Generalization of U-Generator and M-Subgenerator Related to Category σ[M]

By Fitriani Fitriani

Journal of Mathematics Research; Vol. 10, No. 4; August 2018
ISSN 1916-9795 E-ISSN 1916-9809
Published by Canadian Center of Science and Education

Generalization of $\mathcal{U}$-Generator and M-Subgenerator Related to Category $\sigma[M]$

Fitriani[1,2], Indah Emilia Wijayanti[1] & Budi Surodjo[1]

[1] Department of Mathematics, Universitas Gadjah Mada, Yogyakarta, Indonesia

[2] Department of Mathematics, Universitas Lampung, Bandar Lampung, Indonesia

Correspondence: Fitriani, Department of Mathematics, Universitas Gadjah Mada, Yogyakarta, Indonesia.

Received: April 24, 2018 Accepted: May 9, 2018 Online Published: June 28, 2018

doi:10.5539/jmr.v10n4p101 URL: https://doi.org/10.5539/jmr.v10n4p101

Abstract

Let $\mathcal{U}$ be a non-empty set of R-modules. R-module N is generated by $\mathcal{U}$ if there is an epimorphism from $\oplus_\Lambda U_\lambda$ to N, where $U_\lambda \in \mathcal{U}$, for every $\lambda \in \Lambda$. R-module M is a subgenerator for N if N is isomorphic to a submodule of an M-generated module. In this paper, we introduce a $\mathcal{U}_V$-generator, where V be a submodule of $\oplus_\Lambda U_\lambda$, as a generalization of $\mathcal{U}$-generator by using the concept of V-coexact sequence. We also provide a $\mathcal{U}_V$-subgenerator motivated by the concept of M-subgenerator. Furthermore, we give some properties of $\mathcal{U}_V$-generated and $\mathcal{U}_V$-subgenerated modules related to category $\sigma[M]$. We also investigate the existence of pullback and pushout of a pair of morphisms of $\mathcal{U}_V$-subgenerated modules. We prove that the collection of $\mathcal{U}_V$-subgenerated modules is closed under submodules and factor modules.

Keywords: $\mathcal{U}$-generator, $\mathcal{U}_V$-generator, V-coexact sequences, M-subgenerator, $\mathcal{U}_V$-subgenerator

1. Introduction

The concept of exact sequences of R-modules and R-module homomorphisms is a useful tool in the study of modules. A sequence $A \to B \to C$ is exact if $Imf = Kerg(= g^{-1}(0))$. Davvaz and Parnian-Garamaleky (1999) provide the generalization of exact sequences, i.e. quasi-exact sequences. They substitute the submodule $\{0\}$ to any submodule U of C.

Then Anvariyeh dan Davvaz (2005) investigate further results about quasi-exact sequences. They also introduce the generalization of Schanuel's Lemma. Furthermore, Davvaz and ShabaniSolt (2002) give a generalization of some notions in homological algebra. In 2002, Anvariyeh and Davvaz provide U-split sequences. They also establish several connections between U-split sequences and projective modules.

Motivated by the definition of U-exact and V-coexact sequence, Fitriani et al. (2016) provide an X-sub exact sequence, which is a generalization of exact sequence. In 2017, they introduce X-sublinearly independent module by using the concept of X-sub exact sequence.

Let $\mathcal{U}$ be a non-empty set of R-modules. An R-module N is generated by $\mathcal{U}$ if there is an epimorphism from $\oplus_\Lambda U_\lambda$ to N, where $U_\lambda \in \mathcal{U}$, for every $\lambda \in \Lambda$. The trace of $\mathcal{U}$ is defined by $Tr(\mathcal{U}, M) = \sum\{Imh | h : U \to M$, for some $U \in \mathcal{U}\}$. If $\mathcal{U} = \{U\}$ is a singleton, then $Tr(U, M) = \sum\{Imh | h \in Hom_R(U, M)\}$. $Tr(\mathcal{U}, M)$ is the unique largest submodule L of M generated by $\mathcal{U}$ (Wisbauer, 1991). Clearly $Tr(\mathcal{U}, M) = M$ if and only if $\mathcal{U}$ generates M (Anderson & Fuller, 1992). For an indexed set $(M_\alpha)_A$ of modules and class of modules $\mathcal{U}$, the direct sum of the traces $Tr(\mathcal{U}, M)$ is contained in $\oplus_A M_\alpha$. The trace of M in an R-module N is the sum of all M-generated submodules of N (Clark et al., 2006).

Proposition 1 (Wisbauer, 1991) *If $(M_\alpha)_{\alpha\in A}$ is an indexed set of modules, then for each module M*

$$Tr(\mathcal{U}, \oplus_A M_\alpha) = \oplus_A Tr(\mathcal{U}, M_\alpha).$$

Furthermore, an M-subgenerated module is defined as follows.

Definition 2 (Wisbauer, 1991) Let M be an R-module. We say that an R-module N is subgenerated by M, or that M is a subgenerator for N, if N is isomorphic to a submodule of an M-generated module.

A subcategory C of R-MOD is said to be subgenerated by M, or M is a subgenerator for C, if every object in C is subgenerated by M. Category $\sigma[M]$ is the full subcategory of $R - MOD$ whose objects are all R-modules subgenerated by M. This category is a category closely connected to M and hence reflecting properties of M.

