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THE INTEGRAL NORMAL FORM OF A THREE-DIMENSIONAL TRAVELING WAVE SOLUTION MAPPING DERIVED FROM GENERALIZED ΔΔ-mKdV EQUATION

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Abstract

The integrability of traveling wave solution mappings can be obtained as reductions of the discrete generalized ΔΔ modified Korteweg-de Vries (ΔΔ-mKdV) equation. The properties of the integrable discrete dynamical system can be examined through the level set of integral function. In this paper, we show that the integral of a three-dimensional traveling wave solution mapping derived from generalized ΔΔ-mKdV equation can be made in the normal form.

1. Introduction

Discrete integrable systems can be classified into two main classes: integrable partial difference equations and integrable ordinary difference equations. Integrable ordinary difference equations are equivalent to integrable mappings. By imposing a periodicity condition, not only integrable lattice equations can be reduced to ordinary difference equations

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(or mappings/maps) [3, 6, 2, 1] but also there is a connection between the two classes so that many integrable maps can be obtained from integrable partial difference equations [5, 4, 6]. In this paper, we study the integrability of the discrete generalized ΔΔ modified Korteweg-de Vries (ΔΔ-mKdV) equation.

Consider the standard ΔΔ-mKdV equation on the 2D lattice $(\mathbb{Z}^2)$ defined as [6]

$$q(V_{l,m+1}V_{l+1,m+1} - V_{l,m}V_{l+1,m}) = p(V_{l+1,m}V_{l+1,m+1} - V_{l,m}V_{l,m+1}), \quad (1)$$

where the V fields are defined on the lattice sides of the $l, m \in \mathbb{Z}$, which are two discrete variables. Suppose $\xi_{l,m}(k)$ denotes a vector containing a wave function that depends on a spectral parameter k. The above equation can be derived through the following mappings:

$$\xi_{l+1,m}(k) = \frac{1}{p-k} M_{l,m}^{hor} \xi_{l,m}(k),$$

$$\xi_{l,m+1}(k) = \frac{1}{q-k} M_{l,m}^{vert} \xi_{l,m}(k)$$

with

$$M_{l,m}^{hor} = \begin{pmatrix} p & -V_{l+1,m} \\ -\left(\frac{k^2}{V_{l,m}}\right) & p\left(\frac{V_{l+1,m}}{V_{l,m}}\right) \end{pmatrix}$$

and

$$M_{l,m}^{vert} = \begin{pmatrix} q & -V_{l,m+1} \\ -\frac{k^2}{V_{l,m}} & q\frac{V_{l,m+1}}{V_{l,m}} \end{pmatrix}$$

which are Lax pair matrices. These mappings are well defined when the following condition is satisfied:

$$(M_{l+1,m}^{vert}M_{l,m}^{hor} - M_{l,m+1}^{hor}M_{l,m}^{vert})\xi_{l,m} = 0 \tag{2}$$

for all $(l, m) \in \mathbb{Z}^2$. This condition is known as *compatibility condition*.

Tuwankotta and Quispel [7] have made the derived discrete dynamic system through the effort of multiplying the parameters on the Lax matrix system. It is intended that the system's integrity properties are always maintained in examining more dynamics occurring from a discrete dynamic system (see [7] and [8]). By the same procedure, the following shows the attempt to elaborate (1).

Consider the following Lax pair matrices:

$$P_{l,m}^{hor} = \begin{pmatrix} \alpha_1 p & -\alpha_2 V_{l+1,m} \\ -\alpha_3\left(\dfrac{k^2}{V_{l,m}}\right) & \alpha_4 p\left(\dfrac{V_{l+1,m}}{V_{l,m}}\right) \end{pmatrix}$$

and

$$P_{l,m}^{vert} = \begin{pmatrix} \beta_1 q & -\beta_2 V_{l,m+1} \\ -\beta_3 \dfrac{k^2}{V_{l,m}} & \beta_4 q \dfrac{V_{l,m+1}}{V_{l,m}} \end{pmatrix}. \tag{3}$$

