l+1,m+1} V_{l,m} \\ -1 \end{pmatrix} = \mathbf{0}, \tag{2.4}$$

which immediately implies that the determinant of the matrix

$$A = \begin{pmatrix} (\beta_1 - \beta_4)\alpha_2 q k^2 & (\alpha_1 - \alpha_4)\beta_2 p \\ (\alpha_1 - \alpha_4)\beta_3 p & (\beta_1 - \beta_4)\alpha_3 q k^2 \end{pmatrix}$$

is zero. Equation (2.4) also means that the vector

$$\begin{pmatrix} V_{l+1,m+1}V_{l,m} \\ -1 \end{pmatrix}$$

is in the kernel of A for all values of $V_{l+1,m+1}$ and $V_{l,m}$. The kernel of A is either one-dimensional or two-dimensional linear space. In this paper, we restrict ourselves to studying the situation where the kernel is two-dimensional.

If $\ker(A)$ is a two-dimensional linear space, then

$$\begin{cases} (\beta_1 - \beta_4)\alpha_2 = 0 \\ (\alpha_1 - \alpha_4)\beta_2 = 0 \\ (\alpha_1 - \alpha_4)\beta_3 = 0 \\ (\beta_1 - \beta_4)\alpha_3 = 0. \end{cases}$$

Solutions for these equations can be computed easily. Each solution then has to satisfy (2.3). In this paper, we are only going to consider a solution which has the largest number of parameters, i.e.,

$$(\boldsymbol{\alpha}, \boldsymbol{\beta}) = (\alpha_1, \alpha_2, \alpha_3, \alpha_1, \beta_1, \beta_2, \beta_3, \beta_1).$$

As a consequence, the Lax matrices become

$$P_{l,m}^{\mathrm{hor}} = \begin{pmatrix} \alpha_1 p & -\alpha_2 V_{l+1,m} \\ -\alpha_3 \dfrac{k^2}{V_{l,m}} & \alpha_1 p \dfrac{V_{l+1,m}}{V_{l,m}} \end{pmatrix}$$

and

$$P_{l,m}^{\mathrm{vert}} = \begin{pmatrix} \beta_1 q \dfrac{V_{l,m+1}}{V_{l,m}} & -\beta_2 \dfrac{1}{k^2 V_{l,m}} \\ -\beta_3 V_{l,m+1} & \beta_1 q \dfrac{V_{l+1,m}}{V_{l,m}} \end{pmatrix}. \tag{2.5}$$

We conclude that the mappings generalized discrete sine-Gordon equation is a member of the three-parameters family of mappings, i.e.,

$$\theta_1(V_{l,m+1}V_{l+1,m} - V_{l+1,m+1}V_{l,m}) + \theta_2 V_{l+1,m+1}V_{l,m+1}V_{l+1,m}V_{l,m} = \theta_3, \quad (2.6)$$

where $\theta_1 = \alpha_1\beta_1 pq$, $\theta_2 = \alpha_2\beta_3$ and $\theta_3 = \beta_2\alpha_3$. Without loss of generality, we can choose $\alpha_1 = \frac{1}{p}$, $\alpha_2 = 1$, and $\alpha_3 = 1$.

3. Reduction to Ordinary Difference Equation

Let us now turn our attention to the traveling wave solutions of (2.6) which are obtained by setting

$$V_{l,m} = V_n, \text{ where } n = z_1 l + z_2 m, \quad (3.1)$$

with z_1 and z_2 being relatively prime integers. We substitute this into equations (2.6) to derive

$$\theta_1(V_{n+z_2}V_{n+z_1} - V_{n+z_1+z_2}V_n) + \theta_2 V_{n+z_1+z_2}V_{n+z_2}V_{n+z_1}V_n = \theta_3. \quad (3.2)$$

In particular, for $n = 0$ we have:

$$V_{z_1+z_2} = \frac{\theta_3 - \theta_1 V_{z_1}V_{z_2}}{V_0(\theta_2 V_{z_1}V_{z_2} - \theta_1)}.$$

Let us consider the space: $\mathbb{R}^{z_1+z_2}$ with coordinate:

$$(V_{z_1+z_2-1}, V_{z_1+z_2-2}, \ldots, V_0)^T,$$

and a vector field that maps: $(V_{z_1+z_2-1}, V_{z_1+z_2-2}, \ldots, V_0)^T$ to

$$\left(\frac{\theta_3 - \theta_1 V_{z_1}V_{z_2}}{V_0(\theta_2 V_{z_1}V_{z_2} - \theta_1)}, V_{z_1+z_2-1}, \ldots, V_2, V_1\right)^T.$$

Then we can define a discrete dynamical system on $\mathbb{R}^{z_1+z_2}$, by considering the iteration:

$$\begin{cases} \overline{V}_{z_1+z_2-1} = \dfrac{\theta_3 - \theta_1 V_{z_1} V_{z_2}}{V_0(\theta_2 V_{z_1} V_{z_2} - \theta_1)}, \\ \overline{V}_{z_1+z_2-2} = V_{z_1+z_2-1}, \\ \qquad \vdots \\ \overline{V}_1 = V_2, \\ \overline{V}_0 = V_1, \end{cases} \tag{3.3}$$

where the overline denotes the new state of the iteration.

Two explicit formulas for the integrals of (3.3)**.** An integral for the discrete dynamical system (3.3) is the function: $H : \mathbb{R}^{z_1+z_2} \to \mathbb{R}^{z_1+z_2}$ that satisfies:

$$H(\overline{V_{z_1+z_2-1}}, \overline{V_{z_1+z_2-2}}, ..., \overline{V_0}) - H(V_{z_1+z_2-1}, V_{z_1+z_2-2}, ..., V_0) = 0.$$

The following are two explicit formulas for the integrals of system (3.3). These two integrals are derived from the conservation law.

Theorem 3.1. *For all* z_1 *and* z_2, *the function*

$$H_g = \theta_1 \sum_{j=0}^{z_2-1} \left(\frac{V_{z_1+j}}{V_j} + \frac{V_j}{V_{z_1+j}} \right) - \sum_{j=0}^{z_1-1} \left(\theta_2 V_j V_{z_2+j} + \theta_3 \frac{1}{V_j V_{z_2+j}} \right) \tag{3.4}$$

is an integral for the system (3.3).