The properties of $\sigma[M]$ given by the following proposition:

Proposition 3 (Wisbauer, 1991) *For an R-module M we have:*

1. *For N in $\sigma[M]$, all factor modules and submodules of N belong to $\sigma[M]$, i.e. $\sigma[M]$ has kernels and cokernels.*
2. *The direct sum of a family of modules in $\sigma[M]$ belong to $\sigma[M]$ and is equal to the coproduct of these modules in $\sigma[M]$.*
3. *Pullback and pushout of morphisms in $\sigma[M]$ belong to $\sigma[M]$.*

As a generalization of exact sequence of R-modules, Anvanriyeh and Davvaz (1999) defined U-exact sequences as follows: A sequence of R-modules $A \xrightarrow{f} B \xrightarrow{g} C$ if there exists a submodule U of C such that $Im\ f = g^{-1}(U)$. In this case, the sequence is said to be U-exact (at B). If $f(V) = Ker\ g$, where V is a submodule of A, then the sequence is said to be V-coexact.

Let $\mathcal{U}$ be a family of R-modules and V be a submodule of $\oplus_\Lambda U_\lambda$, where $U_\lambda \in \mathcal{U}$, for every $\lambda \in \Lambda$. The aim of this paper is to generalize the concept of $\mathcal{U}$-generator to a $\mathcal{U}_V$-generator, where V is a submodule of $\oplus_\Lambda U_\lambda$. Furthermore, we provide a $\mathcal{U}_V$-subgenerator as a generalization of M-subgenerator. We also investigate the properties of $\mathcal{U}_V$-generated modules and $\mathcal{U}_V$-subgenerated modules related to the properties of the category $\sigma[M]$.

2. Results

2.1 $\mathcal{U}_V$-Generated Modules

Let $\mathcal{U}$ be a family of R-modules. It is possible that an R-module M is not a $\mathcal{U}$-generated module, i.e. there no epimorphism from $\oplus_\Lambda U_\lambda$ to M, but we can define an epimorphism from a submodule V $\oplus_\Lambda U_\lambda$ to M. Therefore we can generalize the concept of a $\mathcal{U}$-generated module to a $\mathcal{U}_V$-generated module by using the definition of V-coexact sequence.

Definition 4 Let $\mathcal{U}$ be a non-empty set of R-modules, V be a submodule of $\oplus_\Lambda U_\lambda$, where $U_\lambda \in \mathcal{U}$, for every $\lambda \in \Lambda$. We say that an R-module N is generated by $\mathcal{U}_V$ if there exists an epimorphism $V \to N \to 0$.

A set $\{U_\lambda\}_\Lambda$ is called $\mathcal{U}_V$-generator for N. Furthermore, the set $\{U_\lambda\}_\Lambda$ is called minimal $\mathcal{U}_V$-generator for N if

$$\Lambda = min\{\Lambda_V | N \text{ is } \mathcal{U}_V - \text{generated}, V \subseteq \oplus_{\Lambda_V} U_\lambda\}.$$

If we take $V = \oplus_\Lambda U_\lambda$, then a $\mathcal{U}_V$-generated module is a $\mathcal{U}$-generated module. Clearly, every $\mathcal{U}$-generated module is $\mathcal{U}_V$-generated. But, a $\mathcal{U}_V$-generated module need not be a $\mathcal{U}$-generated. For example, if we take $\mathcal{U} = \{\mathbb{Q}\}$, then $\mathbb{Z}$-module $\mathbb{Z}$ is a $\mathcal{U}_\mathbb{Z}$-generated module. But, we can not define an epimorphism from $\mathbb{Q}$ to $\mathbb{Z}$ and hence $\mathbb{Z}$-module $\mathbb{Z}$ is not a $\mathcal{U}$-generated module.

Now, we give some examples of $\mathcal{U}_V$-generated modules. *Example 1*

1. Let $\mathcal{U}$ be the set of all free R-modules and P be projective R-module. Since P is projective, P is a direct summand of a free module F. Hence P is $\mathcal{U}_F$-generated module.
2. Let $\mathcal{U} = \{\mathbb{Z}_p | p prime\}$, a family of $\mathbb{Z}$-modules. $\mathbb{Z}$-module $\mathbb{Z}_6$ is a $\mathcal{U}_V$-generated, where $V = \mathbb{Z}_2 \oplus \mathbb{Z}_3$. In general, $\mathbb{Z}$-module $\mathbb{Z}_{pq}$ is a $\mathcal{U}_V$-generated, where $V = \mathbb{Z}_p \oplus \mathbb{Z}_q$, p and q are relative prime.
3. Let $\mathcal{U} = \{\mathbb{Q}\}$. $\mathbb{Z}$-module $\mathbb{Z}_n$, $n \geq 2$, is $\mathcal{U}_V$-generated, where $V = \mathbb{Z}$.
4. Let R be a commutative ring with unit and $\mathcal{U} = \{U_\lambda\}_\Lambda$ be a family of R-modules, where $U_\lambda = Hom_R(R, M_\lambda)$, for every $\lambda \in \Lambda$.
   Based on Adkins & Weintraub (1992), we can define
   $$\phi : Hom_R(R, M) \to M,$$
   where $\phi(f) := f(1)$. Then M_λ is $\mathcal{U}_{U_\lambda}$-generated.
5. Let $\mathcal{U} = \{\mathbb{Z}_n | n \in \mathbb{Z}\}$ be a family of $\mathbb{Z}$-modules. Let $M = \mathbb{Z}_4^{(\mathbb{N})}$ and $N = \mathbb{Z}_2 \oplus M$ be $\mathbb{Z}$-modules. Then M is $\mathcal{U}_N$-generated and N is $\mathcal{U}_M$-generated.

