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By Ahmad Faisol

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The Sufficient Conditions for $R[X]$-module $M[X]$ to be $S[X]$-Noetherian

Ahmad Faisol[1,2,*], Budi Surodjo[1], Sri Wahyuni[1]

[1] *Department of Mathematics, Faculty of Mathematics and Natural Sciences, Universitas Gadjah Mada, Yogyakarta, Indonesia*
[2] *Department of Mathematics, Faculty of Mathematics and Natural Sciences, Universitas Lampung, Bandar Lampung, Indonesia*

Abstract. In this paper, we obtain the necessary and sufficient conditions on a ring R, a multiplicative set $S \subseteq R$, and an R-module M such that the polynomial module, the Laurent polynomial module, the power series module, and the Laurent series module are S-Noetherian. Furthermore, we also obtain the sufficient conditions for these modules to be $S[X]$-Noetherian, $S[X, X^{-1}]$-Noetherian, $S[[X]]$-Noetherian, and $S[[X, X^{-1}]]$-Noetherian, respectively.

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Key Words and Phrases: S-Noetherian module, Polynomial module, Laurent polynomial module, Power series module, Laurent series module

1. Introduction

The Noetherian concept plays an important role in Algebra, especially ring theory and modules. One of the well-known properties in Noetherian ring concept is Hilbert's Basis Theorem, which is a sufficient condition for the polynomial ring to be Noetherian. Varadarajan [5] generalizes the Hilbert Basis Theorem to determine the necessary and sufficient conditions for the polynomial module, the Laurent polynomial module and the power series module to be Noetherian modules.

Anderson and Dumitrescu [1] generalize the concept of Noetherian rings by involving a multiplicatively closed subset of the ring. Moreover, they also determine the sufficient conditions for the polynomial ring $R[X]$ and the power series ring $R[[X]]$ to be S-Noetherian rings, where S is a multiplicative subset of the ring R. On the other hand, Zhongkui [6] provides the sufficient conditions for the Laurent series ring $R[[X, X^{-1}]]$ to be an S-Noetherian ring. Furthermore, Baeck et al. [2] give the sufficient conditions for the Laurent polynomial ring $R[X, X^{-1}]$ to be an S-Noetherian ring.

*Corresponding author.

Email addresses: ahmadfaisol@mail.ugm.ac.id (A. Faisol), b_surodjo@ugm.ac.id (B. Surodjo), swahyuni@ugm.ac.id (S. Wahyuni)

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In this paper we study the necessary and sufficient conditions for $R[X]$-module $M[X]$, $R[X, X^{-1}]$-module $M[X, X^{-1}]$, $R[[X]]$-module $M[[X]]$, and $R[[X, X^{-1}]]$-module $M[[X, X^{-1}]]$ to be S-Noetherian modules. Furthermore, we also determine the sufficient conditions for these modules to be $S[X]$-Noetherian, $S[X, X^{-1}]$-Noetherian, $S[[X]]$-Noetherian, and $S[[X, X^{-1}]]$-Noetherian, respectively.

In Section 2, we will review some definition and properties about S-Noetherian rings follow from [1]. Beside that, we also give the structure of $M[X]$, $M[X, X^{-1}]$, $M[[X]]$, and $M[[X, X^{-1}]]$ follow from [5].

As the main results, in Section 3, we give the necessary and sufficient conditions for the polynomial module to be an S-Noetherian module, list in Theorem 1 and Theorem 3. In similar ways, we also obtain the necessary and sufficient conditions for the Laurent polynomial module, the power series module, and the Laurent series module to be S-Noetherian module, list in Theorem 2, Theorem 4, Theorem 5, and Theorem 6. At the end of Section 3, we give the sufficient conditions for these all modules to be $S[X]$-Noetherian, $S[X, X^{-1}]$-Noetherian, $S[[X]]$-Noetherian, and $S[[X, X^{-1}]]$-Noetherian, respectively, list in Theorem 7 and Theorem 8.

2. Preliminaries

In this section, we review some theory about S-Noetherian rings and modules, and the structure of the polynomial module $M[X]$, the Laurent polynomial module $M[X, X^{-1}]$, the power series module $M[[X]]$, and the Laurent series module $M[[X, X^{-1}]]$, following [1] and [5].

2.1. S-Noetherian Rings and Modules

According to [1], a ring R is called S-Noetherian if each ideal of R is S-finite, i.e., if $SI \subseteq J \subseteq I$ for some finitely generated ideal J of R and some $s \in S$. An R-module M is called S-Noetherian if each submodule of M is an S-finite module, i.e., if $sN \subseteq F$ for some finitely generated submodule F of M and some $s \in S$. A multiplicative set S of a ring R is anti-Archimedean if $\bigcap_{n\geq 1} s^n R \cap S \neq \emptyset$, for every $s \in S$.

The sufficient conditions for $R[X]$ and $R[[X]]$ to be S-Noetherian are given by the following propositions.