Proof. Let

$$\overline{H}_g = \theta_1 \sum_{j=0}^{z_2-1} \left(\frac{\overline{V}_{z_1+j}}{\overline{V}_j} + \frac{\overline{V}_j}{\overline{V}_{z_1+j}} \right) - \sum_{j=0}^{z_1-1} \left(\theta_2 \overline{V}_j \overline{V}_{z_2+j} + \theta_3 \frac{1}{\overline{V}_j \overline{V}_{z_2+j}} \right),$$

and we write $\overline{V}_{z_1+z_2-1} = f$. Then

$$\overline{H}_g - H_g = \theta_1\left(\frac{f}{V_{z_2}} + \frac{V_{z_2}}{f} - \frac{V_{z_1}}{V_0} - \frac{V_0}{V_{z_1}}\right)$$

$$- \theta_2(V_{z_1}f - V_{z_2}V_0) - \theta_3\left(\frac{1}{V_{z_1}f} - \frac{1}{V_0V_{z_2}}\right).$$

By solving $\overline{H}_g - H_g = 0$ for *f*, we found that one of the solutions is

$$f = \frac{\theta_3 - \theta_1 V_{z_1}V_{z_2}}{V_0(\theta_2 V_{z_1}V_{z_2} - \theta_1)}.$$

This completes the proof. □

Let $z_1 = m$ and $z_2 = n$, where *m* and *n* are relatively prime. Then (3.3) defines a dynamical system on $\mathbb{R}^{m+n}$ with integral

$$H_g = \theta_1 \sum_{j=0}^{n-1}\left(\frac{V_{m+j}}{V_j} + \frac{V_j}{V_{m+j}}\right) - \sum_{j=0}^{m-1}\left(\theta_2 V_j V_{n+j} + \theta_3 \frac{1}{V_j V_{n+j}}\right).$$

Let us now consider the case where $z_1 = n$ and $z_2 = m$. Then (3.3) defines a dynamical system on $\mathbb{R}^{m+n}$ which is the same as the case where $z_1 = m$ and $z_2 = n$ (since the system is invariant under interchanging of V_{z_1} and V_{z_2}). The new dynamical system has integral

$$K_g = \theta_1 \sum_{j=0}^{m-1}\left(\frac{V_{n+j}}{V_j} + \frac{V_j}{V_{n+j}}\right) - \sum_{j=0}^{n-1}\left(\theta_2 V_j V_{m+j} + \theta_3 \frac{1}{V_j V_{m+j}}\right).$$

As a consequence of this, we have the following corollary.

Corollary 3.2. *For all z_1 and z_2, the function*

$$K_g = \theta_1 \sum_{j=0}^{z_1-1}\left(\frac{V_{z_2+j}}{V_j} + \frac{V_j}{V_{z_2+j}}\right) - \sum_{j=0}^{z_2-1}\left(\theta_2 V_j V_{z_1+j} + \theta_3 \frac{1}{V_j V_{z_1+j}}\right) \tag{3.5}$$

is an integral for the system (3.3).

4. Dynamics of the Ordinary Difference Equations for $z_1 = 1$ and $z_2 = 1$

Let us consider the case where $z_1 = 1$ and $z_2 = 1$. For this case, the mapping (3.3) is two-dimensional, i.e.,

$$\begin{cases} \overline{V_1} = \dfrac{(\theta_3 - \theta_1 V_1^2)}{V_0(\theta_2 V_1^2 - \theta_1)} \\ \overline{V_0} = V_1 \end{cases}$$

with integral: (see (A.2) in Appendix A). We denote

$$\boldsymbol{\zeta} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} V_1 \\ V_0 \end{pmatrix}$$

and by $\boldsymbol{\theta}$ the parameter vector in $\mathbb{R}^3 : (\theta_1, \theta_2, \theta_3)$. Then the two-dimensional mapping is:

$$\overline{\zeta} = \boldsymbol{f}_{\boldsymbol{\theta}}(\zeta), \tag{4.1}$$

where

$$\boldsymbol{f}_{\boldsymbol{\theta}} : \mathbb{R}^2 \to \mathbb{R}^2,$$

$$(x, y) \mapsto \left(\frac{\theta_3 - \theta_1 x^2}{(\theta_2 x^2 - \theta_1) y}, x \right).$$

The integral (see (A.2) in Appendix A) is rewritten as the function:

$$F(x, y) = \theta_1 \left(\frac{x}{y} + \frac{y}{x} \right) - \left(\theta_2 xy + \theta_3 \frac{1}{xy} \right). \tag{4.2}$$

For all $n \in \mathbb{N}$, the solution $\boldsymbol{\gamma}_n$ of the system (4.1) is contained in a level set of $F(x, y)$.

Since

$$F(x, y) = F(y, x) \text{ and } F(x, y) = F(-y, -x),$$

the level sets are symmetric with respect to the lines $y = x$ and $y = -x$. Furthermore,

$$F(-x, -y) = F(x, y),\ F(-x, y) = -F(x, y) \text{ and } F(x, -y) = -F(x, y).$$

Thus, the level sets of F are symmetric with respect to $x = 0$, $y = 0$ and $(0, 0)$.

Let us assume that $\theta_2 \neq 0$. Then we can write $\theta_1 = \mu\theta_2$ and $\theta_3 = \lambda\theta_2$ and then divide out θ_2 from F. By doing this, the parameter-space is reduced to $\mathbb{R}^2$. Thus, if $\theta_2 \neq 0$, then the integral can be written as:

$$\frac{1}{\theta_2} F(x, y) = \mu\left(\frac{x}{y} + \frac{y}{x}\right) - \left(xy + \lambda\frac{1}{xy}\right). \tag{4.3}$$

Let us consider the case where $\lambda > 0$. Then we can write $\lambda = \delta^4$ (with $\delta > 0$), and then re-scale the variables by $x \mapsto \delta x$, and $y \mapsto \delta y$. Then by rewriting $\mu = \delta^2\overline{\mu}$ and $F_1 = \delta^2\overline{F_1}$, we have

$$\frac{1}{\theta_2\delta^2} F(x, y)(x, y) = \frac{\mu}{\delta^2}\left(\frac{x}{y} + \frac{y}{x}\right) - \left(xy + \frac{1}{xy}\right).$$

If $\lambda < 0$, then we write $\lambda = -\delta^4$ and do the same re-scaling as above. We conclude that we need to consider only $\lambda = -1, 0$ or 1 in (4.3).

