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A Generalization of Basis and Free Modules Relatives to a Family $\mathcal{U}$ of R-Modules

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Abstract. Let $\mathcal{U}$ be a family of R-modules and V be a submodule of a direct sum of some elements in $\mathcal{U}$. The aim of this paper is to generalize basis and free modules. We use the concept of $\mathcal{U}_V$-generated module and X-sublinearly independent to provide the definition of $\mathcal{U}$-basis and $\mathcal{U}$-free module. We construct a $\underline{\mathcal{U}}$-basis of an R-module M as a pair (X, V), which a family $\mathcal{U}$ is X-sub-linearly independent to M and M is a $\mathcal{U}_V$-generated module. Furthermore, we define $\mathcal{U}$-basis of M as a $\underline{\mathcal{U}}$-basis which has the maximal element on the first component and the minimal element on the second component of a pair (X, V). The results show that the first component of (X, V) in $\mathcal{U}$-basis is closed under submodules and intersections. Moreover, we prove that the second component of (X, V) in $\mathcal{U}$-basis is closed under direct sums. We also determine some $\mathcal{U}$-free modules related to a family $\mathcal{U}$ which contains all $\mathbb{Z}$-module $\mathbb{Z}$ modulo p power of n, where p prime and $n \geq 2$.

1. Introduction

Let R be a ring, A, B and C be R-modules and let $A \xrightarrow{f} B \xrightarrow{g} C$ be an exact sequence, i.e. image of f is equal to the kernel of g ($g^{-1}(0)$) [1,2]. Davvaz and Parnian-Garamaleky establish a quasi-exact sequence as a generalization of exact sequence. Let U be a submodule of C. A sequence $A \xrightarrow{f} B \xrightarrow{g} C$ is U-exact in B if $\operatorname{Im} f = g^{-1}(U)$ [3]. For a submodule V of A, they also define a V-coexact sequence as a dual of a U-exact sequence.

Then, Anvariyeh dan Davvaz [4] generalize the Schanuel Lemma by using the quasi-exact sequences. Furthermore, Davvaz and Shabani-Solt [5] give a generalization of homological algebra. In [6], Anvariyeh and Davvaz investigate the connections between projective modules and U-split sequences. Then, Madanshekaf [7] gives some results about quasi-exact sequences. In [8], Amizadeh et al. provide a quasi-exact sequence of S-acts.

Motivated by definition of U-exact sequence, Fitriani et al. [9] introduce a sub exact sequence as a generalization of an exact sequence of modules. As an application of a sub-exact sequence, Fitriani et al. also establish the notion of an X-sub-linearly independent module as a generalization of the linearly independent set in R-modules [10]. Furthermore, Fitriani et al. [11] introduce a $\mathcal{U}_V$-generated module by using coexact sequence. We can say that this notion is a dual of X-sub-linearly independent module. This concept is motivated by the definition of $\mathcal{U}$-generated module from [12-14].

In this paper, we use the concept of $\mathcal{U}_v$-generated module and X-sub-linearly independent module to construct a $\mathcal{U}$-basis and a $\mathcal{U}$-free module which are a basis and a free module relative to a family $\mathcal{U}$ of R-modules. Moreover, we determine some $\mathcal{U}$-free modules, where $\mathcal{U}$ is a family of all $\mathbb{Z}$-modules $\mathbb{Z}$ modulo p^n, p prime and n is an integer greater than 2.

2. Methods

The aim of this paper is to generalize basis and free modules to basis and free modules relative to a family $\mathcal{U}$ of R-modules. If a free module F has a basis X, then $F \cong \oplus_{x\in X} R_x$ with each $R_x \cong R$. We can choose $\mathcal{U} = \{R\}$ so that F is a free module relative to $\mathcal{U}$. In this case, a family $\mathcal{U}$ only contain R as an R-module.

We construct a basis and a free module relative to a family $\mathcal{U} = \{U_\lambda\}_\Lambda$, where U_λ is an R-module, for every $\lambda \in \Lambda$. We use the concept of $\mathcal{U}_v$-generated module and X-sublinearly independent module to provide this concept. We construct a $\underline{\mathcal{U}}$-basis of an R-module M as a pair (X, V), which a family $\mathcal{U}$ is X-sub-linearly independent to M and M is a $\mathcal{U}_v$-generated module. Next, we define $\mathcal{U}$-basis of M as a $\underline{\mathcal{U}}$-basis which has the maximal element on the first component and the minimal element on the second component of a pair (X, V). Furthermore, we determine some $\mathcal{U}$-free module, where $\mathcal{U}$ is a family of all $\mathbb{Z}$ modulo p^n, p prime, $n \in \mathbb{N}$, $n \geq 2$ as a $\mathbb{Z}$-module by using the properties of $\mathbb{Z}_n$ as an Abelian group.