Proposition 1. *[1] Let R be a ring and $S \subseteq R$ an anti-Archimedean multiplicative set. If R is S-Noetherian, then so is the polynomial ring $R[X_1, ..., X_n]$.*

Proposition 2. *[1] Let R be a ring and $S \subseteq R$ an anti-Archimedean multiplicative set of R consisting of nonzerodivisors. If R is S-Noetherian, then so is the power series ring $R[[X_1, ..., X_n]]$.*

Furthermore, the sufficient conditions for $R[X, X^{-1}]$ and $R[[X, X^{-1}]]$ to be S-Noetherian are given by the following propositions.

Proposition 3. *[2] Let S be an anti-Archimedean multiplicative subset of a ring R. If R is S-Noetherian, then so is the Laurent polynomial ring $R[X, X^{-1}]$.*

Proposition 4. *[6] Let R be a ring and $S \subseteq R$ an anti-Archimedean multiplicative set of R consisting nonzerodivisors. If R is S-Noetherian, then so is the Laurent series ring $R[[X, X^{-1}]]$.*

Baeck et al. [2] also give some properties of an S-Noetherian R-module M as given by the following Lemmas.

Lemma 1. *[2] Let R be a ring, S a multiplicative subset of R, and M an R-module. Then the following assertions hold.*

(1) If M is S-Noetherian and N is a submodule of M, then M/N is S-Noetherian.

(2) If N is a S-Noetherian submodule of M such that M/N is S-Noetherian, then M is S-Noetherian.

(3) If R is S-Noetherian and M is finitely generated, then M is S-Noetherian.

Lemma 2. *[2] If $S_1 \subseteq S_2$ are multiplicative subsets of ring R, then S_1-Noetherian R-module is an S_2-Noetherian R-module.*

2.2. The Structure of $M[X]$, $M[X, X^{-1}]$, $M[[X]]$, and $M[[X, X^{-1}]]$

Let $R[X]$ be a polynomial ring and $M[X] = \{\sum_{j=0}^{l} m_j x^j | l \in \mathbb{N} \cup \{0\}, m_j \in M\}$. For any $\sum_{i=0}^{k} r_i x^i \in R[X]$ and $\sum_{j=0}^{l} m_j x^j \in M[X]$, $M[X]$ acquires the structure of an $R[X]$-module with pointwise addition and scalar multiplication defined by

$$(\sum_{i=0}^{k} r_i x^i)(\sum_{j=0}^{l} m_j x^j) = \sum_{\mu=0}^{k+l} c_\mu x^\mu,$$

where $c_\mu = \sum_{i+j=\mu} r_i m_j$. Next, $M[X]$ is called the polynomial module.

Let $R[X, X^{-1}]$ be the Laurent polynomial ring and $M[X, X^{-1}] = \{\sum_{j=-u}^{l} m_j x^j | l, j \in \mathbb{N} \cup \{0\}\}, m_j \in M\}$. For any $\sum_{i=-v}^{k} r_i x^i \in R[X, X^{-1}]$ and $\sum_{j=-u}^{l} m_j x^j \in M[X, X^{-1}]$, $M[X, X^{-1}]$ acquires the structure of an $R[X, X^{-1}]$-module with pointwise addition and scalar multiplication defined by

$$(\sum_{i=-v}^{k} r_i x^i)(\sum_{j=-u}^{l} m_j x^j) = \sum_{\mu=-(v+u)}^{k+l} c_\mu x^\mu,$$

where $c_\mu = \sum_{i+j=\mu} r_i m_j$. Next, $M[X, X^{-1}]$ is called the Laurent polynomial module.

Let $R[[X]]$ be the power series ring and $M[[X]] = \{\sum_{j\geq 0} m_j x^j | j \in \mathbb{N} \cup \{0\}, m_j \in M\}$. For any $\sum_{i\geq 0} r_i x^i \in R[[X]]$ and $\sum_{j\geq 0} m_j x^j \in M[[X]]$, $M[[X]]$ acquires the structure of an $R[[X]]$-module with pointwise addition and scalar multiplication defined by

$$(\sum_{i\geq 0} r_i x^i)(\sum_{j\geq 0} m_j x^j) = \sum_{k\geq 0} c_k x^k,$$

where $c_k = \sum_{i+j=k} r_i m_j$. Next, $M[[X]]$ is called the power series module.

Let $R[[X, X^{-1}]]$ be the Laurent series ring and $M[[X, X^{-1}]] = \{\sum_j m_j x^j | j \in \mathbb{Z}, m_j \in M\}$. For any $\sum_i r_i x^i \in R[[X, X^{-1}]]$ and $\sum_j m_j x^j \in M[[X, X^{-1}]]$, $M[[X, X^{-1}]]$ acquires the structure of an $R[[X]]$-module with pointwise addition and scalar multiplication defined by

$$(\sum_i r_i x^i)(\sum_j m_j x^j) = \sum_k c_k x^k,$$

where $c_k = \sum_{i+j=k} r_i m_j$. Next, $M[[X, X^{-1}]]$ is called Laurent Series Module.