Based on *compatibility condition*, the following four nonlinear equations will be obtained:

$$
\begin{cases}
k^2(\alpha_3\beta_2 - \alpha_2\beta_3)V_{1+l,1+m} & = 0 \\
-k^2(\alpha_3\beta_2 - \alpha_2\beta_3) & = 0 \\
p\alpha_1\beta_2 V_{l,m}V_{l,l+m} - q\alpha_2\beta_1 V_{l,m}V_{1+l,m} & \\
+ q\alpha_2\beta_4 V_{l,1+m}V_{1+l,1+m} - p\alpha_4\beta_2 V_{1+l,m}V_{1+l,1+m} & = 0 \\
-k^2(p\alpha_1\beta_3 V_{l,m}V_{l,1+m} - q\alpha_3\beta_1 V_{l,m}V_{1+l,m}) & \\
-k^2(p(q\alpha_3\beta_4 V_{l,1+m}V_{1+l,1+m}) & \\
-k^2(p\alpha_4\beta_3 V_{1+l,m}V_{1+l,1+m})) & = 0.
\end{cases} \tag{4}
$$

To be consistent with each other, the parameters α_j and β_j with $j = 1, 2, 3, 4$ in equation (4) must be consistent.

As a result, the first two equations are obtained

$$\alpha_3\beta_2 - \alpha_2\beta_3 = 0. \tag{5}$$

In addition, from the last two equations in (4) obtained:

$$q(\alpha_3\beta_2 - \alpha_2\beta_3)(\beta_1 V_{l,m}V_{1+l,m} - \beta_4 V_{l,1+m}V_{1+l,1+m}) = 0. \tag{6}$$

From the relationship (5), equation (6) becomes consistent when $\alpha_2 = \alpha_3$ and $\beta_2 = \beta_3$. Thus, the system with the Lax matrix (3) will be consistent if $\alpha_2 = \alpha_3$ and $\beta_2 = \beta_3$. For example,

$$(\boldsymbol{\alpha}, \boldsymbol{\beta}) = (\alpha_1, \alpha_2, \alpha_2, \alpha_4, \beta_1, \beta_2, \beta_2, \beta_4).$$

As a result, the generalized Lax matrices for the system (1) can be written as

$$P_{l\,m}^{hor} = \begin{pmatrix} \alpha_1 p & -\alpha_2 V_{l+1,m} \\ -\alpha_2\left(\dfrac{k^2}{V_{l,m}}\right) & \alpha_4 p\left(\dfrac{V_{l+1,m}}{V_{l,m}}\right) \end{pmatrix}$$

and

$$P_{l,m}^{vert} = \begin{pmatrix} \beta_1 q & -\beta_2 V_{l,m+1} \\ -\beta_2 \dfrac{k^2}{V_{l,m}} & \beta_4 q \dfrac{V_{l,m+1}}{V_{l,m}} \end{pmatrix}.$$

By substituting $P_{l,m}^{hor}$ and $P_{l,m}^{vert}$ into compatibility condition (2), we get the form of mappings derived from the generalized equation ΔΔ-mKdV which is a part of the four-parameter family mapping:

$$\theta_1 V_{l,m} V_{l,m+1} - \theta_2 V_{l+1,m} V_{l+1,m+1} - \theta_3 V_{l,m} V_{l+1,m} + \theta_4 V_{l,m+1} V_{l+1,m+1} = 0 \tag{7}$$

with $\theta_1 = \alpha_1 \beta_2 p$, $\theta_2 = \alpha_4 \beta_2 p$, $\theta_3 = \alpha_2 \beta_1 q$ and $\theta_4 = \alpha_2 \beta_4 q$.

2. Main Result

A. The traveling wave solution of the generalized ΔΔ-mKdV equation

In 2016, Zakaria and Tuwankotta described a procedure to derive a two-dimensional traveling wave solution mapping of ΔΔ-sine Gordon equation [8]. Using the same procedure, a two-dimensional traveling wave solution mapping of ΔΔ-mKdV equation (7) can be obtained.

