Applied Mathematical Sciences, Vol. 10, 2016, no. 63, 3107 - 3118
HIKARI Ltd, www.m-hikari.com
https://doi.org/10.12988/ams.2016.69238

Empirical Comparison of ML and UMVU Estimators of the Generalized Variance for some Normal Stable Tweedie Models: a Simulation Study

Khoirin Nisa[1]

Department of Mathematics, Lampung University, Indonesia

Célestin C. Kokonendji

Laboratoire de Mathématiques de Besançon
Université Bourgogne Franche-Comté, France

Asep Saefuddin, Aji Hamim Wigena

Department of Statistics, Bogor Agricultural University, Indonesia

I Wayan Mangku

Department of Mathematics, Bogor Agricultural University, Indonesia

Abstract

This paper discuss a comparison of the maximum likelihood (ML) estimator and the uniformly minimum variance unbiased (UMVU) estimator of generalized variance for some normal stable Tweedie models through simulation study. We describe the estimation of some particular cases of multivariate NST models, i.e. normal gamma, normal Poisson

[1]Also affiliated to Bogor Agricultural University, Indonesia and Université Bourgogne Franche-Comté, france

and normal invers-Gaussian. The result shows that UMVU method produces better estimations than ML method on small samples and they both produce similar estimations on large samples.

Mathematics Subject Classification: 62H12

Keywords: Multivariate natural exponential family, variance function, maximum likelihood, uniformly minimum variance unbiased

1 Introduction

Normal stable Tweedie (NST) models were introduced by Boubacar Maïnassara and Kokonendji [3] as the extension of normal gamma [5] and normal inverse Gaussian [4] models. NST models are composed by a fixed univariate stable Tweedie variable having a positive value domain, and the remaining random variables given the fixed one are real independent Gaussian variables with the same variance equal to the fixed component. For a k-dimensional ($k \geq 2$) NST random vector $\mathbf{X} = (X_1, \ldots, X_k)^\top$, the generating σ-finite positive measure $\nu_{\alpha,t}$ is given by

$$\nu_{\alpha,t}(d\mathbf{x}) = \xi_{\alpha,t}(dx_1) \prod_{j=2}^{k} \xi_{2,x_1}(dx_j), \tag{1}$$

where $\xi_{\alpha,t}$ is the well-known probability measure of univariate positive σ-stable distribution generating Lévy process $(X_t^\alpha)_{t>0}$ which was introduced by Feller [7] as follows

$$\xi_{\alpha,t}(dx) = \frac{1}{\pi x} \sum_{r=0}^{\infty} \frac{t^r \Gamma(1+\alpha r) sin(-r\pi\alpha)}{r! \alpha^r (\alpha-1)^{-r} \left[(1-\alpha)x\right]^{\alpha r}} \mathbb{1}_{x>0} dx = \xi_{\alpha,t}(x)dx. \tag{2}$$

Here $\alpha \in (0, 1)$ is the index parameter, $\Gamma(.)$ is the classical gamma function, and $\mathbb{I}_A$ denotes the indicator function of any given event A that takes the value 1 if the event accurs and 0 otherwise. Paremeter α can be extended into $\alpha \in (-\infty, 2]$ [10]. For $\alpha = 2$ in (2), we obtain the normal distribution with density

$$\xi_{2,t}(dx) = \frac{1}{\sqrt{2\pi t}} \exp\left(\frac{-x^2}{2t}\right) dx.$$

In multivariate analysis, including NST models, generalized variance has important roles in descriptive analysis and inferences. In this paper we discuss the ML and UMVU generalized variance estimators of the following NST models:

1. **Normal gamma** (NG). For $\alpha = 0$ in (1) one has the generating measure of normal gamma as follows:

$$\nu_{0,t}(d\boldsymbol{x}) = \frac{x_1^{t-1}}{(2\pi x_1)^{(k-1)/2}\gamma(t)} \exp\left(-x_1 - \frac{1}{2x_1}\sum_{j=2}^{k} x_j^2\right)\mathbb{I}_{x_1>0}dx_1dx_2\cdots x_k. \quad (3)$$

It is a member of simple quadratic natural exponential families (NEFs) [6] and was called as "gamma-Gaussian" which was characterized by Kokonendji and Masmoedi [8].