If $\theta_2 = 0$, then we assume that $\theta_1 \neq 0$. Similar to the previous case, we can rewrite the integral as

$$\frac{1}{\theta_1} F(x, y) = \left(\frac{x}{y} + \frac{y}{x}\right) + \kappa\frac{1}{xy},$$

with $\kappa = \theta_3/\theta_1$. Again, we need only to consider the situation where $\kappa = -1, 0$ or 1. Lastly, if $\theta_1 = 0$, then

$$\frac{1}{\theta_3} F(x, y) = \frac{1}{xy}.$$

The normal forms. We conclude that the level sets of the integral $F(x, y)$ for all values of the parameters are completely determined by the level sets of the following seven functions:

$$F_1(x, y) = \mu\left(\frac{x}{y} + \frac{y}{x}\right) - \left(xy + \frac{1}{xy}\right), \tag{4.4}$$

$$F_2(x, y) = \mu\left(\frac{x}{y} + \frac{y}{x}\right) - xy, \tag{4.5}$$

$$F_3(x, y) = \mu\left(\frac{x}{y} + \frac{y}{x}\right) - \left(xy - \frac{1}{xy}\right), \tag{4.6}$$

$$F_4(x, y) = \frac{x}{y} + \frac{y}{x} + \frac{1}{xy}, \tag{4.7}$$

$$F_5(x, y) = \frac{x}{y} + \frac{y}{x}, \tag{4.8}$$

$$F_6(x, y) = \frac{x}{y} + \frac{y}{x} - \frac{1}{xy} \tag{4.9}$$

and

$$F_7(x, y) = \frac{1}{xy}. \tag{4.10}$$

The level sets of F_1. Let us consider the situation where: $\theta_1 = \mu$, $\theta_2 = \theta_3 = 1$. The dynamics of mapping (4.1) is contained in the level sets of the function F_1. Recall that the level sets are symmetric with respect to:

$$y = x,\ y = -x,\ x = 0,\ y = 0 \text{ and } (0, 0).$$

Writing: $L(x) = F_1(x, x)$, and then solving: $L'(x) = 0$ for x gives us $x = 1$ or $x = -1$. Thus, the critical points of F_1 on the line $y = x$ are: $(1, 1)$ and $(-1, -1)$. Similarly, we found another two critical points on the line $y = -x$ which are: $(-1, 1)$ and $(1, -1)$. This is true for all values of $\mu \in \mathbb{R}$. It is easy to check using (4.1) for $\theta_1 = \mu$, $\theta_2 = 1$ and $\theta_3 = 1$, that the points

$(1, 1)$ and $(-1, -1)$ are fixed points, while $(-1, 1)$ and $(1, -1)$ are period-2 points.

Note that, since:

$$F_1(x, y) = \frac{\mu x^2 + \mu y^2 - (xy)^2 - 1}{xy},$$

for $\mu > 0$ we have four other special points, namely:

$$\left(\frac{1}{\sqrt{\mu}}, 0\right), \left(-\frac{1}{\sqrt{\mu}}, 0\right), \left(0, \frac{1}{\sqrt{\mu}}\right) \text{ and } \left(0, -\frac{1}{\sqrt{\mu}}\right).$$

At these points, both the numerator and the denominator of F_1 (presented as the above written rational function) are zero. These points are the intersection points between level sets of F_1. They are called the *base points*. It is interesting to note that, as $\mu \to 0^+$, then the nontrivial base points go to infinity along the axis at where the base point is located.

In Figure 1, we have plotted a few of the level sets of the functions F_1, for various values of the parameter μ. In the first row, there are three diagrams that correspond to the situation where $\mu = 4, 1,$ and $\frac{1}{4}$ (from left to right, respectively). In the second row, we have presented the diagram for the situation where $\mu = 0$. Note that this corresponds to the situation where the four base points have reached infinity. In the third row, we have plotted three diagrams that correspond to the situation where $\mu = -4, -1,$ and $-\frac{1}{4}$, from left to right, respectively. When $\mu < 0$, apart from the base points disappearing at infinity, the critical points are all elliptic.

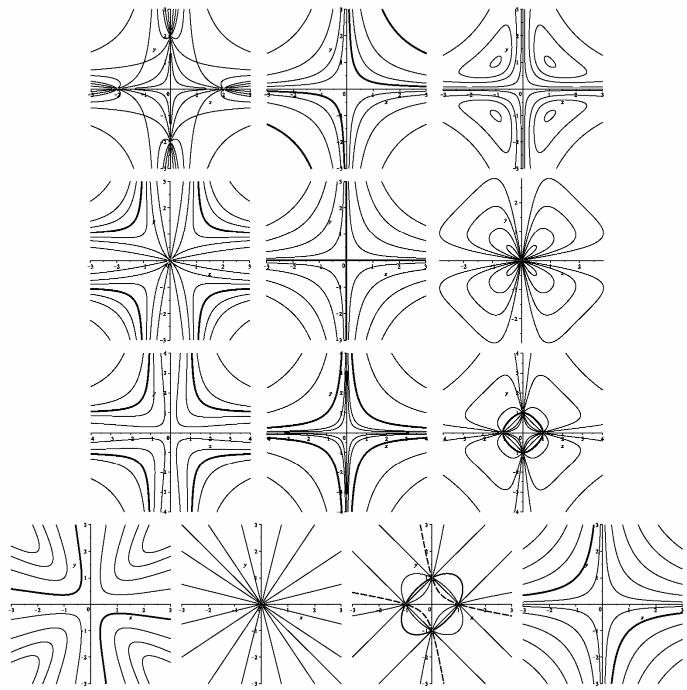

Figure 1. In this figure, we have plotted some level sets of the function F_k, $k = 1, ..., 7$, for various values of μ. The diagrams in the first row are the level sets of F_1 for $\mu = \frac{1}{4}$, 0, and $-\frac{1}{4}$, from left to right, respectively. The diagrams in the second row are the level sets of F_2 for $\mu = 1$, 0, and -1, from left to right, respectively. Lastly, the diagrams in the third row are of F_3 for $\mu = 2$, 0 and $-\frac{1}{4}$. The diagrams in the fourth row are the level sets of F_4, F_5, F_6, and F_7, respectively.

Bifurcations. When μ varies from positive to negative, the critical points of F_1, change from a saddle type to an elliptic type. A known mechanism in the literature, for integrable systems, is through a Saddle-Center bifurcation, where one saddle point becomes degenerate, and breaks into three critical points: two saddles and one elliptic (or also known as center) point. In the case of F_1, the mechanism is different.

Let us concentrate on the domain where $x > 0$ and $y > 0$; the critical point of F_1 is located at $(1, 1)$. In Figure 2, we have plotted three diagrams containing the level sets of F_1 for $\mu = 0.25$, $\mu = 0$ and $\mu = -0.25$, respectively. For $\mu = 0.25$, the critical point of F_1 is of saddle type (see the thickened curve in the most left diagram in Figure 2). As μ approaches 0, the stable and unstable manifolds collapse into each other to form a manifold of critical points:

$$\mathcal{C} = \{(x, y) | x^2 y^2 = 1\},$$

which is exactly the level set: $F_1(x, y) = F_1(1, 1)$. The diagram in the middle of Figure 2 corresponds to the situation where $\mu = 0$. The thickened curve on that diagram is the previously mentioned manifold of critical points $\mathcal{C}$. Consider k not equal but closed to $F_1(1, 1)$. Then the level set $F_1(x, y) = k$ consists of two leaves which are separated by the manifold of critical point $\mathcal{C}$. These two leaves of level set become connected into one closed curve as μ becomes negative. See Figure 2:

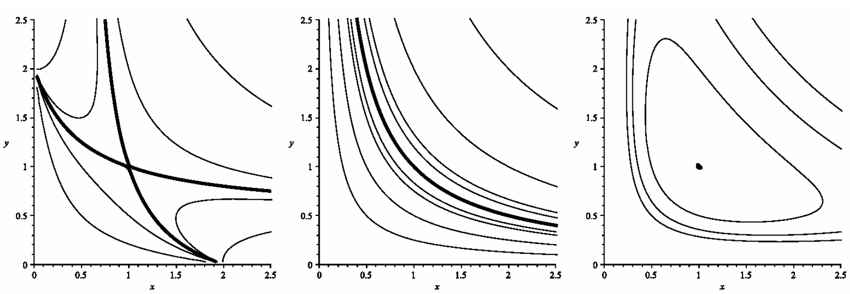

Figure 2. The bifurcation (or change of stability) of the critical point of F_1 as μ passes 0.