If there exists a finite index set $E \subseteq \Lambda$ such that M is $\mathcal{U}_V$-generated and V is a submodule of $\oplus_E U_e$, then we define a finitely $\mathcal{U}_V$-generated module as follows:

Definition 5 Let $\mathcal{U}$ be a non-empty set of R-modules and N be an R-module. If there exists a finite index set $E \subseteq \Lambda$ such that $V \subseteq \oplus_E U_e$ and M is $\mathcal{U}_V$-generated, then R-module N is said to be finitely $\mathcal{U}_V$-generated.

Example 2 Let $\mathcal{U} = \{\mathbb{Z}_p | p \text{ prime}\}$ be a family of $\mathbb{Z}$-modules. $\mathbb{Z}$-module $\mathbb{Z}_{pq}$ is a finitely $\mathcal{U}_V$-generated, where $V = \mathbb{Z}_p \oplus \mathbb{Z}_q$, p and q are relative prime.

Then, we will give some basic properties of $\mathcal{U}_V$-generated modules. Let $\mathcal{U}$ be a non-empty set of R-modules and N be an R-module. We define:

$$\mathcal{U}(N) = \{V \subseteq \oplus_\Lambda U_\lambda, U_\lambda \in \mathcal{U} | N \text{ is } \mathcal{U}_V\text{-generated}\}.$$

In this set, we collect all submodules V of $\oplus_\Lambda U_\lambda$ such that N is a $\mathcal{U}_V$-generated module. In the following proposition, we prove that if $V_\lambda \in \mathcal{U}(N_\lambda)$ for every $\lambda \in \Lambda$, then $\oplus_\Lambda V_\lambda \in \mathcal{U}(\oplus_\Lambda N_\lambda)$.

Proposition 6 *Let $\mathcal{U}$ be a non-empty set of R-modules, V_λ be a submodule of $\oplus_\Lambda U_\lambda$, where $U_\lambda \in \Lambda$ for every $\lambda \in \Lambda$. If N_λ is $\mathcal{U}_{V_\lambda}$-generated, for every $\lambda \in \Lambda$, then $\oplus_\Lambda N_\lambda$ is $\mathcal{U}_{\oplus_\Lambda V_\lambda}$-generated.*

Proof. Since N_λ is $\mathcal{U}_{V_\lambda}$-generated, for every $\lambda \in \Lambda$, the sequences $V_\lambda \to N_\lambda \to 0$ is exact for every $\lambda \in \Lambda$. Therefore, the sequence

$$\oplus_\Lambda V_\lambda \to \oplus_\Lambda N_\lambda \to 0$$

is exact. Hence, $\oplus_\Lambda N_\lambda$ is $\mathcal{U}_{\oplus_\Lambda V_\lambda}$-generated. So, we can say that if $V_\lambda \in \mathcal{U}(N_\lambda)$ for every $\lambda \in \Lambda$, then $\oplus_\Lambda V_\lambda \in \mathcal{U}(\oplus_\Lambda N_\lambda)$.

As a corollary of Proposition 6, we obtain:

Corollary 7 *Let $\mathcal{U}$ be a non-empty set of R-modules. If R-module N_i is $\mathcal{U}_{V_i}$-generated for every $i = 1, 2, ..., n$, then $\oplus_{i=1}^n X_i$ is $\mathcal{U}_{\oplus_{i=1}^n V_i}$-generated, where V_i be submodule of $\oplus_\Lambda U_\lambda$, $U_\lambda \in \Lambda$, for every $i = 1, 2, ..., n$ and $\lambda \in \Lambda$.*

In the following proposition, we will show that if $V \in \mathcal{U}(N)$, for an R-module N, then V is in $\mathcal{U}(N')$, for every homomorphic image N' of N.

Proposition 8 *Let $\mathcal{U}$ be a non-empty set of R-modules. If R-module N is $\mathcal{U}_V$-generated, then N' is $\mathcal{U}_V$-generated, for every homomorphic image N' of N.*

Proof. If R-module N is $\mathcal{U}_V$-generated, then the sequence

$$\oplus_\Lambda U_\lambda \xrightarrow{f} N \to 0$$

is V-coexact. Let N' be homomorphic image of N, then there is an epimorphism $p : N \to N'$. Hence, $g = p \circ f$ is a homomorphism from V to N'. Since f and p are epimorphisms, then g is an epimorphism. So, N' is $\mathcal{U}_V$-generated.

In the next proposition, we will prove that $\mathcal{U}_V(N)$ is closed under direct sum, i.e. if V_λ is in $\mathcal{U}(N)$ for every $\lambda \in \Lambda$, then $\oplus_{\lambda \in \Lambda} V_\lambda$ is in $\mathcal{U}(N)$.