3. Main Results

In this section we provide the necessary and sufficient conditions for $M[X]$, $M[X, X^{-1}]$, $M[[X]]$, and $M[[X, X^{-1}]]$ to be S-Noetherian modules. Besides that, we also obtain the sufficient conditions for $M[X]$ (resp. $M[X, X^{-1}]$, $M[[X]]$, and $M[[X, X^{-1}]]$) to be $S[X]$-Noetherian (resp. $S[X, X^{-1}]$-Noetherian, $S[[X]]$-Noetherian, and $S[[X, X^{-1}]]$-Noetherian).

For any ring R, we denoted $R \oplus \cdots \oplus R$ (n factor), for some $n \geq 1$, by $\bigoplus R^{(n)}$.

Lemma 3. *M is finitely generated R-module if and only if it is isomorphic to a quotient of $\bigoplus R^{(n)}$ for some $n > 0$.*

Proof. Let M be a finitely generated R-module and $m_1, ..., m_n \in M$ generates M. We define a morphism

$$f : \bigoplus R^{(n)} \to M$$

$$(r_1, ..., r_n) \mapsto r_1 m_1 + \cdots + r_n m_n.$$

Since, for any $(r_1, ..., r_n), (t_1, ..., t_n) \in \bigoplus R^{(n)}$ and $r \in R$,

$$f((r_1, ..., r_n) + (t_1, ..., t_n)) = f((r_1, ..., r_n)) + f((t_1, ..., t_n))$$

and

$$f(r(r_1, ..., r_n)) = r f((r_1, ..., r_n)),$$

f is a R-module homomorphism. Since, for any $m \in M$, there exist $(r_1, ..., r_n) \in \bigoplus R^{(n)}$ such that $f((r_1, ..., r_n)) = r_1 m_1 + \cdots + r_n m_n$, then f is a epimorphism. Therefore, $im(f) = M$. Then, by Fundamental Isomorphism Theorem of Modules, we get $M \cong \bigoplus R^{(n)}/Ker(f)$.

Next, we assume $M \cong \bigoplus R^{(n)}/K$, for some submodule $K \subseteq \bigoplus R^{(n)}$. Image of any generating set of $\bigoplus R^{(n)}$ in M, is generates M. Therefore, M is finitely generated. □

The following lemma shows that, If K is any submodule of R-modul M, then $K[X]$ is submodule of $R[X]$-modul $M[X]$. For the next following, $K \leq M$ means K is a submodule of M.

Lemma 4. *Let M be an R-module and $M[X]$ an $R[X]$-module. If $K \leq M$, then $K[X] \leq M[X]$.*

Proof. Let K is a submodule of M, then $rt + su \in K$, for every $r, s \in K$ and $t, u \in M$. For any $k = \sum_{i=0}^{p} r_i x^i, l = \sum_{i=0}^{p} s_i x^i \in K[X]$ and $m = \sum_{j=0}^{q} t_j x^j, n = \sum_{j=0}^{q} u_j x^j \in M[X]$, we get

$$km + ln = (\sum_{i=0}^{p} r_i x^i)(\sum_{j=0}^{q} t_j x^j) + (\sum_{i=0}^{p} s_i x^i)(\sum_{j=0}^{q} u_j x^j)$$

$$= \sum_{\alpha=0}^{p+q}(\sum_{i+j=\alpha} r_i t_j)x^\alpha + \sum_{\alpha=0}^{p+q}(\sum_{i+j=\alpha} s_i u_j)x^\alpha$$

$$= \sum_{\alpha=0}^{p+q}(\sum_{i+j=\alpha} (r_i t_j + s_i u_j))x^\alpha = \sum_{\alpha=0}^{p+q} w_\alpha x^\alpha,$$

where $w_\alpha = \sum_{i+j=\alpha}(r_i t_j + s_i u_j)$.

Since K is a submodule of M, $w_\alpha \in K[X]$. Therefore, $km + ln \in K[X]$. So, $K[X] \leq M[X]$. □

In a similar way on Lemma 4, we obtain the following properties.

Lemma 5. *Let M be an R-module. If $K \leq M$, then:*

(1) $K[X, X^{-1}] \leq M[X, X^{-1}]$

(2) $K[[X]] \leq M[[X]]$

(3) $K[[X, X^{-1}]] \leq M[[X, X^{-1}]]$

The following proposition shows that the polynomial ring over a quotient ring is isomorphic to a quotient ring of the polynomial ring.

Proposition 5. *Let M be an R-module and $M[X]$ be an $R[X]$-module. If $N \leq M$, then $(M/N)[X] \cong M[X]/N[X]$.*

Proof. Define a map $\varphi : M[X] \to (M/N)[X]$, by

$$f = m_0 + m_1 x + \cdots + m_n x^n \mapsto \overline{f} = \overline{m_0} + \overline{m_1} x + \cdots + \overline{m_n} x^n,$$

where $\overline{m_i} = m_i + N$, for $i = 0, 1, 2, ..., n$.

