On X-sub-linearly independent modules

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Abstract. The notion of X-sub-exact sequence of modules is a generalization of exact sequences. Let K, L, M be R-modules and X a submodule of L. The triple (K, L, M) is said to be X-sub-exact at L if $K \to X \to M$ is exact at X. The exact sequence is a special case of X-sub-exact by taking $X = L$. We introduce an X-sub-linearly independent module which is a generalization of linearly independent relative to an R-module M by using the concept of X-sub-exact sequence.

1. Introduction

Let R be a ring and let M be an R-module, A subset $S \subseteq M$ is R-linearly dependent if there exist distinct $x_1, x_2, .., x_n$ in S and elements $a_1, a_2, .., a_n$ of R, not all of which are 0, such that $a_1x_1 + a_2x_2 + ... + a_nx_n = 0$. A set that is not R-linearly dependent is said to be R-linearly independent [1]. Let N be a left R-module, then N is said linearly independent to R (or N is $R-$linearly independent) if there exists a monomorphism $\varphi : R^{(\Lambda)} \to N$ [5].

Suprapto [6] introduced a generalization of linearly independency relative to an R-module M as follows: Let M be an R-module. The family of R-modules $\mathcal{N} = \{N_\lambda\}_\Lambda$ is said to be linearly independent to M if there exist a monomorphism $f : \coprod_\Lambda N_\lambda \to M$. If $\{N_\lambda = N\}_\Lambda$, then $f : N^{(\Lambda)} \to M$. We can say that $\mathcal{N} = \{N_\lambda\}_\Lambda$ is linearly independent to M if the sequence $0 \to \coprod_\Lambda N_\lambda \xrightarrow{f} M$ is exact at $\coprod_\Lambda N_\lambda$.

Let R be a ring and let $A \xrightarrow{f} B \xrightarrow{g} C$ be an exact sequence of R-modules, i.e. $Im\ f = Ker\ g (= g^{-1}(\{0\}))$. We can generalize the submodule $\{0\}$ to any submodule $U \subseteq C$ as we refer to [2] in which Davvaz and Parnian-Garamaleky introduced the concept of quasi-exact sequences. A sequence of R-modules and R-homomorphisms $A \xrightarrow{f} B \xrightarrow{g} C$ is quasi-exact in B or U-exact in B if there exists a submodule U in C such that $Im\ f = g^{-1}(U)$.

Then, Anvariyeh dan Davvaz [7] proved further results about quasi-exact sequences and introduced generalization of Schanuel Lemma. Moreover, they obtained some relationships between quasi-exact sequences and superfluous (or essential) submodules.

Furthermore, Davvaz and Shabani-Solt introduced a generalization of some notions in homological algebra [3]. They gave a generalization of the Lambek Lemma, Snake Lemma, connecting homomorphism and exact triangle and they established new basic properties of the U-homological algebra. In [8], Anvariyeh and Davvaz studied U-split sequences and established several connections between U-split sequences and projective modules.

Let K, L, M be R-modules and X a submodule of L. The triple (K, L, M) is said to be an X-sub-exact at L if

$$K \to X \to M$$

is exact, i.e. $Im\ f = Ker\ g$. The exact sequence is a special case of X-sub-exact by taking $X = L$ [4].

In this paper, we introduce an X-sub-linearly independent module which is a generalization of linearly independent relative to an R-module M by using the concept of X-sub-exact sequence. Let M be an R-module. The family of R-modules $\mathcal{N} = \{N_\lambda\}_\Lambda$ is said to be X-sub-linearly independent to M if the triple $(0, \coprod_\Lambda N_\lambda, M)$ is X-sub-exact (where X is a submodule of $\coprod_\Lambda N_\lambda$). Then, we collect all submodules X of $\coprod_\Lambda N_\lambda$ such that $\mathcal{N}$ is X-sub-linearly independent to M, we denote it by $\sigma(0, \coprod_\Lambda N_\lambda, M)$. In this paper, we give some basic properties of X-sub-linearly independent modules and $\sigma(0, \coprod_\Lambda N_\lambda, M)$. We will show that $\sigma(0, \coprod_\Lambda N_\lambda, M)$ is closed under submodules and intersections. Furthermore, $\sigma(0, \coprod_\Lambda N_\lambda, M)$ always has a maximal element, for every family of R-modules $\mathcal{N}$ and R-module M. In other words, for every family of R-modules $\mathcal{N} = \{N_\lambda\}_\Lambda$ and R-module M, there exist a submodule X maximal such that $\mathcal{N}$ is an X-sublinearly independent.