3. Results and Discussions

We recall the definition of a $\mathcal{U}_v$-generated module as follows: Given a family $\mathcal{U} = \{U_\lambda\}_\Lambda$ of R-modules, V be a submodule of $\oplus_\Lambda U_\lambda$. An R-module N is $\mathcal{U}_v$-generated if there exists a surjective homomorphism from V to N [11]. If we take $V = \oplus_\Lambda U_\lambda$, then a $\mathcal{U}_v$-generated module is a $\mathcal{U}$-generated module. From this fact, we can say that every $\mathcal{U}$-generated module is a $\mathcal{U}_v$-generated module. But, the converse need not be true.

Now, we define the following sets:

$$\sigma(0, \oplus_\Lambda U_\lambda, M) = \{X \subseteq \oplus_\Lambda U_\lambda \mid \mathcal{U} \text{ is } X\text{-sub-linearly independent to } M \quad (1)$$

and

$$\mathcal{U}(M) = \{\, V \subseteq \oplus_\Lambda U_\lambda \mid M \text{ is } U_v\text{-generated}\} \quad (2)$$

The set $\sigma(0, \oplus_\Lambda U_\lambda, M)$ contains all submodules of $\oplus_\Lambda U_\lambda$ which is X-sub-linearly independent to M. Hence, if there is an injective homomorphism from Y to M, where Y is a submodule of $\oplus_\Lambda U_\lambda$, then Y is in the set $\sigma(0, \oplus_\Lambda U_\lambda, M)$.

Suppose that X is in $\sigma(0, \oplus_\Lambda U_\lambda, M)$. Consequently, a family $\mathcal{U}$ is X-sub-linearly independent to M. Therefore, there exists a monomorphism f from X to M. We already know that for every submodule X' of X, we always have a monomorphism i from X' to X. Then $\mathcal{U}$ is also X'-sub-linearly independent to M, for every submodule X' of X [10]. Similarly, if a family $\mathcal{U}$ of R-modules is X_i–sub-linearly independent to an R-module M for every $i \in I$, then a family $\mathcal{U}$ is also $\cap_{i\in I} X_i$–sub-linearly independent to M.

Consider the set $\mathcal{U}(M) = \{\, V \subseteq \oplus_\Lambda U_\lambda \mid M$ is $\mathcal{U}_v$-generated$\}$. In this set, we collect all submodules V of $\oplus_\Lambda U_\lambda$ which M is $\mathcal{U}_v$-generated. If V is in $\mathcal{U}(M)$, we have a surective homomorphism g from V to M. If R-module X_1 is $\mathcal{U}_{v_1}$-generated and R-module X_1 is $\mathcal{U}_{v_2}$-generated, then $X_1 \oplus X_2$ is $\mathcal{U}_{v_1 \oplus v_2}$-generated, where V_1 and V_2 be submodules of $\oplus_\Lambda U_\lambda$, $U_\lambda \in \mathcal{U}$, for every $\lambda \in \Lambda$. Based on [11], we have the set $\mathcal{U}(M)$ is closed under direct sums and homomorphic images. We will use the properties of the set $\sigma(0, \oplus_\Lambda U_\lambda, M)$ and $\mathcal{U}(M)$ to investigate some characteristics of $\mathcal{U}$-basis and $\mathcal{U}$-free module.

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Now, we will construct the definition of $\mathscr{U}$-basis and $\mathscr{U}$-free module by using the concept of X-sub-linearly independent to an R-module M and $\mathscr{U}_V$-generated module.