Proposition 9 *Let $\mathcal{U}$ be a non-empty set of R-modules and V_α be submodules of $\oplus_\Lambda U_\lambda$, $U_\lambda \in \mathcal{U}$ for every $\lambda \in \Lambda$. If R-module M is $\mathcal{U}_{V_\alpha}$-generated, for every $\alpha \in A$, then M is $\mathcal{U}_{\oplus_{\alpha \in A} V_\alpha}$-generated.*

Proof. Since R-module M is $\mathcal{U}_{V_\alpha}$-generated for every $\alpha \in A$, there is an epimorphism f_α such that the sequence: $V_\alpha \xrightarrow{f_\alpha} M \to 0$ is exact for every $\alpha \in A$. We can define $f : \oplus_{\alpha \in A} V_\alpha \to M$, where $f((v_\alpha)_A) = f_{\alpha_i}(v_{\alpha_i})$, $\alpha_i \in A$. From this, we have f is an epimorphism from $\oplus_{\alpha \in A}$ to M. Hence, M is $\mathcal{U}_{\oplus_{\alpha \in A} V_\alpha}$-generated.

As a corollary of Proposition 9, we obtain:

Proposition 10 *Let $\mathcal{U}$ be a non-empty set of R-modules. If R-module M is $\mathcal{U}_{V_i}$-generated for every $i = 1, 2, ..., n$, then M is $\mathcal{U}_{\oplus_{i=1}^n V_i}$-generated, where V_i be submodule of $\oplus_\Lambda U_\lambda$ for every $i = 1, 2, ..., n$.*

If $V_2 \in \mathcal{U}(N)$ and $V_1 \in \mathcal{U}(V_2)$ i.e. N is $\mathcal{U}_{V_1}$-generated and V_2 is $\mathcal{U}_{V_1}$-generated, with modules V_1 and V_2 are submodules of $\oplus_\Lambda U_\lambda$, $U_\lambda \in \mathcal{U}$, then we will show that $V_1 \in \mathcal{U}(N)$, i.e. N is $\mathcal{U}_{V_1}$-generated module.

Proposition 11 *Let $\mathcal{U}$ be a non-empty set of R-modules. If R-module N is $\mathcal{U}_{V_2}$-generated and V_2 is $\mathcal{U}_{V_1}$-generated, then N is $\mathcal{U}_{V_1}$-generated, where V_1, V_2 be submodules of $\oplus_\Lambda U_\lambda$, $U_\lambda \in \Lambda$, for every $\lambda \in \Lambda$.*

Proof. Since N is $\mathcal{U}_{V_2}$-generated and V_2 is $\mathcal{U}_{V_1}$-generated, there exists epimorphisms $\alpha : V_2 \to N$ and $\beta : V_1 \to V_2$. So, we can define $g = \alpha \circ \beta : V_1 \to N$. Since α and β are epimorphisms, g is an epimorphism. Finally, N is $\mathcal{U}_{V_1}$-generated.

As a corollary we obtain:

Corollary 12 *Let $\mathcal{U}$ be a non-empty set of R-modules. If R-module N is $\mathcal{U}_V$-generated and V is $\mathcal{U}$-generated, then N is $\mathcal{U}$-generated, where V be submodule of $\oplus_\Lambda U_\lambda$, $U_\lambda \in \Lambda$, for every $\lambda \in \Lambda$.*

Proof. Since R-module N is $\mathcal{U}_V$-generated and V is $\mathcal{U}$-generated, by Proposition 11, we have N is $\mathcal{U}_{\oplus_\Lambda U_\lambda}$-generated. In other words, N is $\mathcal{U}$-generated.

Corollary 12 *Let $\mathcal{U}$ be a non-empty set of R-modules and $V \subset \oplus_\Lambda U_\lambda$, with modules $U_\lambda \in \mathcal{U}$. If R-module M is $\mathcal{U}_V$-subgenerated and V is a $\mathcal{U}$-generated module, then the sequence*

$$\oplus_\Lambda U_\lambda \to M \to 0$$

is V-coexact.

Proof. Since R-module M is $\mathcal{U}_V$-subgenerated, there is an epimorphism $\alpha : V \to M$. By asumption, V is a $\mathcal{U}$-generated module. So, there is an epimorphism $\pi : \oplus_\Lambda U_\lambda \to V$. Hence, $g = \alpha \circ \pi$ is an epimorphism from $\oplus_\Lambda U_\lambda$ to M such that $g|_V = \alpha$. We have the sequence

$$\oplus_\Lambda U_\lambda \xrightarrow{g} M \to 0$$

is V-coexact.

Corollary 13 *Let $\mathcal{U}$ be a non-empty set of semisimple R-modules. If R-module M is $\mathcal{U}_V$-generated, then M is $\mathcal{U}$-generated, where V is a submodule of $\oplus_\Lambda U_\lambda$.*

Proof. We assume that R-module M is a $\mathcal{U}_V$-generated. Since every submodule of semisimple module $\oplus_\Lambda U_\lambda$ is a direct summand, M is $\mathcal{U}$-generated by using Proposition 11.

2.2 $\mathcal{U}_V$-Subgenerated Modules

We already know that an M-subgenerated module is a generalization of a $\mathcal{U}$-generated module. In the similar way, we can obtain a $\mathcal{U}_V$-subgenerated module as a generalization of $\mathcal{U}_V$-generated module.