2. Main Results

As a generalization of linearly independent relative to an R-module M, we define X-sub-linearly independent by using the concept of X-sub-exact sequence as follows:

Definition 2.1 *Let M be an R-module. The family of R-modules $\mathcal{N} = \{N_\lambda\}_\Lambda$ is said to be X-sublinearly independent to M if the triple $(0, \coprod_\Lambda N_\lambda, M)$ is X-sub-exact (where X is a submodule of $\coprod_\Lambda N_\lambda$), i.e. the sequence*

$$0 \to X \to M$$

is exact.

Example 2.1 Let $\mathcal{N} = \{\mathbb{Z}_2, \mathbb{Z}_5\}$ the family of $\mathbb{Z}$-modules and let $\mathbb{Z}_6$ be $\mathbb{Z}$-module. We define $f : \mathbb{Z}_2 \to \mathbb{Z}_6$, where $f(0) = 0$ and $f(1) = 3$. So, f is a monomorphism. Hence, the sequence

$$0 \to \mathbb{Z}_2 \xrightarrow{f} \mathbb{Z}_6$$

is exact. Therefore, the triple $(0, \mathbb{Z}_2 \oplus \mathbb{Z}_5, \mathbb{Z}_6)$ is $\mathbb{Z}_2$-sub-exact. So, $\mathcal{N}$ is $\mathbb{Z}_2$-sub-linearly independent to $\mathbb{Z}_6$.

Assume f is a monomorphism from $\mathbb{Z}_2 \oplus \mathbb{Z}_5$ to $\mathbb{Z}_6$. Then,

$$0 = f(0, 0) = f(5(0, 1)) = 5f(0, 1).$$

We get $f(0, 1) = f(0, 0) = 0$, a contradiction. So, we can not define a monomorphism from $\mathbb{Z}_2 \oplus \mathbb{Z}_5$ to $\mathbb{Z}_6$. Hence $\mathcal{N}$ is not linearly independent to $\mathbb{Z}_6$.

Example 2.1 shows that if the family of R-modules $\mathcal{N}$ is an X-sub-linearly independent to an R-module M, for some submodule X of $\coprod_\Lambda N_\lambda$, $N_\lambda \in \mathcal{N}$, for all $\lambda \in \Lambda$, then $\mathcal{N}$ is not necessary linearly independent to M.

We already know that any set that containing 0 is linearly dependent since $1.0 = 0$. In the following Proposition, we want to show that the family of R-modules $\mathcal{N}$ is 0-sub-linearly independent to M, for any R-module M.

Proposition 2.1 *Let $\mathcal{N} = \{N_\lambda\}_\Lambda$ be a family of R-modules. Then $\mathcal{N}$ is 0-sub-linearly independent to M, for any R-module M.*

Proof. Since the sequence $0 \to 0 \to M$ is exact, the triple $(0, \coprod_\Lambda N_\lambda, M)$ is 0-sub-exact at $\coprod_\Lambda N_\lambda$. Hence, $\mathcal{N}$ is 0-sub-linearly independent to M. □

In fact, we can define a monomorphism from R-module M to itself. So, Any R-module M is M-sub-linearly independent relative to M. We already know that any subset of a linearly independent set is linearly independent. In the following proposition, we will prove that M is X-sub-linearly independent to M, for every submodule X of M.

Proposition 2.2 *For any R-module M, M is X-sub-linearly independent to M, for every submodule X of M.*

Proof. Let M be an R-module and let X be a submodule of M. We have the inclusion $i : X \to M$ such that the sequence $0 \to X \xrightarrow{i} M$ is exact. Hence, the triple $(0, M, M)$ is X-sub-exact. Therefore M is X-sub-linearly independent to M. □

Then, we will give some properties of X-sub-linearly independent relative to an R-module M.

Clearly, we can define a monomorphism from N_λ to $\coprod_\Lambda N_\lambda$. So, we have the following proposition:

Proposition 2.3 *Let $\mathcal{N} = \{N_\lambda\}_\Lambda$ be a family of R-modules. Then $\mathcal{N}$ is N_λ-sub-linearly independent to $\coprod_\Lambda N_\lambda$, for every $\lambda \in \Lambda$.*

Proof. For every $\lambda \in \Lambda$, we have the inclusion $i : N_\lambda \to \coprod_\Lambda N_\lambda$ such that the sequence $0 \to N_\lambda \xrightarrow{i} \coprod_\Lambda N_\lambda$ is exact. Therefore, the triple $(0, \coprod_\Lambda N_\lambda, \coprod_\Lambda N_\lambda)$ is N_λ-sub-exact at $\coprod_\Lambda N_\lambda$. So, $\mathcal{N}$ is N_λ-sub-linearly independent to $\coprod_\Lambda N_\lambda$, for every $\lambda \in \Lambda$. □

Since for any submodule X of N_λ, we can define a monomorphism from X to $\coprod_\Lambda N_\lambda$, we have the following proposition.