For any $\overline{f} = \overline{m_0} + \overline{m_1} x + \cdots + \overline{m_n} x^n \in (M/N)[X]$, there exist $f = m_0 + m_1 x + \cdots + m_n x^n \in M[X]$ such that $\varphi(f) = \overline{f}$. Then φ is surjective. Therefore, $Im(\varphi) = (M/N)[X]$. Next, if $\varphi(f) = \overline{0}$, then $\overline{f} = \overline{0}$. Therefore, $m_i \in N$, for $i = 0, 1, 2, ..., n$. So, $Ker(\varphi) = N[X]$. Hence, based on the fundamental isomorphism theorem of modules, we get $M[X]/N[X] \cong (M/N)[X]$. □

In similar ways, the properties on Proposition 5 also holds for the Laurent polynomial module, the power series module, and the Laurent series module.

Proposition 6. *Let M be an R-module. If $N \leq M$, then:*

(1) $(M/N)[X, X^{-1}] \cong M[X, X^{-1}]/N[X, X^{-1}]$.

(2) $(M/N)[[X]] \cong M[[X]]/N[[X]]$.

(3) $(M/N)[[X, X^{-1}]] \cong M[[X, X^{-1}]]/N[[X, X^{-1}]]$.

The following proposition shows that the polynomial ring over $\bigoplus R^{(n)}$ is isomorphic to the direct sum of polynomial ring $R[X] \oplus \cdots \oplus R[X]$ (n factor).

Proposition 7. *Let R be a ring and $n \geq 1$. Then* $(\bigoplus R^{(n)})[X] \cong \bigoplus (R[X])^{(n)}$.

Proof. Let $\alpha_i = (r_{i1}, r_{i2}, , ..., r_{in}) \in \bigoplus R^{(n)}$ and $f_j = r_{0j} + r_{1j}x + \cdots + r_{pj}x^p \in R[X]$; where $r_{ij} \in R$ for $i = 0, 1, 2, ..., p$ and $j = 1, 2, ..., n$. Then we get

$$\alpha_0 x^0 = (r_{01}, r_{02}, , ..., r_{0n})$$

$$\alpha_1 x^1 = (r_{11}x, r_{12}x, , ..., r_{1n}x)$$

$$\vdots$$

$$\alpha_p x^p = (r_{p1}x^p, r_{p2}x^p, ..., r_{pn}x^p).$$

Hence, for any $f = \alpha_0 + \alpha_1 x + \cdots + \alpha_p x^p \in (\bigoplus R^{(n)})[X]$ can be written as

$$\begin{aligned} f &= \alpha_0 + \alpha_1 x + \cdots + \alpha_p x^p \\ &= (r_{01}, r_{02}, , ..., r_{0n}) + (r_{11}x, r_{12}x, , ..., r_{1n}x) + \cdots + (r_{p1}x^p, r_{p2}x^p, , ..., r_{pn}x^p) \\ &= (r_{01} + r_{11}x + \cdots + r_{p1}x^p, r_{02} + r_{12}x + \cdots + r_{p2}x^p, ..., r_{0n} + r_{1n}x + \cdots + r_{pn}x^p) \\ &= (f_1, f_2, ..., f_n). \end{aligned}$$

Now, we define a map $\varphi : (\bigoplus R^{(n)})[X] \to \bigoplus (R[X])^{(n)}$, by

$$f = \alpha_0 + \alpha_1 x + \cdots + \alpha_p x^p \mapsto (f_1, f_2, ..., f_n).$$

We will show that φ is a ring isomorphism.

Let $f = \sum_{i=0}^{p} \alpha_i x^i$, $g = \sum_{l=0}^{q} \beta_l x^l \in (\bigoplus R^{(n)})[X]$, where $\alpha_i = (r_{i1}, r_{i2}, , ..., r_{in})$ for $i = 0, 1, 2, ..., p$ and $\alpha_i = 0$ for $i > p$, and $\beta_l = (s_{l1}, s_{l2}, , ..., s_{ln})$ for $l = 0, 1, 2, ..., q$ and $\beta_l = 0$ for $l > q$. And let $f_j = \sum_{k=0}^{p} r_{kj}x^k, g_j = \sum_{k=0}^{q} s_{kj}x^k \in R[X]$ for $j = 1, 2, ..., n$.