The level sets of F_2 *and* F_3. Consider the integral function F_2. This function has no critical point nor base points. Each level set of the function F_2 has four leaves of curve; see the thickened curve in the first diagram of the left of the second row of diagrams in Figure 1. Let us fix our attention on this level set which is plotted using thickened line. This is the level set:

$F_2(x, y) = 0$. As μ approaches zero, the level set $F_2(x, y) = 0$ approaches the x- and y-axes. As μ becomes negative, the level sets of F_2 are all bounded.

The situation for the level sets of F_3 is similar with those of F_2 apart from the fact that the zero level set for $\mu = 0$ is the curve defined by

$$y = \frac{1}{x} \text{ or } y = -\frac{1}{x}.$$

Another difference is, as μ becomes negative, we have four base points coming from infinity through the axis. These base points approach the origin as $\mu \to \infty$.

The level sets of F_3 for negative μ are all bounded closed curve, that intersect each other at the four base points. In Figure 1 in the third row, we have plotted three diagrams containing the level sets of F_3 for $\mu = 1, 0,$ and -1, respectively. The thickened curve is again the zero level set of F_3.

The level sets of F_4, F_5, F_6, *and* F_7. The diagrams in the fourth row of Figure 1 are the level sets of F_4, F_5, F_6, and F_7. We like to note that the F_5 can be seen as the limit of $\mu \to \infty$ of $\frac{1}{\mu} F_k$, $k = 1, 2, 3$.

5. Dynamics of the Ordinary Difference Equations for $z_1 = 1$ and $z_2 = 2$

Let us consider the case where $z_1 = 1$ and $z_2 = 2$. For this case, the mapping (3.3) is three-dimensional:

$$\begin{aligned} \overline{V_2} &= \frac{(\theta_3 - \theta_1 V_1 V_2)}{V_0(\theta_2 V_1 V_2 - \theta_1)}, \\ \overline{V_1} &= V_2, \\ \overline{V_0} &= V_1, \end{aligned} \tag{5.1}$$

with integrals: (A.3) and (A.4) in Appendix A.

This three-dimensional map can be reduced to two-dimensional by defining: $\boldsymbol{\zeta}$ as

$$\boldsymbol{\zeta} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} V_2V_1 \\ V_1V_0 \end{pmatrix}.$$

Similar reduction can be done for the case of even number z_2. Furthermore, let us write $\boldsymbol{\theta} = (\theta_1, \theta_2, \theta_3)$. Let us consider a two-dimensional mapping, defined by:

$$\boldsymbol{\zeta}_{n+1} = \boldsymbol{g}_{\boldsymbol{\theta}}(\boldsymbol{\zeta}_n), \tag{5.2}$$

where

$$\boldsymbol{g}_{\boldsymbol{\theta}} : \mathbb{R}^2 \to \mathbb{R}^2,$$

$$(x, y) \mapsto \left(\frac{(\theta_3 - \theta_1 x)x}{(\theta_2 x - \theta_1)y}, x \right).$$

Consequently, the integral (A.3) can be written as:

$$G(x, y) = \theta_1\left(\frac{x}{y} + \frac{y}{x}\right) - \theta_2(x + y) - \theta_3\left(\frac{1}{x} + \frac{1}{y}\right),$$

while (A.4) can be written as:

$$H_g(x, y, V_2) = \theta_1\left(\frac{x^2}{V_2^2 y} + \frac{V_2^2 y}{x^2} + \frac{V_2^2}{x} + \frac{x}{V_2^2}\right) - \theta_2 V_2^2 \frac{y}{x} - \theta_3 \frac{x}{V_2^2 y}.$$

Thus, the solution of (5.2) is contained in a level set of $G(x, y)$, and by considering a level set of $H_g(x, y, V_2)$, we can reconstruct the full dynamics of (5.1). A similar technique as in the previous section can be applied to derive the seven functions that contain the dynamics of (5.2) for all values of the parameters.

The normal forms. The level sets of the integral $G(x, y)$ for all values of the parameters are completely determined by the level sets of the following seven functions:

$$G_1(x, y) = \mu\left(\frac{x}{y} + \frac{y}{x}\right) - (x + y) - \left(\frac{1}{x} + \frac{1}{y}\right), \tag{5.3}$$

$$G_2(x,\ y) = \mu\left(\frac{x}{y} + \frac{y}{x}\right) - (x + y) + \left(\frac{1}{x} + \frac{1}{y}\right), \tag{5.4}$$

$$G_3(x,\ y) = \mu\left(\frac{x}{y} + \frac{y}{x}\right) - (x + y), \tag{5.5}$$

$$G_4(x,\ y) = \left(\frac{x}{y} + \frac{y}{x}\right) - \left(\frac{1}{x} + \frac{1}{y}\right), \tag{5.6}$$

$$G_5(x,\ y) = \left(\frac{x}{y} + \frac{y}{x}\right) + \left(\frac{1}{x} + \frac{1}{y}\right), \tag{5.7}$$

$$G_6(x,\ y) = \left(\frac{x}{y} + \frac{y}{x}\right) \tag{5.8}$$

and

$$G_7(x,\ y) = \left(\frac{1}{x} + \frac{1}{y}\right). \tag{5.9}$$

The level sets of G_1, G_2 *and* G_3. In contrast with the level sets of F_1, the level sets of the function G_1 for various values of μ are more complex. Note that, since:

$$-G_1(-x,\ -y) = -\mu\left(\frac{x}{y} + \frac{y}{x}\right) - (x + y) - \left(\frac{1}{x} + \frac{1}{y}\right),$$

the level sets of G_1 for $\mu < 0$ is the same as for $\mu > 0$ but reflected with respect to: $(0,\ 0)$. The same holds for G_2 and G_3.

Let us first look at the neighborhood of $\mu = 1$. We define the following critical level sets:

- $\mathcal{C}_{1,1} : G_1(x,\ y) = G_1(1,\ 1)$, plotted using the dashed line curve,
- $\mathcal{C}_{-1,1} : G_1(x,\ y) = G_1(-1,\ 1)$, plotted using the dashed and dotted line curves, and
- $\mathcal{C}_{-1,-1} : G_1(x,\ y) = G_1(-1,\ -1)$, plotted using the solid line curve.