Consider the following discrete traveling wave solution relationship:

$$V_{l,m} = V_n, \text{ where } n = z_1 l + z_2 m$$

with z_1 and z_2 being relatively prime integers. Substituting these terms into (7), we obtain

$$\theta_1 V_n V_{n+z_2} - \theta_2 V_{n+z_1} V_{n+z_1+z_2} - \theta_3 V_n V_{n+z_1} + \theta_4 V_{n+z_2} V_{n+z_1+z_2} = 0. \tag{8}$$

Equation (8) is a form of the traveling wave solution derived from the generalized ΔΔ-mKdV equation. For the fixed values of z_1 and z_2, equation

(8) is a mapping of $\mathbb{R}^{z_1+z_2} \to \mathbb{R}^{z_1+z_2}$. It can be checked that equation (8) is invariant for a transformation $z_1 \to -z_1$, $\theta_1 \to -\theta_2$, and $z_1 \leftrightarrow z_2$. In addition, it also fulfills the periodic nature, namely $(i + z_2,\ j - z_1)$.

Equation (8) is equivalent to the following mapping:

$$\begin{aligned} V'_{z_1+z_2-1} &= \frac{V_0(\theta_3 V_{z_1} - \theta_1 V_{z_2})}{(\theta_4 V_{z_2} - \theta_2 V_{z_1})} \\ V'_{z_1+z_2-2} &= V_{z_1+z_2-1} \\ &\vdots \\ V'_1 &= V_2 \\ V'_0 &= V_1. \end{aligned} \tag{9}$$

Note that the mapping in [6] can be obtained from (9) by setting $\theta_1 = \theta_2 = p$ and $\theta_3 = \theta_4 = q$.

B. The integral normal form for a mapping derived from the generalized ΔΔ-mKdV equation: three-dimensional mapping case

Let us take $z_1 = 1$ and $z_2 = 2$. So from (9), the third order of difference equation will be obtained as follows:

$$\theta_1 V_n V_{n+2} - \theta_2 V_{n+1} V_{n+3} - \theta_3 V_n V_{n+1} + \theta_4 V_{n+2} V_{n+3} = 0$$

which is equivalent to a three-dimensional mapping:

$$\begin{aligned} V'_{n+2} &= \frac{V_n(\theta_3 V_{n+2} - \theta_1 V_{n+1})}{(\theta_4 V_{n+1} - \theta_2 V_{n+2})}, \\ V'_{n+1} &= V_{n+2}, \\ V'_n &= V_{n+1}. \end{aligned} \tag{10}$$

We call equation (10) as *three-dimensional mapping* derived from generalized ΔΔ-mKdV equation.

Consider equation (10). Then we denoted $\boldsymbol{\zeta}_n$ as the sequence in $\mathbb{R}^2$ which is defined as

$$\zeta_n = \begin{pmatrix} \dfrac{V_{n+2}}{V_{n+1}} \\ \dfrac{V_{n+1}}{V_n} \end{pmatrix}$$

and by $\boldsymbol{\theta}$ a parameter vector in $\mathbb{R}^4 : (\theta_1, \theta_2, \theta_3, \theta_4)$.

Therefore, we can express the mapping as the two-dimensional mapping, i.e.:

$$\zeta_{n+1} = \boldsymbol{g}_{\boldsymbol{\theta}}(\zeta_n), \tag{11}$$

where

$$\begin{array}{llll} \boldsymbol{g}_{\boldsymbol{\theta}} : & \mathbb{R}^2 & \to & \mathbb{R}^2 \\ & (W_1, W_0) & \mapsto & \left(\dfrac{-1}{W_1 W_0} \dfrac{(\theta_3 W_1 - \theta_1)}{(\theta_2 W_1 - \theta_4)}, W_1 \right). \end{array}$$

with $W_0 = \dfrac{V_{n+1}}{V_n}$ and $W_1 = \dfrac{V_{n+2}}{V_{n+1}}$.

$\zeta_{n+1} = \boldsymbol{g}_{\boldsymbol{\theta}}(\zeta_n)$ has an integral. In other words, there is a function $S : \mathbb{R}^2 \to \mathbb{R}$ such that $S(\zeta_{n+1}) = S(\zeta_n)$ for all natural number *n*. The procedure to obtain the integrals in more detail is described in [6].