Since,

$$\begin{aligned} \varphi(f + g) &= \varphi(\sum_{i=0}^{p} \alpha_i x^i + \sum_{l=0}^{q} \beta_l x^l) = \varphi(\sum_{k=0}^{max(p,q)} (\alpha_k + \beta_k)x^k) \\ &= (\sum_{k=0}^{max(p,q)} (r_{k1} + s_{k1})x^k, ..., \sum_{k=0}^{max(p,q)} (r_{kn} + s_{kn})x^k) \end{aligned}$$

$$= (\sum_{k=0}^{p} r_{k1}x^k, ..., \sum_{k=0}^{p} r_{kn}x^k) + (\sum_{k=0}^{q} s_{k1}x^k, ..., \sum_{k=0}^{q} s_{kn}x^k)$$
$$= (f_1, ..., f_n) + (g_1, ..., g_n) = \varphi(f) + \varphi(g)$$

and

$$\varphi(fg) = \varphi((\sum_{i=0}^{p} \alpha_i x^i)(\sum_{l=0}^{q} \beta_l x^l)) = \varphi(\sum_{k=0}^{p+q}(\sum_{i+l=k} \alpha_i\beta_l)x^k)$$
$$= (\sum_{k=0}^{p+q}(\sum_{i+l=k} r_{i1}s_{l1})x^k, ..., \sum_{k=0}^{p+q}(\sum_{i+l=k} r_{in}s_{ln})x^k)$$
$$= (\sum_{i=0}^{p} r_{i1}x^i \sum_{l=0}^{q} s_{l1}x^l, ..., \sum_{i=0}^{p} r_{in}x^i \sum_{l=0}^{q} s_{ln}x^l)$$
$$= (\sum_{i=0}^{p} r_{i1}x^i, ..., \sum_{i=0}^{p} r_{in}x^i)(\sum_{l=0}^{q} s_{l1}x^l, ..., \sum_{l=0}^{q} s_{ln}x^l)$$
$$= (f_1, ..., f_n)(g_1, ..., g_n) = \varphi(f)\varphi(g)$$

Then, φ is a ring homomorphism.

Next, we wil show that φ is injective and surjective. Let $\varphi(f) = \varphi(g)$, then $(f_1, ..., f_n) = (g_1, ..., g_n)$. So, $f_j = g_j$ implies $r_{kj} = s_{kj}$ and $p = q$ for $k = 0, 1, 2, ..., p$ and $j = 1, 2, ...n$. Hence $\alpha_i = \beta_l$ for $i, l = 0, 1, 2, ..., p$. So, we get $f = \sum_{i=0}^{p} \alpha_i x^i = \sum_{l=0}^{q} \beta_l x^l = g$. In the other word, φ is injective.

Furthermore, since for every $(f_1, f_2, ..., f_n) \in \bigoplus (R[X])^{(n)}$, there exist $f = \sum_{i=1}^{p} \alpha_i x^i = (f_1, f_2, ..., f_n)$ such that $\varphi(f) = (f_1, f_2, ..., f_n)$, then φ is surjective.

So, φ is a ring isomorphism. Therefore, $(\bigoplus R^{(n)})[X] \cong \bigoplus (R[X])^{(n)}$. □

In similar ways, the conditions on Proposition 7 also holds for $R[X, X^{-1}]$, $R[[X]]$, and $R[X, X^{-1}]$.

Proposition 8. *Let R be a ring and $n \geq 1$, then:*

(1) $(\bigoplus R^{(n)})[X, X^{-1}] \cong \bigoplus (R[X, X^{-1}])^{(n)}$.

(2) $(\bigoplus R^{(n)})[[X]] \cong \bigoplus (R[[X]])^{(n)}$.

(3) $(\bigoplus R^{(n)})[[X, X^{-1}]] \cong \bigoplus (R[[X, X^{-1}]])^{(n)}$.

Proposition 7 and Proposition 8 can be obtained by the different way, i.e. by generalizing Proposition 2.7. in [4] as follows.

Proposition 9. *For $i = 1, 2, ..., n$, let R_i be rings, $(S, \leq)$ a strictly ordered monoid, and $\omega^{(i)} : S \to End(R_i)$ monoid homomorphisms. Then*

$$(\bigoplus_{i=1}^{n} R_i)[[S, \bigoplus_{i=1}^{n} \omega^{(i)}]] \cong \bigoplus_{i=1}^{n} (R_i[[S, \omega^{(i)}]])$$

.

By taking $S = \mathbb{N} \cup 0$ with usual addition, trivial order $\leq$ and $\omega_s^{(i)} = id_{R_i}$ for every $s \in S$ and every i, we get Proposition 7.

If $S = \mathbb{Z}$ with usual addition, trivial order $\leq$ and $\omega_s^{(i)} = id_{R_i}$, for every $s \in S$ and every i, we get Proposition 8(1). Next, by taking $S = \mathbb{N} \cup 0$ with usual addition, usual order $\leq$ and $\omega_s^{(i)} = id_{R_i}$, for every $s \in S$ and every i, we get Proposition 8(2). Finally, if $S = \mathbb{Z}$ with usual addition, usual order $\leq$ and $\omega_s^{(i)} = id_{R_i}$, for every $s \in S$ and every i, we get Proposition 8(3). With the different approach, Proposition 9 above came to similar result to Proposition 1.15. on Mazurek and Ziembowski [3].

The sufficient condition for $R[X]$-module $M[X]$ to be finitely generated module is given in the following proposition.