Definition 2.1. *Given a family $\mathscr{U}$ of R-modules. A pair of submodules (X, V) of $\oplus_\Lambda U_\lambda$ is said to be a $\mathscr{U}$-basis of* R-*module M if $\mathscr{U}$ is an X-sublinearly independent to M and M is a $\mathscr{U}_V$-generated module.*

From Definition 2.1, $\mathscr{U}$-basis of R-module M is a pair of two submodules X and V of $\oplus_\Lambda U_\lambda$ which $\mathscr{U}$ is an X-sublinearly independent to M and M is a $\mathscr{U}_V$-generated module. In other words, if (X, V) is a $\mathscr{U}$-basis of R-module M, then there are a monomorphism f from X to M and an epimorphism g from V to M. Then we will give some examples of $\mathscr{U}$-basis of an R-module.

Example 2.2. Let $\mathscr{U} = \{\mathbb{Z}_p \mid p \text{ prime}\}$, a family of $\mathbb{Z}$-modules, where $\mathbb{Z}$ is a set of integers. We consider $\mathbb{Z}_6$ as a $\mathbb{Z}$-module. We will find $\mathscr{U}$-basis of $\mathbb{Z}$-module $\mathbb{Z}_6$. We can define monomorphisms from 0, $\mathbb{Z}_2$, $\mathbb{Z}_3$ and $\mathbb{Z}_2 \oplus \mathbb{Z}_3$ to $\mathbb{Z}_6$. Also, we can define an epimorphism from $\mathbb{Z}_2 \oplus \mathbb{Z}_3$ to $\mathbb{Z}_6$. Therefore, we have some $\mathscr{U}$-basis of $\mathbb{Z}$-module $\mathbb{Z}_6$ as follows: $(0, \mathbb{Z}_2\oplus\mathbb{Z}_3)$, $(\mathbb{Z}_2, \mathbb{Z}_2\oplus\mathbb{Z}_3)$, $(\mathbb{Z}_3, \mathbb{Z}_2\oplus\mathbb{Z}_3)$, $(\mathbb{Z}_2\oplus\mathbb{Z}_3, \mathbb{Z}_2\oplus\mathbb{Z}_3)$.

Example 2.3. Let $\mathscr{U} = \{\mathbb{Z}\}$, a family of $\mathbb{Z}$-module, where $\mathbb{Z}$ is a set of integers. We will find $\mathscr{U}$-basis of $\mathbb{Z}$-module $\mathbb{Z}_4$. Clearly, there is a monomorphism from 0 to $\mathbb{Z}_4$ and hence a family $\mathscr{U}$ is 0-sub-linearly independent to $\mathbb{Z}_4$. Furthermore, we can define an epimorphism from $\mathbb{Z}$ to $\mathbb{Z}_4$. As a consequence, $(0, \mathbb{Z})$ is a $\mathscr{U}$-basis of $\mathbb{Z}$-module $\mathbb{Z}_4$. In general, we can show that $(0, \mathbb{Z})$ is a $\mathscr{U}$-basis of $\mathbb{Z}$-module $\mathbb{Z}_n$, for every $n \geq 2$.

Now, we will give some properties of $\mathscr{U}$-basis of an R-module M. We already know that the set $\sigma(0, \oplus_\Lambda U_\lambda, M)$ is closed under intersections [10]. We will use this property to show the following proposition.

Proposition 2.4. *Given a family $\mathscr{U}$ of R-modules. If (X_α, V) is a $\mathscr{U}$-basis of an R-module M, for every $\alpha \in A$, then $(\cap_\alpha X_\alpha, V)$ is $\mathscr{U}$-basis of M.*

Proof. Suppose that (X_α, V) is a $\mathscr{U}$-basis of an R-module M, for every $\alpha \in A$. Consequently, a family $\mathscr{U}$ is X_α-sub-linearly independent to M, for every $\alpha \in A$. Hence, $X_\alpha \in \sigma(0, \oplus_\Lambda U_\lambda, M)$, for every $\alpha \in A$. Based on [10], the set $\sigma(0, \oplus_\Lambda U_\lambda, M)$ is closed under intersections. As a consequence, we have $(\cap_\alpha X_\alpha, V)$ is in $\sigma(0, \oplus_\Lambda U_\lambda, M)$. In other words, a family $\mathscr{U}$ is $\cap_\alpha X_\alpha$-sub-linearly independent to M and hence we have $(\cap_\alpha X_\alpha, V)$ is a $\mathscr{U}$-basis of M. QED

Next, we will use the fact that the set $\sigma(0, \oplus_\Lambda U_\lambda, M)$ is closed under submodules to proof the following property of $\mathscr{U}$-basis of M.