2. **Normal invers Gaussian** (NIG). For $\alpha = 1/2$ in (1) we can write the normal inverse Gaussian generating measure as follows

$$\nu_{1/2,t}(d\boldsymbol{x}) = \frac{tx_1^{-(k+2)/2}}{(2\pi)^{k/2}} \exp\left[\frac{-1}{2x_1}\sum_{j=2}^{k} x_j^2\left(t^2 + \sum_{j=2}^{k} x_j^2\right)\right]\mathbb{I}_{x_1>0}dx_1dx_2\cdots x_k. \quad (4)$$

It was introduced as a variance-mean mixture of a univariate inverse Gaussian with multivariate Gaussian distribution [4] and has been used in finance (see e.g. [1, 2]).

3. **Normal Poisson** (NP). For the limit case $\alpha = -\infty$ in (1) we have the so-called normal Poisson generating measure

$$\nu_{-\infty,t}(d\boldsymbol{x}) = \frac{t^{x_1}(x_1!)^{-}1}{(2\pi x_1)^{(k-1)/2}} \exp\left(-t - \frac{1}{2x_1}\sum_{j=2}^{k} x_j^2\right)\mathbb{I}_{x_1\in\mathbb{N}^*}\delta_{x_1}(dx_1)dx_2\cdots x_k. \quad (5)$$

Since it is also possible to have $x_1 = 0$ in the Poisson part, the corresponding normal Poisson distribution is degenerated as δ_0. This model is recently characterized by Nisa et al. [9]

2 Generalized Variance of NST Models

The cumulant function $\mathbf{K}_{\nu_{\alpha,t}}(\boldsymbol{\theta}) = \log\int_{\mathbb{R}^k}\exp\left(\boldsymbol{\theta}^T\boldsymbol{x}\right)\nu_{\alpha,t}(d\boldsymbol{x})$ of NST models is given by

$$\mathbf{K}_{\nu_{\alpha,t}}(\boldsymbol{\theta}) = K_{\xi_{\alpha,t}}\left(\theta_1 + \frac{1}{2}\sum_{j=2}^{k}\theta_j^2\right) \quad (6)$$

where $K_{\xi_{\alpha,t}} = \log\int_{\mathbb{R}^k}\exp(\theta x)\xi_{\alpha,t}(dx)$ is the cumulant function of the associated univariate stable Tweedie distribution $\xi_{\alpha,t}$. Then for each distribution we discuss here the corresponding cumulant function is given by

$$\mathbf{K}_{\nu_{\alpha,t}}(\boldsymbol{\theta}) = \begin{cases} t\exp\left(\theta_i + \frac{1}{2}\sum_{j=2}^{k}\theta_j^2\right), & \text{for NG} \\ -t\left[\frac{1}{2}\left(-\theta_i - \frac{1}{2}\sum_{j=2}^{k}\theta_j^2\right)\right], & \text{for NIG} \\ -t\log\left(-\theta_i - \frac{1}{2}\sum_{j=2}^{k}\theta_j^2\right), & \text{for NP} \end{cases} \tag{7}$$

(see [3, Section 2]). The cumulant function is finite for $\boldsymbol{\theta}$ in canonical domain $\Theta(\nu_{\alpha,t}) = \{\boldsymbol{\theta} \in \mathbb{R}^k; \theta_1 + \frac{1}{2}\sum_{j=2}^{k}\theta_j^2 \in \Theta(\xi_{\alpha,1})\}$ with

$$\Theta(\xi_{\alpha,1}) = \begin{cases} (-\infty, 0) & \text{for NG} \\ (-\infty, 0] & \text{for NIG} \\ \mathbb{R} & \text{for NP.} \end{cases}$$