Proposition 10. *Let M be an R-module and $M[X]$ an $R[X]$-module. If M is finitely generated, then so is $M[X]$.*

Proof. Based on Lemma 3, it is enough to show $M[X] \cong \bigoplus (R[X])^{(n)}/N$, for some submodule $N \subseteq \bigoplus (R[X])^{(n)}$. Since M is finitely generated, then by Lemma 3, $M \cong \bigoplus R^{(n)}/K$, for some submodule $K \subseteq \bigoplus R^{(n)}$. Since $K \leq \bigoplus R^{(n)}$, then by Lemma 4, we get $K[X] \leq (\bigoplus R^{(n)})[X]$. Furthermore, base on Proposition 7, we have $K[X] \leq \bigoplus (R[X])^{(n)}$. Hence, we can choose $N = K[X]$.

Now, we will show, $(\bigoplus R^{(n)}/K)[X] \cong \bigoplus (R[X])^{(n)}/K[X]$. By using Proposition 5, we get $(\bigoplus R^{(n)}/K)[X] \cong (\bigoplus R^{(n)})[X]/K[X]$. Furthermore, by using Proposition 7, we get $(\bigoplus R^{(n)})[X] \cong \bigoplus (R[X])^{(n)}$. So, $(\bigoplus R^{(n)}/K)[X] \cong (\bigoplus R^{(n)})[X]/K[X] \cong \bigoplus (R[X])^{(n)}/K[X]$. In other word, $M[X] \cong \bigoplus (R[X])^{(n)}/N$, or $M[X]$ is finitely generated as an $R[X]$-module. □

In similar ways, we obtain the sufficient conditions for $R[X, X^{-1}]$-module $M[X, X^{-1}]$, $R[[X]]$-module $M[[X]]$, and $R[[X, X^{-1}]]$-module $M[[X, X^{-1}]]$ to be finitely generated.

Proposition 11. *If M is finitely generated as an R-module, then*

(1) $M[X, X^{-1}]$ is finitely generated as an $R[X, X^{-1}]$-module.

(2) $M[[X]]$ is finitely generated as an $R[[X]]$-module.

(3) $M[[X, X^{-1}]]$ is finitely generated as an $R[[X, X^{-1}]]$-module.

Now, we give the necessary conditions for $R[X]$-module $M[X]$ to be an S-Noetherian module.

Theorem 1. *Let R be a ring, $S \subseteq R$ a multiplicative set, and M an R-module. If $R[X]$-module $M[x]$ is S-Noetherian, then so is R-modul M.*

Proof. Let $N = \{\sum_{i=0}^{k} m_i x^i \in M[X] | m_0 = 0\}$. For any $\sum_{i=0}^{k} m_i x^i, \sum_{i=0}^{l} n_i x^i \in N$ and $\sum_{j=0}^{p} r_j x^j, \sum_{j=0}^{q} s_j x^j \in R[X]$, we have

$$\begin{aligned}\sum_{i=0}^{k} m_i x^i \sum_{j=0}^{p} r_j x^j + \sum_{i=0}^{l} n_i x^i \sum_{j=0}^{q} s_j x^j &= \sum_{u=0}^{k+p} (\sum_{i+j=u} m_i r_j) x^u + \sum_{u=0}^{l+q} (\sum_{i+j=u} n_i s_j) x^u \\ &= \sum_{u=0}^{max(k+p,l+q)} (\sum_{i+j=u} (m_i r_j + n_i s_j)) x^u \\ &= \sum_{u=0}^{d} c_u x^u,\end{aligned}$$

where $c_u = \sum_{i+j=u} (m_i r_j + n_i s_j)$ and $d = max(k+p, l+q)$.

Now, we will show that N is a submodule of $M[X]$. To do this, it is enough to show that $c_0 = 0$. Since $m_0 = n_0 = 0$, $c_0 = \sum_{i+j=0} (m_i r_j + n_i s_j) = m_0 r_0 + n_0 s_0 = 0$. So, $\sum_{u=0}^{d} c_u x^u = \sum_{i=0}^{k} m_i x^i \sum_{j=0}^{p} r_j x^j + \sum_{i=0}^{l} n_i x^i \sum_{j=0}^{q} s_j x^j \in N$, that is, N is a submodule of $M[X]$.

Next, we define $\psi : M \to M[X]/N$ by $\psi(m) = m + N$, for every $m \in M$. Since, for any $m, m' \in M$ and $r \in R$, $\psi(m + m') = (m + m') + N = (m + N) + (m' + N) = \psi(m) + \psi(m')$ and $\psi(rm) = rm + N = r(m + N) = r\psi(m)$, then ψ is a module homomorphism. For any $m + N \in M[X]/N$, there exist $m \in M$ such that $\psi(m) = m + N$. So, ψ is surjective. Furthermore, if $\psi(m) = 0$, then $m + N = 0$, that is $m \in N$. Then by the definition of N, we get $m = 0$. Therefore, $Ker(\psi) = 0$. So, ψ is injective. In other word, ψ is a module isomorphism. So, $M \cong M[X]/N$.