Proposition 2.5. *Given a family $\mathscr{U}$ of R-modules. If (X, V) is a $\mathscr{U}$-basis of R-module M, then a pair (X', V) is a $\mathscr{U}$-basis of M, for every submodule X' of X.*

Proof. Let a pair (X, V) is a $\mathscr{U}$-basis of an R-module M. Then a family $\mathscr{U}$ is X-sub-linearly independent to M. This implies that $X \in \sigma(0, \oplus_\Lambda U_\lambda, M)$. Based on [10], $\sigma(0, \oplus_\Lambda U_\lambda, M)$ is closed under submodules. So, for every submodule X' of X, X' is in $\sigma(0, \oplus_\Lambda U_\lambda, M)$. Therefore, (X', V) is a $\mathscr{U}$-basis of M, for every submodule X' of X. QED

In the next proposition, we focus on the second component of a pair (X, V) which is $\mathscr{U}$-basis of an R-module M. We will use the properties of the set $\mathscr{U}(M)$ to proof the next property of $\mathscr{U}$-basis.

Proposition 2.6. *Given a family $\mathscr{U}$ of R-modules. If (X, V_β) is a $\mathscr{U}$-basis of R-module M, for every $\beta \in B$, then $(X, \oplus_B V_\beta)$ is a $\mathscr{U}$-basis of M.*

Proof. Suppose that (X, V_β) is a $\mathscr{U}$-basis of R-module M, for every $\beta \in B$. Then M is a $\mathscr{U}_{V_\beta}$-generated module. This implies $V_\beta \in \mathscr{U}(M)$, for every $\beta \in B$. Based in [11], we already know that the set $\mathscr{U}(M)$ is closed under direct sums. Therefore, we have $\oplus_B V_\beta \in \mathscr{U}(M)$. In other words, we can say M is a $\mathscr{U}_{\oplus_B V_\beta}$-generated module. Hence, a pair $(X, \oplus_B V_\beta)$ is a $\mathscr{U}$-basis of M. QED

Proposition 2.7. *Given a family $\mathscr{U}$ of R-modules. If (X_γ, V_γ) is a $\mathscr{U}$-basis of R-module M_γ, for every $\gamma \in \Gamma$, then $(\oplus_\Gamma X_\gamma, \oplus_\Gamma V_\gamma)$ is $\mathscr{U}$-basis of $\oplus_\Gamma M_\gamma$.*

Proof. Suppose that (X_γ, V_γ) is a $\mathcal{U}$-basis of R-module M, for every $\gamma \in \Gamma$. From this we have a family $\mathcal{U}$ is X_γ-sub-linearly independent to M_γ and M_γ is $\mathcal{U}_{V_\gamma}$ -generated. This implies $X_\gamma \in \sigma(0, \oplus_\Lambda U_\lambda, M)$ and $V_\gamma \in \mathcal{U}(M)$, for every $\gamma \in \Gamma$. Therefore, we have a monomorphism from X_γ to M and an epimorphism from V_γ to M_γ for every $\gamma \in \Gamma$. Clearly, we can construct a monomorphism from $\oplus_\Lambda X_\lambda$ to $\oplus_\Lambda M_\lambda$. Also, we can define an epimorphism from $\oplus_\Lambda V_\lambda$ to $\oplus_\Lambda M_\lambda$. Hence $\oplus_\Gamma X_\gamma \in \sigma(0, \oplus_\Lambda U_\lambda, \oplus_\Lambda M_\lambda)$ and $\oplus_\Gamma V_\gamma \in \mathcal{U}(\oplus_\Lambda M_\lambda)$. Therefore, $(\oplus_\Gamma X_\gamma, \oplus_\Gamma V_\gamma)$ is $\mathcal{U}$-basis of $\oplus_\Gamma M_\gamma$. QED

We can see from Example 2.2 that $\mathcal{U}$-basis of an R-module M is not uniquely determined. From this fact, we will choose a maximal element in first part of $\mathcal{U}$-basis of M and a minimal element in second part of $\mathcal{U}$-basis of M. In other words, we will find a maximal element of the set $\sigma(0, \oplus_\Lambda U_\lambda, M)$ and a minimal element of $\mathcal{U}(M)$. Based on [10], the set $\sigma(0, \oplus_\Lambda U_\lambda, M)$ always has a maximal element. We will denote $\mathcal{U}$-basis of M which has a maximal element in the first component and a minimal element in the second component of a pair (X, V), a $\mathcal{U}$-basis of M. Now, we give the formal definition of $\mathcal{U}$-basis of M and $\mathcal{U}$-free module.