Definition 14 *Let $\mathcal{U}$ be a non-empty set of R-modules, V be a submodule of $\oplus_\Lambda U_\lambda$. We say that an R-module N is subgenerated* by $\mathcal{U}_V$ *if N isomorphic to a submodule of a $\mathcal{U}_V$-generated module*.

M-subgenerated module is a special case of $\mathcal{U}_V$-subgenerated modules by taking $\mathcal{U} = \{M\}$ and $V = M^{(\Lambda)}$. By Definition 14, every $\mathcal{U}_V$-generated module is a $\mathcal{U}_V$-subgenerated module. But the converse need not be true. For example, let $\mathcal{U}$ the set of all $\mathbb{Z}$-modules. $\mathbb{Z}$-module $\mathbb{Z}$ is $\mathcal{U}_\mathbb{Q}$-subgenerated. But, $\mathbb{Z}$-module $\mathbb{Z}$ is not $\mathcal{U}_\mathbb{Q}$-generated.

Proposition 15 *Let $\mathcal{U}$ be a non-empty set of R-modules and V be a submodule of $\oplus_\Lambda U_\lambda$. If R-module N is $\mathcal{U}_V$-subgenerated and N is a direct summand of a $\mathcal{U}_V$-generated module, then N is $\mathcal{U}_V$-generated module.*

Let $\mathcal{U}$ be a non-empty set of R-modules and N be an R-module. In $\sigma[M]$, Wisbauer (1991) collect all R-modules subgenerated by M. In the similar way, we will collect all R-modules subgenerated by $\mathcal{U}_V$, we denote it by $\sigma_V(\mathcal{U})$:

$$\sigma_V(\mathcal{U}) = \{N|N \text{ is } \mathcal{U}_V\text{-subgenerated}\}.$$

The full subcategory $\sigma[M]$ of $R-MOD$ is a special case of $\sigma_V(\mathcal{U})$ by taking $\mathcal{U} = \{M\}$ and $V = M^{(\Lambda)}$. Next, we will show that $\sigma_V(\mathcal{U})$ is closed under submodules and factor modules.

Proposition 16 *Let $\mathcal{U}$ be a non-empty set of R-modules and V be a submodule of $\oplus_\Lambda U_\lambda$. If R-module N is $\mathcal{U}_V$-subgenerated, then N' is a $\mathcal{U}_V$-subgenerated module, for every submodule N' of N.*

Proof. Since N is a $\mathcal{U}_V$-subgenerated, then N isomorphic to a submodule of a $\mathcal{U}_V$-generated module. So, there is an epimorphism:

$$V \xrightarrow{f} K \to 0$$

and N is isomorphic to a submodule of K. Let N' be a submodule of N. We have N' is somorphic to a submodule of K and N' is a $\mathcal{U}_V$-subgenerated module.

Proposition 17 *Let $\mathcal{U}$ be a non-empty set of R-modules and V be a submodule of $\oplus_\Lambda U_\lambda$. If R-module N is $\mathcal{U}_V$-subgenerated, then N/L is $\mathcal{U}_V$-subgenerated module, for every factor module N/L of N.*

Proof. Since N is a $\mathcal{U}_V$-subgenerated, there is a $\mathcal{U}_V$-generated module K and an epimorphism:

$$V \xrightarrow{f} K \to 0$$

and N is isomorphic to a submodule of K. Let L be a submodule of N. We have L is isomorphic to a submodule of K and hence N/L is is isomorphic to a submodule of K/L', where $L \cong L'$. Since K/L' is a $\mathcal{U}_V$-generated module, we get N/L is a $\mathcal{U}_V$-subgenerated module.

As a corolarry of Proposition 16 and 17, we obtain:

Corollary 18 *Let $\mathcal{U}$ be a non-empty set of R-modules, V be a submodule of $\oplus_\Lambda U_\lambda$ and*

$$0 \to K \to L \to M \to 0$$

be an exact sequence of R-modules. If L is a $\mathcal{U}_V$-subgenerated module, then K and M are $\mathcal{U}_V$-subgenerated modules.

If R-module N_1 and N_2 are $\mathcal{U}_V$-subgenerated, then we have two exact sequences: $V \to M_1 \to 0$ and $V \to M_1 \to 0$. Furthermore, N_1 and N_2 are isomorphic to submodules of M_1 and M_2, respectively. Hence $Tr(V, M_1) = M_1$ and $Tr(V, M_2) = M_2$. By Proposition 1, we have $Tr(V, M_1 \oplus M_2) = Tr(V, M_1) \oplus Tr(V, M_2) = M_1 \oplus M_2$. But, $N_1 \oplus N_2$ need not be a $\mathcal{U}_V$-subgenerated module. By Proposition 6, we have $N_1 \oplus N_2$ is a $\mathcal{U}_{V_1 \oplus V_2}$-subgenerated module.

In the following proposition, we will show the existence of pullback and pushout of a pair of morphisms of $\mathcal{U}_V$-subgenerated modules.