Since $M[X]$ is a S-Noetherian module, then based on Lemma 1(1), $M[X]/N$ is S-Noetherian. Therefore, M is S-Noetherian. □

In similar ways, we also obtain the necessary conditions for $M[X, X^{-1}]$, $M[[X]]$, and $M[[X, X^{-1}]]$ to be S-Noetherian.

Theorem 2. *Let R be a ring, $S \subseteq R$ a multiplicative set, and M an R-module.*

(1) If $M[X, X^{-1}]$ is S-Noetherian, then M is S-Noetherian.

(2) If $M[[X]]$ is S-Noetherian, then M is S-Noetherian.

(3) If $M[[X, X^{-1}]]$ is S-Noetherian, then M is S-Noetherian.

Next, the sufficient conditions for $R[X]$-module $M[X]$ to be an S-Noetherian module are given by the following theorem.

Theorem 3. *Let R be a ring, $S \subseteq R$ an anti-Archimedean multiplicative set, and M an R-module. If R is S-Noetherian and M is finitely generated, then $R[X]$-module $M[X]$ is an S-Noetherian module.*

Proof. Since R is S-Noetherian and $T \subseteq R$ is an anti-Archimedean multiplicative set, based on Proposition 1, $R[X]$ is S-Noetherian. Next, based on Proposition 10, $M[X]$ is a finitely generated $R[X]$-module. Then, by using Lemma 1(3), we get $M[X]$ is S-Noetherian $R[X]$-module. □

Now, we give the sufficient conditions for $R[X, X^{-1}]$-module $M[X, X^{-1}]$ to be an S-Noetherian module.

Theorem 4. *Let R be a ring, $S \subseteq R$ an anti-Archimedean multiplicative set, and M an R-module. If R is S-Noetherian and M is finitely generated, then $R[X, X^{-1}]$-module $M[X, X^{-1}]$ is an S-Noetherian module.*

Proof. Since R is S-Noetherian and $T \subseteq R$ is an anti-Archimedean multiplicative set, based on Proposition 3, $R[X, X^{-1}]$ is S-Noetherian. Next, based on Proposition 11(1), $M[X, X^{-1}]$ is a finitely generated $R[X, X^{-1}]$-module. Then, by using Lemma 1(3), we get $M[X, X^{-1}]$ is an S-Noetherian $R[X, X^{-1}]$-module. □

Next, we give the sufficient conditions for $R[[X]]$-module $M[[X]]$ to be an S-Noetherian module.

Theorem 5. *Let R be a ring, $S \subseteq R$ an anti-Archimedean multiplicative set of R consisting nonzerodivisors, and M an R-module. If R is S-Noetherian and M is finitely generated, then $R[[X]]$-module $M[[X]]$ is an S-Noetherian module.*

Proof. Since R is S-Noetherian and $T \subseteq R$ is an anti-Archimedean multiplicative set consisting nonzerodivisors, based on Proposition 2, $R[[X]]$ is S-Noetherian. Next, based on Proposition 11(2), $M[[X]]$ is a finitely generated $R[[X]]$-module. Then, by using Lemma 1(3), we get $M[[X]]$ is an S-Noetherian $R[[X]]$-module. □

Next, the sufficient conditions for $R[[X, X^{-1}]]$-module $M[[X, X^{-1}]]$ to be an S-Noetherian module are given by the following theorem.

Theorem 6. *Let R be a ring, $S \subseteq R$ an anti-Archimedean multiplicative set of R consisting nonzerodivisors, and M an R-module. If R is S-Noetherian and M is finitely generated, then $R[[X, X^{-1}]]$-module $M[[X, X^{-1}]]$ is an S-Noetherian module.*

Proof. Since R is S-Noetherian and $T \subseteq R$ is an anti-Archimedean multiplicative set consisting nonzerodivisors, based on Proposition 3, $R[[X, X^{-1}]]$ is S-Noetherian. Next, based on Proposition 11(3), $M[[X, X^{-1}]]$ is a finitely generated $R[[X, X^{-1}]]$-module. Then, by using Lemma 1(3), we get $M[[X, X^{-1}]]$ is an S-Noetherian $R[[X, X^{-1}]]$-module. □

For any multiplicative subset S of a ring R, we define the set

$$S[X] = \{\sum_{i=0}^{n} s_i x^i \in R[X] | s_i \in S \text{ for every } i\}.$$

The following lemma shows that $S[X]$ is a multiplicative subset of $R[X]$ when S is additively closed.

Lemma 6. *Let S be a multiplicative subset of a ring R. If S is additively closed, then $S[X]$ is a multiplicative subset of polynomial ring $R[X]$.*

Proof. For any $f, g \in S[X]$, we will show that $fg \in S[X]$. Let $f = \sum_{i=0}^{p} s_i x^i$ and $g = \sum_{j=0}^{q} t_j x^j$, then we have $fg = (\sum_{i=0}^{p} s_i x^i)(\sum_{j=0}^{q} t_j x^j) = \sum_{k=0}^{p+q} c_k x^k$, where $c_k = \sum_{i+j=k} s_i t_j$.