Definition 2.8. *Given a family $\mathcal{U}$ of R-modules. A pair (X, V) is said to be a $\mathcal{U}$-basis if (X, V) is $\mathcal{U}$-basis of M, X is a maximal element of $\sigma(0, \oplus_\Lambda U_\lambda, M)$ and V is a minimal element of $\mathcal{U}(M)$. An R-module M is called $\mathcal{U}$-free if M has $\mathcal{U}$-basis.*

Example 2.9. Given a family $\mathcal{U}$ of R-modules. Then a pair $(0,0)$ is $\mathcal{U}$-basis of R-module 0.

Example 2.10. Given a family $\mathcal{U} = \{\mathbb{Z}_n \mid n \text{ prime}\}$ of $\mathbb{Z}$-modules. From Example 2.2, we have some $\mathcal{U}$ -basis of $\mathbb{Z}$-module $\mathbb{Z}_6$, i.e. $(0, \mathbb{Z}_2\oplus\mathbb{Z}_3)$, $(\mathbb{Z}_2, \mathbb{Z}_2\oplus\mathbb{Z}_3)$, $(\mathbb{Z}_3, \mathbb{Z}_2\oplus\mathbb{Z}_3)$, $(\mathbb{Z}_2\oplus\mathbb{Z}_3, \mathbb{Z}_2\oplus\mathbb{Z}_3)$. Therefore, we have $\mathbb{Z}_2\oplus\mathbb{Z}_3$ is a maximal element in the first component of $\mathcal{U}$ -basis and also a minimal element of the second component of $\mathcal{U}$–basis of $\mathbb{Z}_6$. Hence, a pair $(\mathbb{Z}_2\oplus\mathbb{Z}_3, \mathbb{Z}_2\oplus\mathbb{Z}_3)$ is a $\mathcal{U}$-basis of $\mathbb{Z}$-module $\mathbb{Z}_6$.

Example 2.11. Given a family $\mathcal{U}$ of $\mathbb{Z}$-module. Based on Example 2.3, we have a pair $(0, \mathbb{Z})$ is a $\mathcal{U}$-basis of $\mathbb{Z}$-module $\mathbb{Z}_n$, where $n \geq 2$.

We already know that $\mathbb{Z}$-module $\mathbb{Z}_n$ is not a free module. But, from Example 2.11 we have $\mathbb{Z}$-module $\mathbb{Z}_n$ is a $\mathcal{U}$-free module relative to a family $\mathcal{U} = \{\mathbb{Z}\}$ of $\mathbb{Z}$-module, where $n \geq 2$.

In the Proposition 2.7, we have proved that a $\mathcal{U}$–basis of an R-module M is closed under direct sums. A similar result holds for an $\mathcal{U}$-basis of an R-module M.

Proposition 2.12. *Given a family $\mathcal{U}$ of R-modules. If (X_γ, V_γ) is a $\mathcal{U}$-basis of R-module M_γ, for every $\gamma \in \Gamma$, then $(\oplus_\Gamma X_\gamma, \oplus_\Gamma V_\gamma)$ is $\mathcal{U}$-basis of $\oplus_\Gamma M_\gamma$.*

In the previous examples, a submodule X and V of $\oplus_\Lambda U_\lambda$ which is a $\mathcal{U}$-basis of an R-module M need not be isomorphic. In case X is isomorphic to V, we will introduce a $\mathcal{U}$-strictly basis and a $\mathcal{U}$-strictly free module as follows:

Definition 2.13. *Given a family $\mathcal{U}$ of R-modules. A pair (X, V) is said to be a $\mathcal{U}$-strictly basis if (X, V) is $\mathcal{U}$-basis of M and X is isomorphic to V. An R-module M is called a $\mathcal{U}$-strictly free if M has $\mathcal{U}$-strictly basis.*

Since X is isomorphic to V, we simply write X instead (X, V) as a $\mathcal{U}$-strictly basis of an R-module M. We will determine a family $\mathcal{U}$ of R-modules to regard a free module as a $\mathcal{U}$-strictly free module. We already know that if a free module F has a basis X, then $F \cong \oplus_{x\in X} R_x$ with each $R_x \cong R$. We can choose $\mathcal{U} = \{R\}$ as a family of R-module. Hence, we have $\oplus_{x\in X} R_x = R^{(X)}$ is a $\mathcal{U}$-strictly basis of F. This implies F is $\mathcal{U}$-strictly free. From this fact, we can say that every free R-module F is a $\mathcal{U}$-strictly free module. Based on Example 2.9, a pair $(0,0)$ is $\mathcal{U}$-basis of R-module 0, for any family $\mathcal{U}$ of R-modules. As a consequence, R-module 0 is $\mathcal{U}$-strictly free.