In Figure 3, we have plotted nine diagrams that illustrate the level sets of G_1; the values of μ for the diagrams in each column (from left to right) are for $\mu = 1.1, 1,$ and 0.9. From the diagrams in the first row, we can see the evolution of the critical level set: $\mathcal{C}_{1,1}$, while in the second row: $\mathcal{C}_{-1,1}$. As μ varies from 1.1 to 0.9, the critical level sets $\mathcal{C}_{1,1}$ and $\mathcal{C}_{-1,1}$ coalesce at $\mu = 1$ and break up again. There is neither change of stability nor the location of the critical points of the function G_1, but the positions of base points are a bit shifted.

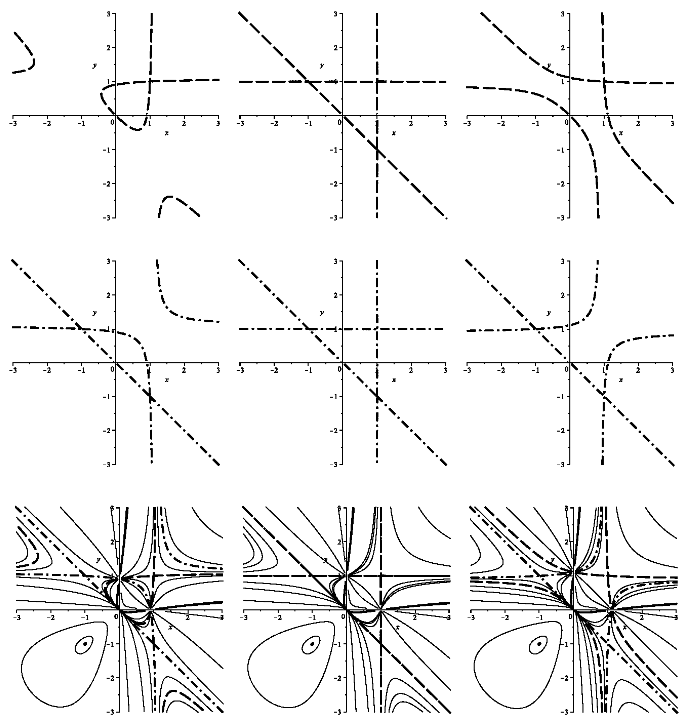

Figure 3. In this figure, we plotted the bifurcations of the critical level sets of the function G_1, for μ in the neighborhood of 1. The diagrams in the first row are the graphs of $\mathcal{C}_{1,1}$ for $\mu = 1.1, 1.0,$ and 0.9 (respectively, from left to right). The diagrams in the second row are the graphs of $\mathcal{C}_{-1,1}$, while the diagrams in the third row are the graphs of various level sets of G_2.

It is interesting to note that the critical point $(1, 1)$ is of saddle type. Moreover, for $\mu = 1.1,$ its stable and unstable manifolds are connected in a homoclinic loop. However, this homoclinic loop also contains two base points which are located in the positive part of the *x*-axis, and *y*-axis (see the upper left diagram in Figure 3). For the same value of $\mu,$ the critical points $(-1, 1)$ and $(1, -1)$ are connected with each other in a heteroclinic cycle. Note that this connection also contains the previously mentioned base points (see the middle left diagram in Figure 3). At $\mu = 1,$ the three critical points are connected in a heteroclinic loop, as the level sets $\mathcal{C}_{1,1}$ and $\mathcal{C}_{-1,1}$ coalesce. For $\mu = 0.9,$ all of these connections disappear. A detailed study on the dynamics of (5.2) will be a subject of investigation in the future. Interesting question such as the time behavior of solution on the level set $C_{1,1}$ forms a homoclinic loop.

In the neighborhood of $\mu = \frac{1}{2},$ the critical point at $(1, 1)$ changes its stability. As μ varies from 0.505 to 0.495, the critical point $(1, 1)$ changes from a saddle type critical point to an elliptic critical point, through the usual period-doubling bifurcation, where another two saddle type critical points are created.

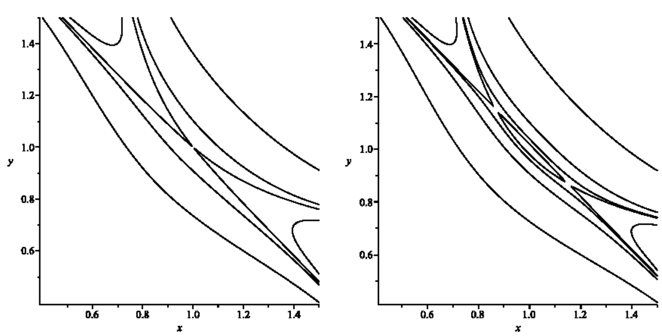

Figure 4. The period-doubling bifurcation of the critical point of G_1 in the neighborhood of $\mu = 0.5.$ The values of μ are 0.505 (the diagram on the left) and 0.495 (the diagram on the right), respectively.

The level sets of G_2 and G_3 for $\mu \geq 0$ are plotted in the five diagrams in Figure 5. There is no interesting bifurcation to note in this situation. The diagrams in the first row of Figure 5 are the level sets of G_2 for $\mu = 2, 0.4,$ and 0, respectively. In the two diagrams in the second row, we plotted the level sets of G_3 for $\mu = 2$ and 0. As $\mu \to 0^+$, the critical points and the base points go to infinity.

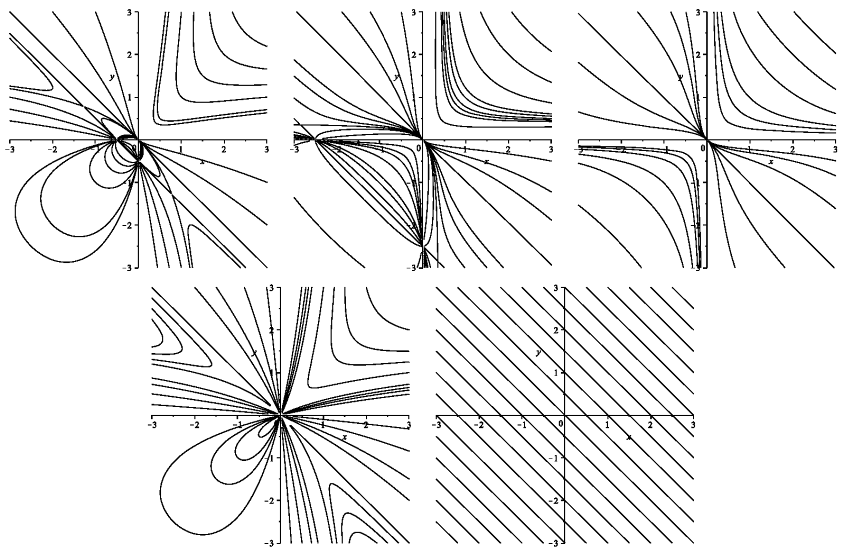

Figure 5. In this figure, the level sets of G_2 and G_3 are presented. The three diagrams in the first row are for G_2 with $\mu = 2, 0.4,$ and 0 (from left to right). The second row is for G_3 with $\mu = 0.5$ and 0.