Let $\mathbf{G}(\nu_{\alpha,t}) = \left\{P(\boldsymbol{\theta};\alpha,t); \boldsymbol{\theta} = (\theta_1,\ldots,\theta_k)^\top \in \Theta(\nu_{\alpha,t})\right\}$ be the set of probability distributions $P(\boldsymbol{\theta};\alpha,t)(d\mathbf{x}) = \exp\left[\boldsymbol{\theta}^\top\mathbf{x} - \mathbf{K}_{\nu_{\alpha,t}}(\boldsymbol{\theta})\right]\nu_{\alpha,t}(d\mathbf{x})$. The variance function which is the variance-covariance matrix in term of mean parameterization; $P(\boldsymbol{\mu};\mathbf{G}_{\alpha,t}) := P[\boldsymbol{\theta}(\boldsymbol{\mu});\nu_{\alpha,t}]$; is obtained through the second derivative of the cumulant function, i.e. $\mathbf{V}_{\mathbf{G}_{\alpha,t}}(\boldsymbol{\mu}) = \mathbf{K}''_{\nu_{\alpha,t}}[\boldsymbol{\theta}(\boldsymbol{\mu})]$ where $\boldsymbol{\mu} = \mathbf{K}'_{\nu_{\alpha,t}}(\boldsymbol{\theta})$. Then calculating the determinant of the variance function will give the generalized variance. We summarize the variance function and the generalized variance of NG, NIG and NP models in Table 1.

Table 1: Variance Function and Generalized Variance

Model	$\mathbf{V}_{\mathbf{G}_{\alpha,t}}(\boldsymbol{\mu})$	$\psi = \det\mathbf{V}_{\mathbf{G}_{\alpha,t}}(\boldsymbol{\mu})$
NG	$(1/t)\boldsymbol{\mu}\boldsymbol{\mu}^\top + \mathbf{diag}_k(0,\mu_1,\ldots,\mu_1)$	$(1/t)\mu_1^{k+1}$
NIG	$(\mu_1/t^2)\boldsymbol{\mu}\boldsymbol{\mu}^\top + \mathbf{diag}_k(0,\mu_1,\ldots,\mu_1)$	$(1/t^2)\mu_1^{k+2}$
NP	$(1/\mu_1)\boldsymbol{\mu}\boldsymbol{\mu}^\top + \mathbf{diag}_k(0,\mu_1,\ldots,\mu_1)$	μ_1^k

The ML and UMVU estimators of the generalized variance in Table 1 are stated in the following proposition.

Proposition 1 *Let $X_1,\ldots,X_n$ be random vectors with distribution $\mathbf{P}(\boldsymbol{\theta};\alpha,t) \in \mathbf{G}(\nu_{p,t})$ in a given NST family. Denoting $\overline{X} = (X_1+\ldots+X_n)/n = (\overline{X}_1,\ldots,\overline{X}_k)^T$ the sample mean with positive first component $\overline{X}_1$, the ML estimator of the generalized variance of NG, NP and NIG models is given by:*

$$T_{n;k;t} = \det \boldsymbol{V}_{\mathbf{G}_{p,t}}(\overline{X}) = \begin{cases} (1/t)\overline{X}_1^{k+1}, & \textit{for NG} \\ \overline{X}_1^{k}, & \textit{for NP} \\ (1/t^2)\overline{X}_1^{k+2}, & \textit{for NIG} \end{cases}$$