Moreover, in case X is an element of $\mathcal{U}$. X is $\mathcal{U}$-strictly free. Furthermore, we consider the result of Proposition 2.12. If M_i is a $\mathcal{U}$-strictly free module for every $i = 1, 2, \ldots, n$, then $\oplus_{i=1}^{n} M_i$ is also a $\mathcal{U}$-strictly free module. Now, we will give some examples of $\mathcal{U}$-strictly free modules.

Example 2.14. Let R be a commutative ring with unit and $\mathcal{U} = \{U_\lambda\}_\Lambda$ be a family of R-modules, where $U_\lambda = \mathrm{Hom}_R(R, M_\lambda)$ for every $\lambda \in \Lambda$. Based on [1], we can define a homomorphism φ from $\mathrm{Hom}_R(R, M_\lambda)$ to M_λ, where $\varphi(f) := f(1)$. We can show that φ is an isomorphism. This implies that a family $\mathcal{U}$ is U_λ-sub-linearly independent to M_λ and M_λ is $\mathcal{U}_{v_\gamma}$ -generated. Therefore, we can conclude that M_λ is $\mathcal{U}$-strictly free.

Example 2.15. Given a family $\mathcal{U} = \{\mathbb{Z}_n \mid n \in \mathbb{Z}, n \geq 2\}$ of $\mathbb{Z}$-modules. Let $M = \mathbb{Z}_4^{(\mathbb{N})}$ and $N = \mathbb{Z}_2 \oplus M$ be $\mathbb{Z}$-modules. Since a family $\mathcal{U}$ is M-sub-linearly independent to M and N is $\mathcal{U}_N$-generated, (M, N) is $\mathcal{U}$-basis of M. We will show that M is not isomorphic to N. Assume that there is an isomorphism f from M to N. Since $(1,0,0,\ldots) \in N$, there is $0 \neq (a_i) \in M$ such that $f((a_i)) = (1,0,0,..)$. Then $f(2(a_i)) = 2f((a_i)) = 2(1,0,0,\ldots) = 0$. By hypothesis, f is a monomorphism. So, we have $2(a_i) = 0$. Therefore, $a_1 = 0$ or 1 and $a_i = 0$ or 2, for $i \geq 2$. So, there is $(b_i) \in M$ such that $(a_i) = 2(b_i)$. This implies $b_1 = 0$ or 1 and $b_i = 0, 1$ or 3 for $i \geq 2$. Hence, $0 = 2f((b_i)) = f(2(b_i)) = f((a_i))$. But $1 = f((a_i))_1 = 0$, a contradiction. We can conclude that M is not isomorphic to N and hence (M, N) is not $\mathcal{U}$-strictly basis of M.

Now, we consider the following properties of $\mathbb{Z}_n$ as an Abelian group.

Theorem 2.16. [15] *Let m and n be positive integers. If gcd(m,n)=1 (i.e. m and n are relative prime), then $\mathbb{Z}_m \times \mathbb{Z}_n$ is cyclic and is isomorphic to $\mathbb{Z}_{mn}$ and* $(1,1)$ *is a generator of $\mathbb{Z}_m \times \mathbb{Z}_n$.*

Theorem 2.17. [15] *The group $\Pi^n_{i=1}\mathbb{Z}_{mi}$ is cyclic and isomorphic to $\mathbb{Z}_{m_1 m_2 \ldots m_n}$ if and only if the numbers m_i, for $i = 1, \ldots, n$ are pairwise relative prime, that is, the gcd of two of them is* 1.

Therefore, by using Theorem 2.16 and 2.17, we can determine some $\mathcal{U}$-strictly free modules as follows.