Proposition 2.4 *Let $\mathcal{N} = \{N_\lambda\}_\Lambda$ be a family of R-modules. Then, for every $\lambda \in \Lambda$, $\mathcal{N}$ is X-sub-linearly independent to N_λ for any submodule X of N_λ.*

Proof. Let X be a submodule of $N_\lambda \subset \coprod_\Lambda N_\lambda$. We have the inclusion $i : X \to N_\lambda$ such that the sequence $0 \to X \to N_\lambda$ is exact. This implies the triple $(0, \coprod_\Lambda N_\lambda, N_\lambda)$ is X-sub-exact sequence at $\coprod_\Lambda N_\lambda$. Hence $\mathcal{N}$ is X-sublinearly independent to N_λ. □

Let K, L, M be R-modules. We define

$$\sigma(K, L, M) = \{X \leq L | (K, L, M) \ X\text{-sub-exact at } L\}.$$

Then $\sigma(K, L, M) \neq \emptyset$ since $0 \in \sigma(K, L, M)$.

Let $\mathcal{N}$ be a family of R-modules. If we take $K = 0$, $L = \coprod_\Lambda N_\lambda$ and $K = M$, then

$$\begin{aligned}\sigma(0, L, M) &= \{X \leq L | (0, L, M) \text{ is } X\text{-sub-exact at } L\} \\ &= \{X \leq L | \mathcal{N} \text{ is } X\text{-sublinearly independent to } M\}.\end{aligned}$$

We recall the properties of $\sigma(0, L, M)$ as follows:

Proposition 2.5 *[4] Let L, M be two R-modules and X_λ be a submodule of L, for every $\lambda \in \Lambda$. If $X_\lambda \in \sigma(0, L, M)$, for every $\lambda \in \lambda$, then $\cap_\Lambda X_\lambda \in \sigma(0, L, M)$.*

In the following Proposition, we will prove that $\sigma(0, \coprod_\Lambda N_\lambda, M)$ is closed under intersections, i.e. if $X_i \in \sigma(0, \coprod_\Lambda N_\lambda, M)$, for every $i \in I$, then $\cap_{i\in I} X_i \in \sigma(0, \coprod_\Lambda N_\lambda, M)$.

Proposition 2.6 *Let $\mathcal{N} = \{N_\lambda\}_\Lambda$ be a family of R-modules and let M be an R-module. If $\mathcal{N}$ is X_i-sub-linearly independent to M, for every $i \in I$, then $\mathcal{N}$ is $\cap_{i\in I} X_i$-sub-linearly independent to M. In other words, $\sigma(0, \coprod_\Lambda N_\lambda, M)$ is closed under intersections.*

Proof. Since $\mathcal{N}$ is X_i-sub-linearly independent to M, for every $i \in I$, then the triple $(0, \coprod_\Lambda N_\lambda, M)$ is X_i-sub-exact. Therefore, $X_i \in \sigma(0, \coprod_\Lambda N_\lambda, M)$, for every $i \in I$. By Proposition 2.5, we get $\cap_{i\in I} X_i \in \sigma(0, \coprod_\Lambda N_\lambda, M)$. So, $\mathcal{N}$ is $\cap_{i\in I} X_i$-sub-linearly independent to M. □

Furthermore, in the following proposition, we want to show that $\sigma(0, \coprod_\Lambda N_\lambda, M)$ is closed under submodules.

Proposition 2.7 *Let $\mathcal{N} = \{N_\lambda\}_\Lambda$ be a family of R-modules and M be an R-module. If $\mathcal{N}$ is X-sub-linearly independent to M, then $\mathcal{N}$ is X'-sub-linearly independent to M, for every submodule X' of X. In other words, $\sigma(0, \coprod_\Lambda N_\lambda, M)$ is closed under submodules.*

Proof. Since $\mathcal{N}$ is X-sub-linearly independent to M, then there is a monomorphism $f : X \to M$. Let X' be a submodule of X. Then, we can define the inclusion $i : X' \to X$. So, $f \circ i : X' \to M$ is a monomorphism. Hence, $\mathcal{N}$ is X'-sub-linearly independent to M. Therefore, if $X \in \sigma(0, \coprod_\Lambda N_\lambda, M)$, then $X' \in \sigma(0, \coprod_\Lambda N_\lambda, M)$, for every submodule X' of X. □

We already know that a basis for a free R-module F is a maximal linearly independent set in R-module F. So, we will investigate the maximal element of $\sigma(0, \coprod_\Lambda N_\lambda, M)$, i.e the maximal subset X of $\coprod_\Lambda N_\lambda$ such that $\mathcal{N}$ is X-sub-linearly independent to an R-module M. If there is a monomorphism $f : \coprod_\Lambda N_\lambda \to M$, then $\coprod_\Lambda N_\lambda$ is the maximal element in $\sigma(0, \coprod_\Lambda N_\lambda, M)$, i.e. for every $X \in \sigma(0, \coprod_\Lambda N_\lambda, M)$, if $\coprod_\Lambda N_\lambda \subseteq X$, then $\coprod_\Lambda N_\lambda = X$. But, $\coprod_\Lambda N_\lambda$ is not necessary belong to $\sigma(0, \coprod_\Lambda N_\lambda, M)$.