and the UMVU estimator is given by

$$U_{n;k,t} = \begin{cases} t^k\Gamma(nt)[\Gamma(nt+k+1)]^{-1}\Sigma_{i=1}^n x_{(1i)}^{k+1}, & \textit{for NG} \\ n^{-k}[\Sigma_{i=1}^n x_{(1i)}][\Sigma_{i=1}^n x_{(1i)}-1]\cdots[\Sigma_{i=1}^n x_{(i1)}-k+1],\ n\geq k & \textit{for NP} \\ t^k 2^{-1-k/2}[\Gamma(1+k/2)]^{-1}\Sigma_{i=1}^n x_{(1i)}^{3/2}\exp\left\{(nt)^2/[2\Sigma_{i=1}^n x_{(1i)}]\right\}\times & \\ \int_0^{\Sigma_{i=1}^n x_{(1i)}} y_1^{k/2}[\Sigma_{i=1}^n x_{(1i)}-y_1]^{-3/2}\times & \\ \exp\left\{-y_1-[(nt)^2/2[\Sigma_{i=1}^n x_{(1i)}-y_1]]\right\}dy_1, & \textit{for NIG} \end{cases}$$

(see Boubacar Maïnassara and Kokonendji , [3])

3 Simulation Study

In order to examine the behavior of ML and UMVU estimators empirically we carried out a simulation study. We run Monte-Carlo simulations using R software. We set several sample sizes (n) varied from 3 to 1000 and we generated 1000 samples for each n. We consider $k = 2, 4, 6$ to see the effects of k on generalized variance estimations. For simplicity we set $\mu_1 = 1$. Moreover, to see the effect of zero values proportion within X_1 in the case of normal Poisson, we also consider small mean values on the Poisson component i.e. $\mu_1 = 0.5$ because $\mathbb{P}(X_1 = 0) = \exp(-\mu_1)$.

We report the numerical results of the generalized variance estimations for each model, i.e. the empirical expected value of the estimators with its standard errors (Se) and the empirical mean square error (MSE). We calculated the mean square error (MSE) of each method over 1000 data sets using the following formula:

$$MSE(\widehat{\psi}) = \frac{1}{1000}\sum_{i=1}^{1000}\left\{\widehat{\psi}_i - \det \mathbf{V}_{\mathbf{G}_{\alpha,t}}(\boldsymbol{\mu})\right\}^2 \tag{8}$$

where $\widehat{\psi}$ is the estimate of $\det \mathbf{V}_{\mathbf{G}_{\alpha,t}}(\boldsymbol{\mu})$ using ML and UMVU estimators.

3.1 Normal gamma

We generated normal gamma distribution samples using the generating σ-finite positive measure $\nu_{\alpha,t}$ of normal gamma in (1). Table 2 show the expected values of generalized variance estimates with their standard errors (in parentheses) and the means square error values of both ML and UMVU methods in case of normal gamma.

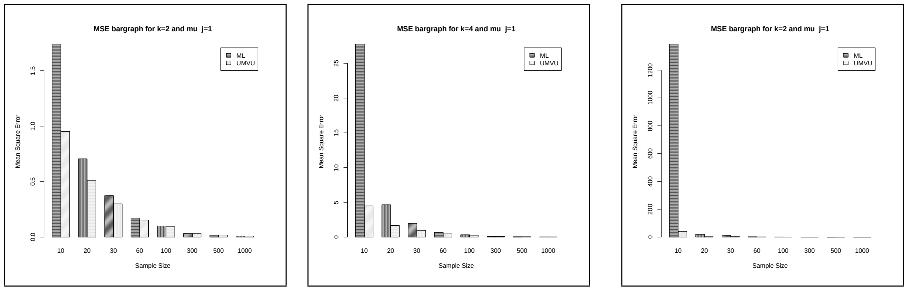
From the result in Table 2 we can observe different performances of ML estimator ($T_{n;k,t}$) and UMVU estimator ($U_{n;k,p,t}$) of the generalized variance. The expected values of $T_{n;k,t}$ converge while the values of $U_{n;k,t}$ do not, but $U_{n;k,t}$ is always closer to the parameter than $T_{n;k,t}$ for small sample sizes, i.e.

for $n \leq 30$, this shows that UMVU is an unbiased estimator while ML is an asymtotically unbiased estimator. For the two methods, the standar error of the estimates decreases when the sample size increase.