Proposition 2.18. *Given a family $\mathcal{U} = \{\mathbb{Z}_p \mid p$ prime$\}$ of $\mathbb{Z}$-modules and q, r be two distinct primes. Then $\mathbb{Z}$-module $\mathbb{Z}_{qr}$ is $\mathcal{U}$-strictly free.*

Proof. Since q and r are relative primes, $\mathbb{Z}_q \oplus \mathbb{Z}_r$ is $\mathcal{U}$-strictly basis of $\mathbb{Z}_{qr}$. Hence, $\mathbb{Z}$-modules $\mathbb{Z}_{qr}$ is a $\mathcal{U}$-strictly free. QED

Proposition 2.19. *Given a family $\mathcal{U} = \{\mathbb{Z}_{p^n} \mid p$ prime, $n \in \mathbb{N}\}$ of $\mathbb{Z}$-modules. Then $\mathbb{Z}_n$ is a $\mathcal{U}$-strictly free module, for every positive integer $n \geq 2$.*

Proof. We already know that every positive integer n can be uniquely factorized as a product of distinct prime number $n = p_1^{n_1} p_2^{n_2} \ldots p_r^{n_r}$, where p_i prime and $n_i \in \mathbb{N}$ for $i = 1, 2, \ldots, r$. By Theorem 2.17, we have:

$$\mathbb{Z}_n \cong \mathbb{Z}_{p_1^{n_1}} \times \mathbb{Z}_{p_2^{n_2}} \times \ldots \times \mathbb{Z}_{p_r^{n_r}}$$

Therefore, we have $\mathbb{Z}_{p_1^{n_1}} \times \mathbb{Z}_{p_2^{n_2}} \times \ldots \times \mathbb{Z}_{p_r^{n_r}}$ is a $\mathcal{U}$-strictly basis of $\mathbb{Z}_n$. This implies that $\mathbb{Z}_n$ is a $\mathcal{U}$-strictly free module, for every positive integer n. QED

From Proposition 2.18, we have some $\mathcal{U}$-strictly free modules, where $\mathcal{U}$ is a family of $\mathbb{Z}$-modules $\mathbb{Z}$ modulo p, p prime. $\mathbb{Z}$-module $\mathbb{Z}_{qr}$ is $\mathcal{U}$-strictly free for every two distinct primes q and r. Moreover, based on Proposition 2.19, we have $\mathbb{Z}$-modules $\mathbb{Z}_n$ are $\mathcal{U}$-strictly free module relative to a family $\mathcal{U}$ which contains all $\mathbb{Z}$-modules $\mathbb{Z}_{p^n}$, p prime, for every positive integer $n \geq 2$.

We already know that since $\mathbb{Z}$-module $\mathbb{Z}_n$ is not linearly independent, $\mathbb{Z}_n$ is not a free module, for every positive integer n greater than 2. But, this module is $\mathcal{U}$-strictly free module relative to a family $\mathcal{U} = \{\mathbb{Z}_p \mid p$ prime$\}$ of $\mathbb{Z}$-modules. Consequently, $\mathcal{U}$-strictly free module is a generalization of a free module. If we take $\mathcal{U} = \{R\}$, where R is a ring, then an R-module M is $\mathcal{U}$-strictly free if and only if R-module M is free. But, if $\mathcal{U}$ is another family of R-module, then not every $\mathcal{U}$-strictly free module is a free module.

4. Conclussions

A $\mathscr{U}$-basis and a $\mathscr{U}$-free modules are a basis and a free module relative to a family $\mathscr{U}$ of R-module. These notions are the generalization of the concept of a basis and a free module. Every free module F is a $\mathscr{U}$-free module, where $\mathscr{U} = \{R\}$ as a family of R-module. But not every $\mathscr{U}$-free module is a free module. For example, $\mathbb{Z}$-module $\mathbb{Z}_n$ is a $\mathscr{U}$-strictly free module, but $\mathbb{Z}$-module $\mathbb{Z}_n$ is not a free module.

If $\mathscr{U}$ be a family of all $\mathbb{Z}$-module $\mathbb{Z}_p$, where p prime, then $\mathbb{Z}$-module $\mathbb{Z}_{qr}$ is a $\mathscr{U}$-strictly free module, where q and r be distinct primes. Furthermore, if $\mathscr{U}$ be a family of all $\mathbb{Z}$ modulo p power of n, where p prime and n positive integer larger than 2, $\mathbb{Z}$-module $\mathbb{Z}_n$ is a $\mathscr{U}$-strictly free module, for every positive integer $n \geq 2$.

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