The most important criterion for the existence of maximal elements in a partially ordered set is Zorn's lemma. We recall Zorn's lemma as follows:

Proposition 2.8 ***(Zorn's Lemma)****[1] Let X be a partially ordered set and assume that every chain in X has an upper bound. Then X has a maximal element.*

By using Zorn's lemma, we want to show that there exist a submodule X in $\coprod_\Lambda N_\lambda$ maximal such that $\mathcal{N}$ is an X-sublinearly independent to M, for every family of R-modules $\mathcal{N}$ and R-module M.

Theorem 2.1 *Let $\mathcal{N} = \{N_\lambda\}_\Lambda$ be a family of R-modules and M be an R-module. Then there exist a submodule X in $\coprod_\Lambda N_\lambda$ maximal such that $\mathcal{N}$ is an X-sublinearly independent to M. In other words, $\sigma(0, \coprod_\Lambda N_\lambda, M)$ always has a maximal element.*

Proof. Let $\mathcal{X} = \{X | X \subseteq \coprod_\Lambda N_\lambda \text{ and } X \subseteq M\}$. The set $\mathcal{X}$ is not empty since $0 \in \mathcal{X}$. Let $\{X_i\}_{i\in I}$ be a chain (totally ordered set) in $\mathcal{X}$. Let $Y = \bigcup_{i\in I} X_i$, where $X_i \in \mathcal{X}$, for all $i \in I$. As a set, Y certainly contains all the X_i's. Since a union of submodules is not usually a submodule, we will show that Y is a submodule of $\coprod_\Lambda N_\lambda$.

If x and y are in Y, then $x \in X_i$ and $y \in X_j$, for two of the submodules X_i and X_j of $\coprod_\Lambda N_\lambda$. Since the set of submodules $\{X_i\}_i \in I$ is totally ordered,

$$X_i \subset X_j \text{ or } X_j \subset X_i.$$

Without loss of generality, $X_i \subset X_j$. Therefore x and y are in X_j, so $x + y \in X_j \subset Y$ and $rx \in X_j \subset Y$, for every $r \in R$. We can conclude that $Y = \cup_{i\in I} X_i$ is a submodule of $\coprod_\Lambda N_\lambda$. Similarly, we obtain Y is a submodule of M.

Since Y contains every X_i, for all $i \in I$, Y is an upper bound on the totally ordered set $\{X_i\}_{i\in I}$. By Zorn's lemma, $\mathcal{X}$ contains a maximal element. This maximal element is a submodule of $\coprod_\Lambda N_\lambda$ and M that is maximal for inclusion among all submodule of $\coprod_\Lambda N_\lambda$ and M. We can conclude that there exist a submodule X in $\coprod_\Lambda N_\lambda$ maximal such that $\mathcal{N}$ is an X-sublinearly independent to M or $\sigma(0, \coprod_\Lambda N_\lambda, M)$ always has a maximal element. □

3. Conclusion

The family of R-modules $\mathcal{N} = \{N_\lambda\}_\Lambda$ is an X-sub-linearly independent to M if the triple $(0, \coprod_\Lambda N_\lambda, M)$ is X-sub-exact (where X is a submodule of $\coprod_\Lambda N_\lambda$). If we take $X = \coprod_\Lambda N_\lambda$, then $\mathcal{N} = \{N_\lambda\}_\Lambda$ is linearly independent. Hence, sub-linearly independent module is a generalization of linearly independent module.

Then, we collect all submodules X of $\coprod_\Lambda N_\lambda$ such that $\mathcal{N}$ is X-sub-linearly independent to M, we denote it by $\sigma(0, \coprod_\Lambda N_\lambda, M)$. We have proved that $\sigma(0, \coprod_\Lambda N_\lambda, M)$ is closed under submodules and intersections. Furthermore, for every family of R-modules $\mathcal{N} = \{N_\lambda\}_\Lambda$ and R-module M, there exist X maximal such that $\mathcal{N}$ is an X-sublinearly independent. In other words, $\sigma(0, \coprod_\Lambda N_\lambda, M)$ always has a maximal element, for every family of R-modules $\mathcal{N}$ and R-module M.

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