Table 2: The expected values (with empirical standard errors) and MSE of $T_{n;k,t}$ and $U_{n;k,t}$ for normal-gamma with 1000 replications for given target value $\mu_1^{k+1} = 1$ with $k \in \{2, 4, 6\}$.

		Expected values and Standard errors		MSE	
k	n	$T_{n;k,t}$	$U_{n;k,t}$	$T_{n;k,t}$	$U_{n;k,t}$
2	3	1.9805 (3.7192)	0.8912 (1.6736)	14.7935	2.8128
	10	1.2878 (1.2875)	0.9756 (0.9754)	1.7405	0.9520
	20	1.1648 (0.8236)	1.0085 (0.7131)	0.7054	0.5085
	30	1.0998 (0.6031)	0.9978 (0.5471)	0.3736	0.2994
	60	1.0380 (0.4115)	0.9881 (0.3917)	0.1708	0.1536
	100	1.0231 (0.3152)	0.9931 (0.3060)	0.0999	0.0937
	300	1.0036 (0.1774)	0.9936 (0.1757)	0.0315	0.0309
	500	1.0076 (0.1365)	1.0016 (0.1357)	0.0187	0.0184
	1000	1.0110 (0.0953)	1.0080 (0.0950)	0.0092	0.0091
4	5	4.2191 (13.3899)	0.8720 (2.7674)	189.6509	7.6750
	10	2.3799 (5.0869)	0.9906 (2.1174)	27.7810	4.4837
	20	1.6461 (2.0572)	1.0328 (1.2906)	4.6494	1.6668
	30	1.3831 (1.3505)	1.0066 (0.9828)	1.9707	0.9660
	60	1.1904 (0.8014)	1.0117 (0.6811)	0.6784	0.4640
	100	1.0869 (0.5706)	0.9849 (0.5171)	0.3332	0.2676
	300	1.0293 (0.2938)	0.9957 (0.2842)	0.0872	0.0808
	500	1.0286 (0.2296)	1.0083 (0.2251)	0.0535	0.0507
	1000	1.0137 (0.1610)	1.0036 (0.1594)	0.0261	0.0254
6	7	13.7175 (103.5833)	1.3062 (9.8634)	10891.2275	97.3811
	10	6.6118 (36.8236)	1.1467 (6.3866)	1387.4736	40.8103
	20	2.2455 (4.3052)	0.8670 (1.6622)	20.0860	2.7806
	30	1.9055 (3.4774)	0.9905 (1.8076)	12.9123	3.2676
	60	1.4151 (1.5070)	1.0092 (1.0748)	2.4434	1.1553
	100	1.2248 (0.8843)	0.9972 (0.7199)	0.8325	0.5183
	300	1.0606 (0.4416)	0.9894 (0.4119)	0.1986	0.1698
	500	1.0182 (0.3160)	0.9765 (0.3030)	0.1002	0.0924
	1000	1.0228 (0.2311)	1.0016 (0.2263)	0.0539	0.0512

To examine the consistency of the estimators we have to look at their MSE. The result shows that when n increases the MSE of the two methods become more similar and they both produced almost the same result for $n = 1000$. The MSE values for $n \geq 10$ in the table are presented graphically in Figure 1. In the figure we can easily see that all estimators become more similar when the sample size increase. For small sample sizes, UMVU always has smaller MSE, in this situation UMVU is preferable than ML. The figure also shows that the difference between ML and UMVU for small sample sizes increases when the dimension increases.