Since S is multiplicative subset of R, we obtain $s_i t_j \in S$ for every i, j. Furthermore, since S is additively closed, we have $c_k = \sum_{i+j=k} s_i t_j \in S$. In other words, $fg \in S[X]$. Thus, $S[X]$ is a multiplicative subset of $R[X]$. □

Now, for any multiplicative subset S of a ring R, we also define the sets

$$S[X, X^{-1}] = \{\sum_{i=-u}^{n} s_i x^i \in R[X, X^{-1}] | s_i \in S \text{ for every } i\},$$

$$S[[X]] = \{\sum_{i \geq 0} s_i x^i \in R[[X]] | s_i \in S \text{ for every } i\}, \text{ and}$$

$$S[[X, X^{-1}]] = \{\sum_{i} s_i x^i \in R[[X, X^{-1}]] | s_i \in S \text{ for every } i\}.$$

In similar ways on Lemma 6, we obtain the sufficient conditions for $S[X, X^{-1}]$, $S[[X]]$, and $S[[X, X^{-1}]]$ to be multiplicative subset of $R[X, X^{-1}]$, $R[[X]]$, and $R[[X, X^{-1}]]$, respectively.

Lemma 7. *Let S be a multiplicative subset of a ring R. If S is additively closed, then*

(1) $S[X, X^{-1}]$ is a multiplicative subset of Laurent polynomial ring $R[X, X^{-1}]$.

(2) $S[[X]]$ is a multiplicative subset of power series ring $R[[X]]$.

(3) $S[[X, X^{-1}]]$ is a multiplicative subset of Laurent series ring $R[[X, X^{-1}]]$.

Finally, we give the sufficient conditions for $M[X]$, $M[X, X^{-1}]$, $M[[X]]$, and $M[[X, X^{-1}]]$ to be $S[X]$-Noetherian, $S[X, X^{-1}]$-Noetherian, $S[[X]]$-Noetherian, and $S[[X, X^{-1}]]$-Noetherian, respectively.

Theorem 7. *Let R be an S-Noetherian ring, $S \subseteq R$ an additive and multiplicative set with anti-Archimedean properties, and M a finitely generated R-module. Then,*

(1) $R[X]$-module $M[X]$ is an $S[X]$-Noetherian module.

(2) $R[X, X^{-1}]$-module $M[X, X^{-1}]$ is an $S[X, X^{-1}]$-Noetherian module.

Proof.

(1) Based on Theorem 3, $M[X]$ is an S-Noetherian $R[X]$-module. Furthermore, $S[X]$ is a multiplicative subset of $R[X]$ by Lemma 6. It is clear that $S \subseteq S[X]$. Therefore, by using Lemma 2, we get $M[X]$ is an $S[X]$-Noetherian $R[X]$-module.

(2) Based on Theorem 4, $M[X, X^{-1}]$ is an S-Noetherian $R[X, X^{-1}]$-module. Furthermore, $S[X, X^{-1}]$ is a multiplicative subset of $R[X, X^{-1}]$ by Lemma 7(1). It is clear that $S \subseteq S[X, X^{-1}]$. Therefore, by using Lemma 2, we get $M[X, X^{-1}]$ is an $S[X, X^{-1}]$-Noetherian $R[X, X^{-1}]$-module. □

Theorem 8. *Let R be an S-Noetherian ring, $S \subseteq R$ an additive and multiplicative set consisting nonzerodivisors with anti-Archimedean properties, and M a finitely generated R-module. Then,*

(1) $R[[X]]$-module $M[[X]]$ is an $S[[X]]$-Noetherian module.

(2) $R[[X, X^{-1}]]$-module $M[[X, X^{-1}]]$ is an $S[[X, X^{-1}]]$-Noetherian module.

Proof.

(1) Based on Theorem 5, $M[[X]]$ is an S-Noetherian $R[[X]]$-module. Furthermore, $S[[X]]$ is a multiplicative subset of $R[[X]]$ by Lemma 7(2). It is clear that $S \subseteq S[[X]]$. Therefore, by using Lemma 2, we get $M[[X]]$ is an $S[[X]]$-Noetherian $R[[X]]$-module.

(2) Based on Theorem 6, $M[[X, X^{-1}]]$ is an S-Noetherian $R[[X, X^{-1}]]$-module. Furthermore, $S[[X, X^{-1}]]$ is a multiplicative subset of $R[[X, X^{-1}]]$ by Lemma 7(3). It is clear that $S \subseteq S[[X, X^{-1}]]$. Therefore, by using Lemma 2, we get $M[[X, X^{-1}]]$ is an $S[[X, X^{-1}]]$-Noetherian $R[[X, X^{-1}]]$-module. □

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