The level sets of G_4, G_5, G_6 *and* G_7. Using a similar argument as for G_1, G_2 and G_3, i.e.,

$$G_5(-x, -y) = \left(\frac{x}{y} + \frac{y}{x}\right) - \left(\frac{1}{x} + \frac{1}{y}\right) = G_4(x, y),$$

we conclude that the level sets of G_5 are the same with G_4 but reflected with respect to $(0, 0)$. The graph of some level sets of G_4 is plotted in the first diagram in Figure 6.

The function G_6 is the same with the function F_5. Thus, we refer to the second diagram in the fourth row in Figure 1 for the level sets of G_6. The level sets of G_7 are presented as the second diagram in Figure 6:

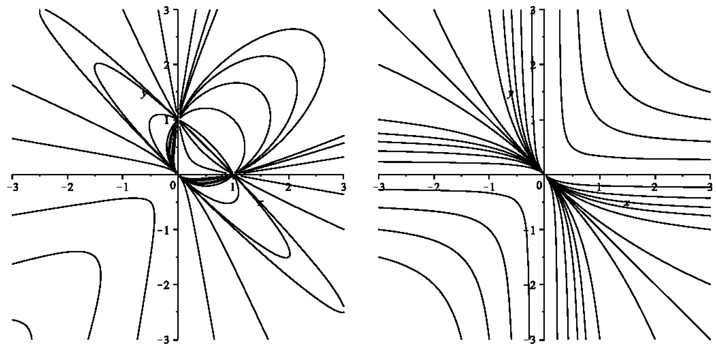

Figure 6. In this figure, we have plotted the level sets of G_4 and G_7, for left and right, respectively.

6. Concluding Remarks

As is indicated in the previous section, there are still some aspects which have not been analyzed regarding dynamics of the system (4.1) or (5.2). We know that the system has an integral and that solutions are confined in a level set of that integral function. However, the integral function has singularities at where level sets for different values intersect. It is interesting to study the behavior of solutions in the neighborhood of these singular points.

For example, consider a solution ξ_n which starts at a particular point (x_0, y_0) on a level set $F(x, y) = C_0$. After N iterations, the solution arrives at one particular singular point.

How can we modify the system such that the solution can get out of that singular point and go to the $n + 1$ iteration. If the system cannot be modified as such, then it means that every point in $\{\xi_1, ..., \xi_N\}$ is eventually singular. Generally speaking, it is very well possible that these eventually singular points are dense subset of the level set $F(x, y) = C_0$.

During the numerical experiments, we have done so far, we have observed some degenerate situation. For example, for system (5.2), we have found a manifold in the parameter space at where all solutions of the system are period-6. A description of a complete unfolding of this situation is instructive.

Appendix A. Computation of Explicit Formulas for the Integrals Using the Staircase Method

For a general setting of the staircase method, see [10, 7]. To illustrate the staircase method for periodic reduction of a generalized $\Delta\Delta$-sine-Gordon equation (2.6), let us consider the situation for $z_1 = 3$ and $z_2 = 7$. Equation (3.3) for this case becomes:

$$\begin{cases} \overline{V}_9 = \dfrac{\theta_3 - \theta_1 V_3 V_7}{V_0(\theta_2 V_3 V_7 - \theta_1)}, \\ \overline{V}_8 = V_9, \\ \vdots \\ \overline{V}_1 = V_2, \\ \overline{V}_0 = V_1. \end{cases} \tag{A.1}$$

For simplicity of the notation, we denote $z = (z_1, z_2)^T$ and $(l, m) \cdot z = lz_1 + mz_2$.

Let us start at an arbitrary point on a two-dimensional lattice at where we have labelled that point as $(0, 0)$. Note that by using the formula in (3.1), we have $V_{0,0} = V_{(0,0)\cdot z} = V_0$. Then going to the right direction on the lattice is the point labeled by $(1, 0)$, which corresponds to: $V_{(1,0)\cdot z} = V_3$. We carry on going to the right direction on the lattice (twice the step) to have: $V_{(2,0)\cdot z} = V_6$, and $V_{(3,0)\cdot z} = V_9$. If we go further to the right, then

$$(4, 0) \cdot z = 12 > 3 + 7 = 10.$$

Instead, we go downward one step on the lattice to have: $V_{(3,-1)\cdot z} = V_2$. From this point, we can take two steps to the right to have: $V_{(4,-1)\cdot z} = V_5$ and $V_{(5,-1)\cdot z} = V_8$. From this point, by the same argument as before, we go downward instead of going to the right on the lattice to have: $V_{(5,-2)\cdot z} = V_1$. Taking another two steps to the right, we have: $V_{(6,-2)\cdot z} = V_4$ and $V_{(7,-2)\cdot z} = V_7$. Lastly, by going downward, we get back V_0. See Figure 7 for a graphical illustration. The monodromy matrix is computed as:

$$(P^{\text{vert}}_{7,-3})^{-1} P^{\text{hor}}_{6,-2} P^{\text{hor}}_{5,-2} (P^{\text{vert}}_{5,-2})^{-1} P^{\text{hor}}_{4,-1} P^{\text{hor}}_{3,-1} (P^{\text{vert}}_{3,-1})^{-1} P^{\text{hor}}_{2,0} P^{\text{hor}}_{1,0} P^{\text{hor}}_{0,0}.$$

This monodromy matrix is constructed by following the staircase illustrated in Figure 7:

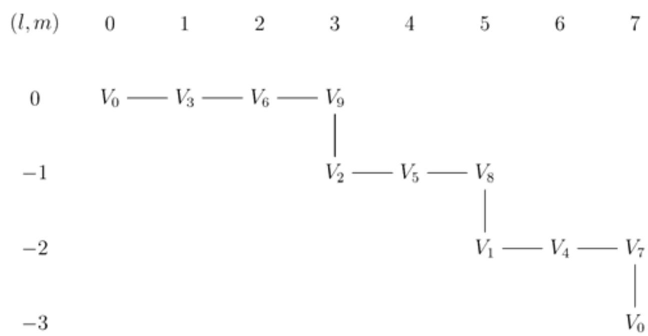

Figure 7. For the case where $z_1 = 3$ and $z_2 = 7$.

The entries of the Lax matrices (2.5) depend on the parameter k^2. Then, in general, the trace of the monodromy matrix can be written as:

$$\sum_{j \in \mathcal{J}} H_{2j} k^{2j},$$

where $\mathcal{J}$ is a finite subset of $\mathbb{Z}$. As a consequence, H_{2j}, $j \in \mathcal{J}$, are the integrals of (3.3).