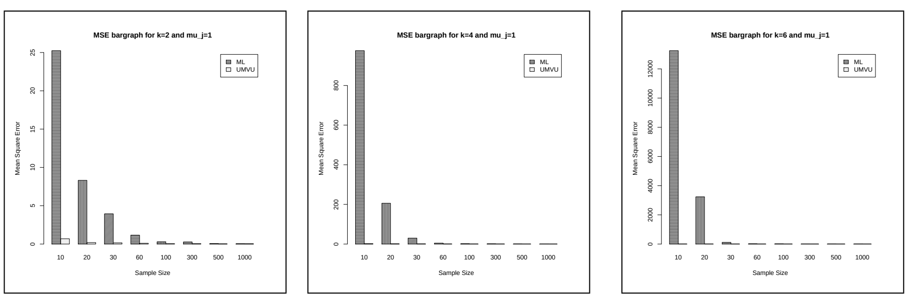


(a) k=2 (b) k=4 (c) k=6

Figure 1: Bargraphs of the mean square errors of $T_{n;k,t}$ and $U_{n;k,t}$ for normal-gamma with $n \in \{10, 20, 30, 60, 100, 300, 500, 1000\}$ and $k \in \{2, 4, 6\}$.

3.2 Normal inverse-Gaussian

The result for normal inverse-Gaussian is presented in Table 3. Similar with normal gamma, the result for normal inverse-Gaussian shows that UMVU method produced better estimates than ML method for small sample sizes. From the result we can conclude that the two estimators are consistent. The bargraph of MSE values for $n \geq 10$ in Table 3 is presented in Figure 2. Notice that the result for this case is similar to the normal gamma case, i.e. for small sample sizes the difference between the MSEs of ML and UMVU estimators for normal inverse-Gaussian also increases when k increases.



(a) k=2 (b) k=4 (c) k=6

Figure 2: Bargraphs of the mean square errors of $T_{n;k,t}$ and $U_{n;k,t}$ for normal inverse Gaussian with $n \in \{10, 20, 30, 60, 100, 300, 500, 1000\}$ and $k \in \{2, 4, 6\}$.

3.3 Normal Poisson

The simulation results for normal Poisson are presented in Table 4 and Table 5 for $\mu_1 = 1$ and $\mu_1 = 0.5$ respectively. In this simulation, the proportion of zero values in the samples increases when the mean of the Poisson component becomes smaller. For normal-Poisson distribution with $\mu_j = 0.5$, we have many zero values in the samples. However, this situation does not affect the

Table 3: The expected values (with standar errors) and MSE of $T_{n;k,t}$ and $U_{n;k,t}$ for normal inverse-Gaussian with 1000 replications for given target value $\mu_1^{k+2} = 1$ and $k \in \{2, 4, 6\}$.