For the mapping (10), the integral can be explicitly computed from the monodromy matrix below:

$$\begin{pmatrix} \beta_1 q & -\beta_2 V_2 \\ -\dfrac{\beta_2 k^2}{V_0} & \dfrac{\beta_4 q V_2}{V_0} \end{pmatrix}^{-1}.$$

$$\begin{pmatrix} \alpha_1 p & -\alpha_2 V_2 \\ -\dfrac{\alpha_2 k^2}{V_1} & \dfrac{\alpha_4 p V_2}{V_1} \end{pmatrix} \cdot \begin{pmatrix} \alpha_1 p & -\alpha_2 V_1 \\ -\dfrac{\alpha_2 k^2}{V_0} & \dfrac{\alpha_4 p V_1}{V_0} \end{pmatrix}. \tag{12}$$

The trace of the monodromy matrix in (12) is

$$\begin{aligned} S(W_0, W_1) = {} & p^2 q(\alpha_4^2 \beta_1 + \alpha_1^2 \beta_4) \\ & - k^2 \left\{ \theta_1 \left(\frac{1}{W_0} + \frac{1}{W_1} \right) + \theta_2 (W_0 + W_1) \right\} \\ & - k^2 \left\{ \theta_3 \left(\frac{1}{W_0 W_1} \right) - \theta_4 (W_0 W_1) \right\}. \end{aligned} \tag{13}$$

For all $n \in \mathbb{N}$, solution ζ_n from system (11) is a level set of $S(W_0, W_1)$.

C. The normalization processing for the integral

Assume that $\theta_2 \neq 0$. Then the integral (13) can be written as

$$\hat{S}(W_0, W_1) = \mu_1 \left(\frac{1}{W_0} \frac{1}{W_1} \right) + (W_0 + W_1) - \mu_2 (W_0 W_1) - \mu_3 (W_0 W_1) \, [0.5cm] \tag{14}$$

with $\mu_1 = \dfrac{\theta_1}{\theta_2}$, $\mu_2 = \dfrac{\theta_3}{\theta_2}$, $\mu_3 = \dfrac{\theta_4}{\theta_2}$, and

$$\hat{S}(W_0, W_1) = \frac{S(W_0, W_1) - p^2 q(\alpha_4^2 \beta_1 + \alpha_1^2 \beta_4)}{k^2}.$$

Condition: $\mu_2 = \dfrac{1}{\delta^3} > 0$ **where** $\delta > 0$**.**

Let $\mu_2 = \dfrac{1}{\delta^3} > 0$ and $\delta > 0$. Then choose scaling $W_0 \mapsto \dfrac{W_0}{\delta}$ and $W_1 \mapsto \dfrac{W_1}{\delta}$. If $\mu_3 = \delta$, $\mu_3 = -\delta$, and $\mu_3 = 0$, then based on (14), we have, respectively:

$$S_1(W_0, W_1) = \mu\left(\frac{1}{W_0} + \frac{1}{W_1}\right) + (W_0 + W_1) - \left(\frac{1}{W_0W_1}\right) - (W_0W_1), \tag{15}$$

$$S_2(W_0, W_1) = \mu\left(\frac{1}{W_0} + \frac{1}{W_1}\right) + (W_0 + W_1) - \left(\frac{1}{W_0W_1}\right) + (W_0W_1), \tag{16}$$

$$S_3(W_0, W_1) = \mu\left(\frac{1}{W_0} + \frac{1}{W_1}\right) + (W_0 + W_1) - \left(\frac{1}{W_0W_1}\right). \tag{17}$$