Proposition 19 *Let $\mathcal{U}$ be a non-empty set of R-modules. If N_1 is $\mathcal{U}_{V_1}$-subgenerated and N_2 is $\mathcal{U}_{V_2}$-subgenerated, then pullback of $f_1 : N_1 \to N$ and $f_2 : N_2 \to N$ is $\mathcal{U}_{V_1 \oplus V_2}$-subgenerated module, where V_1, V_2 are submodules of $\oplus_\Lambda U_\lambda$.*

Proof. Since N_1 is $\mathcal{U}_{V_1}$-subgenerated and N_2 is $\mathcal{U}_{V_2}$-subgenerated, N_1 and N_2 are $\mathcal{U}_{V_1 \oplus V_2}$-subgenerated. Let $f_1 : N_1 \to M$, $f_2 : N_2 \to M$ be a pair of morphisms of $\mathcal{U}_{V_1 \oplus V_2}$-subgenerated modules. We have $N_1 \oplus N_2$ is $\mathcal{U}_{V_1 \oplus V_2}$-subgenerated module. Based on Wisbauer (1991), pullback of (f_1, f_2) is a submodule of $N_1 \oplus N_2$. Since every submodule of $\mathcal{U}_{V_1 \oplus V_2}$-subgenerated module is a $\mathcal{U}_{V_1 \oplus V_2}$-subgenerated, the pullback of (f_1, f_2) is a $\mathcal{U}_{V_1 \oplus V_2}$-subgenerated module.

Proposition 20 *Let $\mathcal{U}$ be a non-empty set of R-modules. If N_1 is $\mathcal{U}_{V_1}$-subgenerated and N_2 is $\mathcal{U}_{V_2}$-subgenerated, then pushout of $g_1 : X \to N_1$ and $g_2 : X \to N_2$ is $\mathcal{U}_{V_1 \oplus V_2}$-subgenerated module, where V_1, V_2 are submodules of $\oplus_\Lambda U_\lambda$.*

Proof. Since N_1 is $\mathcal{U}_{V_1}$-subgenerated and N_2 is $\mathcal{U}_{V_2}$-subgenerated, N_1 and N_2 are $\mathcal{U}_{V_1 \oplus V_2}$-subgenerated. Let $g_1 : X \to N_1$, $g_2 : X \to N_2$ be a pair of morphisms of $\mathcal{U}_{V_1 \oplus V_2}$-subgenerated module. We have $N_1 \oplus N_2$ is $\mathcal{U}_{V_1 \oplus V_2}$-subgenerated modules. Based on Wisbauer (1991), pushout of (g_1, g_2) is a factor module of $N_1 \oplus N_2$. Since every factor module of $\mathcal{U}_{V_1 \oplus V_2}$-subgenerated module is a $\mathcal{U}_{V_1 \oplus V_2}$-subgenerated, the pushout of (g_1, g_2) is a $\mathcal{U}_{V_1 \oplus V_2}$-subgenerated module.

A submodule N of R-module M is called fully invariant if $f(N)$ is contained in N for every R-endomorphism f of M. M is called a duo module provided every submodule of M is fully invariant (Özcan et al., 2006).

The following theorem shows that the properties of R-modules in $\sigma_V\mathcal{U}$ are reflecting the properties of V.

Theorem 21 *Let $\mathcal{U}$ be a non-empty set of R-modules and V be a submodule of $\oplus_\Lambda U_\lambda$, $U_\lambda \in \mathcal{U}$, for every $\lambda \in \Lambda$.*

1. *If R-module U is V-injective (V-projective), then U is N-injective (N-projective), for every $N \in \sigma_V(\mathcal{U})$.*
2. *If V is semisimple, then every module in $\sigma_V(\mathcal{U})$ is semisimple.*
3. *If V is Noetherian (Artinian), then N is Noetherian (Artinian), for every $N \in \sigma_V(\mathcal{U})$.*
4. *If V is a duo module, quasi-injective and quasi-projective, then N is a duo module, V-projective and V-injective, for every $N \in \sigma_V(\mathcal{U})$.*

Proof.

1. Let $N \in \sigma_V\mathcal{U}$. Then N is isomorphic to a submodule of $\mathcal{U}_V$-generated module, say M. We have the following exact sequence:

$$0 \to Ker\ f \to V \xrightarrow{f} M \to 0.$$

Based on Wisbauer (1991), if U is V-injective, then U is M-injective. Therefore by Wisbauer (1991) 16.3, U is N-injective.
2 and 3 can be shown in a similar way to 1.

4 Based on Özcan et. al. (2006), if V is a duo module and quasi-injective, then every submodule of V is a duo module. Futhermore, if V is a duo module and quasi-projective, then every homomorphic image of V is a duo module. From 1, we have N is V-projective and V-injective, for every N in $\sigma_V(\mathcal{U})$.

3. Conclusions

A $\mathcal{U}_V$-generator is a generalization of $\mathcal{U}$-generator. If an R-module N is $\mathcal{U}_V$-generated, then every homomorphic image of N is also $\mathcal{U}_V$-generated. Furthermore, direct sums of $\mathcal{U}_V$-generated R-modules are $\mathcal{U}_{V'}$-generated, for some submodules V' of $\oplus_\Lambda U_\lambda$. In the set $\mathcal{U}(N)$, we collect all submodules V of $\oplus_\Lambda U_\lambda$ such that N is a $\mathcal{U}_V$-generated module and we have $\mathcal{U}(N)$ is closed under direct sums.

In the set $\sigma_V(\mathcal{U})$, we collect all R-modules subgenerated by $\mathcal{U}_V$. The full subcategory $\sigma[M]$ of $R - MOD$ is a special case of $\sigma_V(\mathcal{U})$ by taking $\mathcal{U} = \{M\}$ and $V = M^{(\Lambda)}$. The set $\sigma_V(\mathcal{U})$ is closed under submodules and factor modules. Furthermore, the properties of R-modules in $\sigma_V(\mathcal{U})$ are reflecting the properties of V.