Let us now present some explicit formulas for the integrals of the mapping (3.3), for the case where $z_1 = 1$ and various choices of z_2. In these cases, the integrals of the mapping are linear in the parameters: θ_1, θ_2, and θ_3.

The case where $z_2 = 1$**.** The mapping (3.3) has integral:

$$H_0 = \theta_1\left(\frac{V_0}{V_1} + \frac{V_1}{V_0}\right) - \theta_2 V_0 V_1 - \theta_3 \frac{1}{V_1 V_0}. \tag{A.2}$$

Clearly, in this case, the integral H_g in (3.4) and the integral K_g in (3.5) are the same.

The case where $z_2 = 2$**.** The mapping (3.3) has integral:

$$H_0 = \theta_1\left(\frac{V_0}{V_2} + \frac{V_2}{V_0}\right) - \theta_2(V_0V_1 + V_1V_2) - \theta_3\left(\frac{1}{V_0V_1} + \frac{1}{V_1V_2}\right). \tag{A.3}$$

This integral is the same with K_g. Thus, in this case, we have another integral which is H_g, i.e.:

$$H_g = \theta_1\left(\frac{V_1}{V_0} + \frac{V_0}{V_1} + \frac{V_2}{V_1} + \frac{V_1}{V_2}\right) - \left(\theta_2 V_0 V_2 + \theta_3 \frac{1}{V_0 V_2}\right). \tag{A.4}$$

The case where $z_2 = 3$**.** In this case, the mapping (3.3) is defined on $\mathbb{R}^4$. Computing the trace of the monodromy matrix gives us two integrals, i.e.,

$$H_0 = \theta_1\left(\frac{V_0}{V_3} + \frac{V_3}{V_0}\right) - \theta_2(V_0V_1 + V_1V_2 + V_2V_3) - \theta_3\left(\frac{1}{V_0V_1} + \frac{1}{V_1V_2} + \frac{1}{V_2V_3}\right)$$

and

$$H_2 = \theta_1\left(\frac{V_0}{V_1} + \frac{V_1}{V_2} + \frac{V_2}{V_3} + \frac{V_1}{V_0} + \frac{V_2}{V_1} + \frac{V_3}{V_2}\right) - \theta_2 V_0 V_3 - \frac{\theta_3}{V_0 V_3}.$$

One can see that $H_g = H_2$ while $K_g = H_0$.

The case where $z_2 = 4$. For $z_2 = 4$, the mapping is defined on $\mathbb{R}^5$. There exist two integrals:

$$H_0 = \theta_1\left(\frac{V_0}{V_4} + \frac{V_4}{V_0}\right) - \theta_2(V_0V_1 + V_1V_2 + V_2V_3 + V_3V_4)$$
$$- \theta_3\left(\frac{1}{V_0V_1} + \frac{1}{V_1V_2} + \frac{1}{V_2V_3} + \frac{1}{V_3V_4}\right)$$

and

$$H_2 = \theta_1\left(\frac{V_0}{V_2} + \frac{V_1}{V_3} + \frac{V_2}{V_4} + \frac{V_2}{V_0} + \frac{V_3}{V_1} + \frac{V_4}{V_2} + \frac{V_0V_3}{V_4V_1}\right.$$
$$\left. + \frac{V_3V_2}{V_4V_1} + \frac{V_0V_3}{V_2V_1} + \frac{V_2V_1}{V_0V_3} + \frac{V_4V_1}{V_3V_2} + \frac{V_4V_1}{V_0V_3}\right)$$
$$- \theta_2\left(\frac{V_0V_4V_3}{V_2} + V_0V_3 + \frac{V_0V_4V_1}{V_2} + V_4V_1\right)$$
$$- \theta_3\left(\frac{V_2}{V_0V_4V_1} + \frac{1}{V_0V_3} + \frac{V_2}{V_0V_4V_3} + \frac{1}{V_4V_1}\right).$$

Just as in the case where $z_2 = 2$, none of the integrals above is the same with H_g.

The case where $z_2 = 5$. For $z_2 = 5$, (3.3) is a mapping on $\mathbb{R}^6$ which has three integrals. The first one is

$$H_0 = \theta_1\left(\frac{V_0}{V_5} + \frac{V_5}{V_0}\right) - \sum_{j=0}^{4}\left(\theta_2 V_j V_{j+1} + \theta_3 \frac{1}{V_j V_{j+1}}\right).$$

The second integral can be written as:

$$H_2 = \theta_1 H_2^1 - \theta_2 H_2^2 - \theta_3 H_2^3,$$

with

$$H_2^1 = \frac{V_0}{V_3} + \frac{V_4}{V_1} + \frac{V_5}{V_2} + \frac{V_2}{V_5} + \frac{V_1}{V_4} + \frac{V_3}{V_0} + \frac{V_0V_3}{V_5V_1} + \frac{V_0V_4}{V_5V_2} + \frac{V_3V_2}{V_0V_4}$$

$$+ \frac{V_0V_4}{V_2V_1} + \frac{V_3V_2}{V_5V_1} + \frac{V_4V_3V_0}{V_5V_2V_1} + \frac{V_0V_4}{V_3V_2} + \frac{V_4V_3}{V_5V_1} + \frac{V_2V_1}{V_0V_4}$$

$$+ \frac{V_5V_1}{V_4V_3} + \frac{V_5V_2V_1}{V_4V_3V_0} + \frac{V_5V_2}{V_0V_4} + \frac{V_5V_1}{V_3V_2} + \frac{V_5V_1}{V_0V_3},$$

$$H_2^2 = V_0V_3 + V_5V_2 + V_4V_1 + \frac{V_5V_4V_0}{V_2} + \frac{V_0V_5V_1}{V_3} + \frac{V_4V_3V_0}{V_2}$$

$$+ \frac{V_5V_2V_1}{V_3} + \frac{V_5V_4V_1}{V_3} + \frac{V_4V_1V_0}{V_2} + \frac{V_0V_5V_4V_1}{V_3V_2}$$

and

$$H_2^3 = \frac{1}{V_5V_2} + \frac{1}{V_4V_1} + \frac{1}{V_0V_3} + \frac{V_3V_2}{V_0V_5V_4V_1}$$

$$+ \frac{V_3}{V_0V_5V_1} + \frac{V_2}{V_4V_1V_0} + \frac{V_3}{V_5V_2V_1} + \frac{V_2}{V_5V_4V_0} + \frac{V_3}{V_5V_4V_1} + \frac{V_2}{V_4V_3V_0}.$$

Lastly, the third integral is

$$H_4 = \theta_1\left(\frac{V_0}{V_1} + \frac{V_3}{V_2} + \frac{V_2}{V_3} + \frac{V_3}{V_4} + \frac{V_1}{V_0} + \frac{V_5}{V_4} + \frac{V_2}{V_1} + \frac{V_1}{V_2} + \frac{V_4}{V_3} + \frac{V_4}{V_5}\right)$$

$$- \theta_2V_0V_5 - \frac{\theta_3}{V_0V_5}.$$

This last integral H_4 is equal to H_g for $z_1 = 1$ and $z_2 = 5$.