Condition: $\mu_2 = -\frac{1}{\delta^3} < 0$ **and** $\delta > 0$**.**

Let $\mu_2 = -\frac{1}{\delta^3} < 0$ and $\delta > 0$. Then choose scaling $W_0 \mapsto \frac{W_0}{\delta}$ and $W_1 \mapsto \frac{W_1}{\delta}$. If $\mu_3 = -\delta$, $\mu_3 = \delta$, and $\mu_3 = 0$, then based on (14), we have, respectively:

$$S_4(W_0, W_1) = \mu\left(\frac{1}{W_0} + \frac{1}{W_1}\right) + (W_0 + W_1) + \left(\frac{1}{W_0W_1}\right) - (W_0W_1), \tag{18}$$

$$S_5(W_0, W_1) = \mu\left(\frac{1}{W_0} + \frac{1}{W_1}\right) + (W_0 + W_1) + \left(\frac{1}{W_0W_1}\right) + (W_0W_1), \tag{19}$$

$$S_6(W_0, W_1) = \mu\left(\frac{1}{W_0} + \frac{1}{W_1}\right) + (W_0 + W_1) + \left(\frac{1}{W_0W_1}\right). \tag{20}$$

Condition: $\mu_2 = 0$**.**

$$S_7(W_0, W_1) = \mu\left(\frac{1}{W_0} + \frac{1}{W_1}\right) + (W_0 + W_1) - (W_0W_1), \tag{21}$$

$$S_8(W_0, W_1) = \mu\left(\frac{1}{W_0} + \frac{1}{W_1}\right) + (W_0 + W_1) + (W_0W_1), \tag{22}$$

$$S_9(W_0, W_1) = \mu\left(\frac{1}{W_0} + \frac{1}{W_1}\right) + (W_0 + W_1). \tag{23}$$

Condition: $\theta_2 = 0$ **and** $\theta_4 \neq 0$**.**

If $\theta_2 = 0$ and $\theta_4 \neq 0$, then the integral (14) can be written as:

$$\hat{S}(W_0, W_1) = \mu_4\left(\frac{1}{W_0} + \frac{1}{W_1}\right) - \mu_5\left(\frac{1}{W_0 W_1}\right) - (W_0 W_1), \tag{24}$$

where $\mu_4 = \frac{\theta_1}{\theta_4}$, and $\mu_5 = \frac{\theta_3}{\theta_4}$.

Let $\mu_5 = \frac{1}{\delta_4}$ and $\delta \neq 0$. Then choose scaling $W_0 \mapsto \frac{W_0}{\delta}$ and $W_1 \mapsto \frac{W_1}{\delta}$. Therefore, we have the following integrals:

$$S_{10}(W_0, W_1) = \mu\left(\frac{1}{W_0} + \frac{1}{W_1}\right) - \left(\frac{1}{W_0 W_1}\right) - (W_0 W_1), \tag{25}$$

$$S_{11}(W_0, W_1) = \mu\left(\frac{1}{W_0} + \frac{1}{W_1}\right) + \left(\frac{1}{W_0 W_1}\right) - (W_0 W_1), \tag{26}$$

$$S_{12}(W_0, W_1) = \mu\left(\frac{1}{W_0} + \frac{1}{W_1}\right) - (W_0 W_1). \tag{27}$$

Condition: $\theta_2 = \theta_4 = 0$**.**

For the case of $\theta_2 = \theta_4 = 0$, we have two normal forms of integrals normal. Firstly, if $\theta_3 = \frac{1}{\delta^4}$ and $\delta \neq 0$, then the integral (14) can be written as:

$$S_{13}(W_0, W_1) = \mu\left(\frac{1}{W_0} + \frac{1}{W_1}\right) - \left(\frac{1}{W_0 W_1}\right). \tag{28}$$

Secondly, if $\theta_3 = 0$, we have other normal integral function, i.e.

$$S_{14}(W_0, W_1) = \left(\frac{1}{W_0} + \frac{1}{W_1}\right). \tag{29}$$

Note that the level sets of the integral $S(W_0, W_1)$ in (13) for all values of the parameters are completely determined by the level sets of $S_j(W_0, W_1)$, where $j = 1, 2, ..., 14$.

3. Concluding Remark

Based on the above discussion, we conclude that the integral in three-dimensional maps derived from a ΔΔ-mKdV equation for all values of the parameters is completely determined by fourteen integral forms. After some derivations, three-dimensional traveling wave solution mapping is expressed into two-dimensional mapping, having solution on the level sets of fourteen integral functions $S_j(W_0, W_1)$, where $j = 1, 2, ..., 14$.

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