Acknowledgements

The authors thank the Ministry of Research, Technology and the Higher Education Republic of Indonesia, due to the funding of this work through the scheme of Research of Doctoral Dissertation with contract number 385/UN26.21/PN/2018. The authors also thank the referees for useful comments and suggestions.

References

Adkins, W. A., & Weintraub S. H. (1992). *Algebra "An Approach via Module Theory"*. New York : Springer-Verlag. https://doi.org/10.1007/978-1-4612-0923-2

Anderson, F. W., & Fuller, K. R. (1992). *Rings and categories of Modules*. New York: Springer-Verlag. https://doi.org/10.1007/978-1-4612-4418-9

Anvariyeh, S. M., & Davvaz, B. (2005). On Quasi-Exact Sequences. *Bull. Korean Math. Soc., 42*(1), 149-155. https://doi.org/10.4134/BKMS.2005.42.1.149

Anvariyeh, S. M., & Davvaz, B. (2002). U-Split-Exact Sequences. *Far East J. Math. Sci.(FJMS), 4*(2), 209-219.

Clark, J., Lomp, C., Vanaja, N., & Wisbauer, R. (2006). *Lifting Modules*. Switzerland: Birkhauser Verlag. https://doi.org/10.1007/3-7643-7573-6

Davvaz, B., & Parnian-Garamaleky, Y. A. (1999). A Note on Exact Sequences, *Bull.Malays. Math. Sci. Soc. 22*(1), 53-56.

Davvaz, B., & Shabani-Solt, H. A. (2002). Generalization of Homological Algebra. *J. Korean Math. Soc. 39*(6), 881-898.

Fitriani, Surodjo, B., & Wijayanti, I. E. (2016). On X-sub-exact Sequences. *Far East J. Math. Sci.(FJMS) 100*(7), 1055-1065. http://dx.doi.org/10.17654/MS100071055

Fitriani, Surodjo, B., & Wijayanti, I. E. (2017). On X-sub-linearly Independent Modules. *J. Phys.: Conf. Ser. 893*, 012008. https://doi.org/10.1088/1742-6596/893/1/012008

Özcan, A. Ç., Harmanci, A., & Smith, P. F. (2006). Duo Modules. *Glasgow Math. J., 48*, 533-545. https://doi.org/10.1017/S0017089506003260

Wisbauer, R. (1991). *Foundation of Module and Ring Theory*. Philadelphia: Gordon and Breach.

Generalization of U-Generator and M-Subgenerator Related to Category σ[M]

ORIGINALITY REPORT

22%

SIMILARITY INDEX

PRIMARY SOURCES

1	Abdoul DIALLO, Papa DIOP, Mamadou BARRY. "Modules Whose Nonzero Endomorphisms Have E-small Kernels", Journal of Mathematics Research, 2018 Crossref	59 words — 2%
2	www.iwnest.com Internet	47 words — 1%
3	www.tandfonline.com Internet	41 words — 1%
4	jmre.dlut.edu.cn Internet	36 words — 1%
5	Jawad Y. Abuhlail. "Rational Modules for Corings", Communications in Algebra, 2003 Crossref	33 words — 1%
6	KHORAMDEL, Medhi and HESARİ, Saboura Dolati Hesari. "Some notes on Dedekind modules", TUBITAK, 2011. Publications	31 words — 1%
7	Najmeh Dehghani, Fatma A. Ebrahim, S. Tariq Rizvi. "On the Schröder–Bernstein property for modules", Journal of Pure and Applied Algebra, 2019 Crossref	31 words — 1%
8	openaccess.iyte.edu.tr Internet	31 words — 1%

Algebra, 2015.

9	Crossref	30 words — 1%
10	www.math.nthu.edu.tw Internet	28 words — 1%
11	Inaam Mohammed Ali Hadi, Rana Noori Majeed. "On P-duo modules", International Journal of Algebra, 2014 Crossref	26 words — 1%
12	"Algebra and its Applications", Springer Science and Business Media LLC, 2016 Crossref	26 words — 1%
13	JAWAD ABUHLAIL, CHRISTIAN LOMP. "ON THE NOTION OF STRONG IRREDUCIBILITY AND ITS DUAL", Journal of Algebra and Its Applications, 2013 Crossref	24 words — 1%
14	www.ccsenet.org Internet	24 words — 1%
15	Mauricio Gabriel Medina-Bárcenas, Luis Angel Zaldívar-Corichi, Martha Lizbeth Shaid Sandoval-Miranda. "A generalization of quantales with applications to modules and rings", Journal of Pure and Applied Algebra, 2016 Crossref	18 words — < 1%
16	A. Çiğdem Özcan. " The Torsion Theory Cogenerated by δ– -Small Modules and GCO-Modules ", Communications in Algebra, 2007 Crossref	17 words — < 1%
17	Algebra and Applications, 2003. Crossref	17 words — < 1%
18	Joseph J. Rotman. "Special Modules", An Introduction to Homological Algebra, 2009 Crossref	16 words — < 1%
19	scialert.net Internet	15 words — < 1%