The case where $z_2 = 6$**.** For $z_2 = 6$, the mapping is seven-dimensional with three integrals. The first integral is

$$H_0 = \theta_1\left(\frac{V_0}{V_6} + \frac{V_6}{V_0}\right) - \sum_{j=0}^{5}\left(\theta_2V_jV_{j+1} + \theta_3\frac{1}{V_jV_{j+1}}\right).$$

The second integral can be written as:

$$H_2 = \theta_1H_2^1 - \theta_2H_2^2 - \theta_3H_2^3,$$

where

$$
\begin{aligned}
H_2^1 = {} & \frac{V_0}{V_4} + \frac{V_5}{V_1} + \frac{V_2}{V_6} + \frac{V_6}{V_2} + \frac{V_1}{V_5} + \frac{V_4}{V_0} + \frac{V_0V_5}{V_6V_3} + \frac{V_2V_1}{V_0V_5} \\
& + \frac{V_3V_2}{V_0V_5} + \frac{V_6V_1}{V_0V_3} + \frac{V_6V_3}{V_0V_5} + \frac{V_6V_1}{V_4V_3} + \frac{V_6V_1}{V_5V_4} + \frac{V_6V_1}{V_3V_2} + \frac{V_0V_3}{V_6V_1} + \frac{V_3V_2}{V_6V_1} \\
& + \frac{V_4V_3}{V_6V_1} + \frac{V_0V_4}{V_6V_2} + \frac{V_5V_4}{V_6V_1} + \frac{V_6V_2}{V_0V_4} + \frac{V_0V_5}{V_3V_2} + \frac{V_0V_5}{V_4V_3} + \frac{V_4V_3}{V_0V_5} + \frac{V_0V_5}{V_2V_1} \\
& + \frac{V_6V_2V_1}{V_0V_5V_4} + \frac{V_6V_3V_2}{V_0V_5V_4} + \frac{V_4V_3V_0}{V_6V_2V_1} + \frac{V_6V_2V_1}{V_4V_3V_0} + \frac{V_0V_5V_4}{V_6V_2V_1} + \frac{V_0V_5V_4}{V_6V_3V_2},
\end{aligned}
$$

$$
\begin{aligned}
H_2^2 = {} & \frac{V_5V_4V_1}{V_3} + \frac{V_0V_5V_4}{V_2} + V_5V_2 + V_4V_1 + \frac{V_0V_5V_1}{V_3} + \frac{V_5V_2V_1}{V_3} + \frac{V_6V_5V_1}{V_3} \\
& + \frac{V_4V_3V_0}{V_2} + \frac{V_0V_6V_5V_1}{V_3V_2} + \frac{V_0V_6V_5V_1}{V_4V_3} + \frac{V_6V_5V_2}{V_4} + \frac{V_6V_5V_2V_1}{V_4V_3} + V_6V_3 \\
& + V_0V_3 + \frac{V_6V_2V_1}{V_4} + \frac{V_0V_6V_1}{V_4} + \frac{V_0V_6V_5}{V_2} + \frac{V_0V_5V_4V_1}{V_3V_2} + \frac{V_0V_4V_1}{V_2} + \frac{V_6V_3V_2}{V_4},
\end{aligned}
$$

$$
\begin{aligned}
H_2^3 = {} & \frac{V_2}{V_0V_4V_1} + \frac{V_3}{V_5V_4V_1} + \frac{V_2}{V_4V_3V_0} + \frac{V_3V_2}{V_6V_5V_1V_0} + \frac{V_4}{V_6V_5V_2} + \frac{1}{V_6V_3} \\
& + \frac{V_3}{V_5V_2V_1} + \frac{1}{V_0V_3} + \frac{V_4}{V_6V_2V_1} + \frac{V_4V_3}{V_6V_5V_2V_1} + \frac{V_2}{V_0V_5V_4} + \frac{V_2}{V_0V_6V_5} \\
& + \frac{V_4V_3}{V_6V_5V_1V_0} + \frac{V_4}{V_6V_1V_0} + \frac{V_3V_2}{V_0V_5V_4V_1} + \frac{1}{V_5V_2} + \frac{1}{V_4V_1} + \frac{V_4}{V_6V_3V_2} \\
& + \frac{V_3}{V_6V_5V_1} + \frac{V_3}{V_5V_1V_0}
\end{aligned}
$$

and

$$
\begin{aligned}
H_4 = {} & \theta_1\Bigg(\frac{V_4}{V_2} + \frac{V_3}{V_1} + \frac{V_3}{V_5} + \frac{V_4V_1}{V_5V_2} + \frac{V_0V_5}{V_4V_1} + \frac{V_5V_4}{V_6V_3} + \frac{V_2V_1}{V_0V_3} + \frac{V_4V_1}{V_0V_5} + \frac{V_4V_3}{V_5V_2} \\
& + \frac{V_3V_2}{V_4V_1} + \frac{V_6}{V_4} + \frac{V_0}{V_2} + \frac{V_1}{V_3} + \frac{V_2}{V_0} + \frac{V_5}{V_3} + \frac{V_4}{V_6} + \frac{V_5V_2}{V_6V_1}
\end{aligned}
$$

$$+\frac{V_0V_5}{V_6V_1}+\frac{V_6V_3}{V_5V_4}+\frac{V_6V_1}{V_5V_2}+\frac{V_6V_3}{V_5V_2}+\frac{V_6V_1}{V_0V_5}+\frac{V_0V_3}{V_4V_1}$$

$$\left.+\frac{V_5V_2}{V_6V_3}+\frac{V_4V_1}{V_3V_2}+\frac{V_4V_1}{V_0V_3}+\frac{V_5V_2}{V_4V_1}+\frac{V_5V_2}{V_4V_3}+\frac{V_0V_3}{V_2V_1}+\frac{V_2}{V_4}\right)$$

$$-\theta_2\left(V_6V_1+\frac{V_0V_6V_5}{V_4}+\frac{V_0V_6V_3}{V_4}+\frac{V_6V_1V_0}{V_2}+V_0V_5+\frac{V_0V_6V_3}{V_2}\right)$$

$$-\theta_3\left(\frac{V_4}{V_0V_6V_5}+\frac{V_4}{V_0V_6V_3}+\frac{V_2}{V_0V_6V_3}+\frac{1}{V_0V_5}+\frac{1}{V_6V_1}+\frac{V_2}{V_6V_1V_0}\right).$$

Again, in this case, none of the integrals above is the same with H_g.

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