20	users.metu.edu.tr Internet	14 words — < 1%
21	www.ams.org Internet	13 words — < 1%
22	Nışanci Türkmen. "A generalization of modules with the property (P*)", Mathematica Moravica, 2017 Crossref	13 words — < 1%
23	I Rosyida, Widodo, D Indriati. "On total irregularity strength of caterpillar graphs with two leaves on each internal vertex", Journal of Physics: Conference Series, 2018 Crossref	13 words — < 1%
24	cadjds.org Internet	12 words — < 1%
25	Dimensions of Ring Theory, 1987. Crossref	12 words — < 1%
26	J. Escoriza, B. Torrecillas. "Multiplication objects in commutative grothendieck categories", Communications in Algebra, 1998 Crossref	11 words — < 1%
27	Jawad Y. Abuhlail. "Dual Entwining Structures and Dual Entwined Modules", Algebras and Representation Theory, 2005 Crossref	10 words — < 1%
28	Constantin Nastasescu, Freddy Van Oystaeyen. "Methods of Graded Rings", Springer Science and Business Media LLC, 2004 Crossref	10 words — < 1%
29	repository.lppm.unila.ac.id Internet	10 words — < 1%
30	Rachid Tribak. "Dual CS-Rickart Modules over Dedekind Domains", Algebras and Representation	9 words — < 1%

Theory, 2019
Crossref

31 Ana M. De Viola-Prioli, Jorge E. Viola-Prioli, Robert Wisbauer. "Module categories with linearly ordered closed subcategories", Communications in Algebra, 1994
Crossref
9 words — < 1%

32 K-Monographs in Mathematics, 1996.
Crossref
8 words — < 1%

33 BA, Alhousseynou, and Oumar Diankha. "On FGS-Modules", Journal of Mathematics Research, 2013.
Crossref
8 words — < 1%

34 Hwankoo Kim. " Module-Theoretic Characterizations of -Linkative Domains ", Communications in Algebra, 2008
Crossref
8 words — < 1%

35 ejmathsci.org
Internet
8 words — < 1%

36 Maria Elisa Antunes Sim[otilde]es, Patrick F. Smith. "Dirct products satisfying the ascending chain condition for submodules with a bounded number of generators", Communications in Algebra, 1995
Crossref
8 words — < 1%

37 N. Vanaja. "Lifting Property of Direct Sums of Hollow Modules", Communications in Algebra, 2007
Crossref
8 words — < 1%

38 www.sci-int.com
Internet
8 words — < 1%

39 Reza Aminizadeh, Hamid Rasouli, Abolfazl Tehranian. "Quasi-exact Sequences of S-Acts", Bulletin of the Malaysian Mathematical Sciences Society, 2018
Crossref
8 words — < 1%

40 S Wahyuni, I E Wijayanti, N Hijriati. "On representation of polynomial ring on a vector space via a linear

transformation", Journal of Physics: Conference Series, 2017
Crossref
8 words — < 1%

41 cnf.sttu.ac.ir
Internet
8 words — < 1%

42 koreascience.or.kr
Internet
8 words — < 1%

43 Diompy, Mankagna Albert, Oumar Diankha, and Mamadou Sanghare. "Some Characterizations of I-modules", Journal of Mathematics Research, 2012.
Crossref
7 words — < 1%

44 A. Çirdem Özcan. "The Torsion Theory Generated by M-Small Modules", Algebra Colloquium, 06/2003
Crossref
7 words — < 1%

45 "Preliminaries", Algebra and Applications, 2008
Crossref
7 words — < 1%

46 Edgar E. Enochs, Overtoun M. G. Jenda. " - Gorenstein Projective and Flat Covers and - Gorenstein Injective Envelopes ", Communications in Algebra, 2004
Crossref
7 words — < 1%

47 N. Vanaja. " All finitely generated -subgenerated modules are extending ", Communications in Algebra, 1996
Crossref
7 words — < 1%

48 Anwar, Yunita Septriana, and Indah Emilia Wijayanti. "Injective Envelope", Proceedings of the International Conference on Algebra 2010 - Advances in Algebraic Structures, 2011.
Crossref
7 words — < 1%

49 Yongduo Wang. "Strongly lifting modules and strongly dual Rickart modules", Frontiers of Mathematics in China, 2016
Crossref
6 words — < 1%

50	"From lifting to perfect modules", Frontiers in Mathematics, 2006 Crossref	6 words — < 1%
51	B. N. Türkmen. "A Generalization of Semiperfect Modules", Ukrainian Mathematical Journal, 2017 Crossref	6 words — < 1%
52	Rogelio Fernández-Alonso, Janeth Magaña. "Galois Connections Between Lattices of Preradicals Induced by Adjoint Pairs Between Categories of Modules", Applied Categorical Structures, 2015 Crossref	6 words — < 1%
53	A. Ghorbani, M. Naji Esfahani, Z. Nazemian. "Some new dimensions of modules and rings", Communications in Algebra, 2016 Crossref	6 words — < 1%
54	Marubayashi, H, K Masaike, K Oshiro, and M Sato. "ON NON-δ-M-COSINGULAR COMPLETELY ⊕-δM-SUPPLEMENTED MODULES", Ring Theory 2007, 2008. Crossref	6 words — < 1%

EXCLUDE QUOTES	ON	EXCLUDE MATCHES	OFF
EXCLUDE BIBLIOGRAPHY	ON		