		Expected values and Standard errors		MSE	
k	n	$T_{n;k,t}$	$U_{n;k,t}$	$T_{n;k,t}$	$U_{n;k,t}$
2	3	2.0068 (4.9227)	0.9135 (0.8235)	25.2469	0.6856
	10	1.4249 (2.8513)	1.0316 (0.4388)	8.3103	0.1935
	20	1.5936 (1.8951)	1.1340 (0.3718)	3.9439	0.1562
	30	1.3677 (1.0155)	1.1641 (0.2668)	1.1664	0.0981
	60	1.0846 (0.5341)	1.1104 (0.1856)	0.2924	0.0466
	100	1.0819 (0.5166)	1.1102 (0.1675)	0.2735	0.0402
	300	1.0006 (0.2570)	1.0843 (0.0919)	0.0660	0.0156
	500	1.0356 (0.1890)	1.1374 (0.0727)	0.0370	0.0242
	1000	1.0156 (0.1219)	1.0116 (0.0670)	0.0151	0.0115
4	5	9.3836 (30.0947)	1.3196 (1.1323)	975.9726	1.3843
	10	4.6547 (13.8643)	1.2837 (0.8153)	205.5754	0.7452
	20	2.7487 (5.1845)	1.2963 (0.6189)	29.9373	0.4709
	30	1.4822 (2.1166)	1.1854 (0.4572)	4.7125	0.2434
	60	1.3095 (1.1051)	1.2560 (0.3054)	1.3170	0.1588
	100	1.1673 (0.8467)	1.2264 (0.2671)	0.7449	0.1226
	300	1.0849 (0.4296)	1.2542 (0.1520)	0.1918	0.0877
	500	1.0350 (0.2839)	1.0762 (0.0914)	0.0818	0.0416
	1000	1.0107 (0.2080)	1.0102 (0.1137)	0.0434	0.0337
6	7	20.4865 (113.4633)	0.9423 (0.9984)	12056.9414	1.0001
	10	12.1032 (55.7841)	1.0596 (0.8610)	2329.5787	0.7449
	20	3.4498 (10.3056)	1.0054 (0.5933)	112.2060	0.3520
	30	2.1422 (3.2262)	1.0246 (0.4970)	11.7130	0.2476
	60	1.8236 (2.6064)	1.0587 (0.3744)	7.4717	0.1436
	100	1.2468 (1.1599)	1.0129 (0.2643)	1.4062	0.1170
	300	1.0781 (0.4953)	1.0568 (0.1596)	0.2514	0.0929
	500	1.0815 (0.4065)	1.0230 (0.1110)	0.1719	0.0922
	1000	1.0207 (0.2816)	1.0204 (0.0775)	0.0798	0.0760

generalized variance estimation as we can see that $T_{n;k,t}$ and $U_{n;k,t}$ have the same behavior for both values of μ_1.

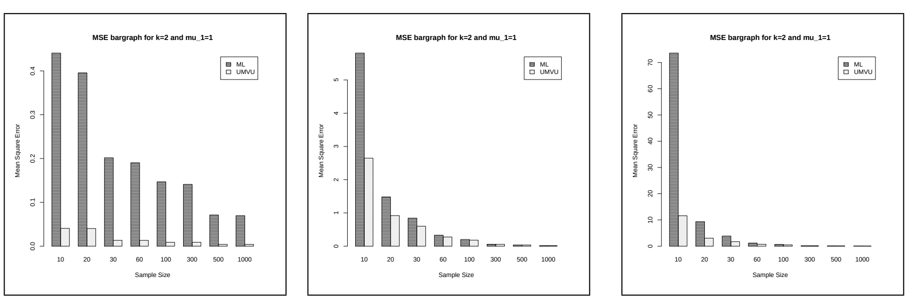
The MSE in Table 4 and 5 for $n \geq 10$ are displayed as bargraphs presented in Figure 3 and Figure 4. From those figures we see that UMVU is preferable than ML because it always has smaller MSE values when sample sizes are small ($n \leqslant 30$).

4 Conclusion

In this paper we have discussed the generalized variance estimator of normal gamma, normal inverse-Gaussian and normal Poisson models using ML and UMVU methods. The simulation studies of the generalized variance estimators

Table 4: The expected values (with standar errors) and MSE of $T_{n;k,t}$ and $U_{n;k,t}$ for normal Poisson with 1000 replications for given target value $\mu_1^k = 1$ and $k \in \{2, 4, 6\}$.

		Expected values and Standard errors		MSE	
k	n	$T_{n;k,t}$	$U_{n;k,t}$	$T_{n;k,t}$	$U_{n;k,t}$
2	3	1.3711 (1.4982)	1.0349 (1.3130)	2.3824	1.7252
	10	1.0810 (0.6589)	0.9817 (0.6286)	0.4407	0.3955
	20	1.0424 (0.4471)	0.9925 (0.4363)	0.2017	0.1904
	30	1.0329 (0.3817)	0.9996 (0.3756)	0.1468	0.1411
	60	1.0184 (0.2661)	1.0017 (0.2639)	0.0711	0.0697
	100	1.0066 (0.2016)	0.9966 (0.2006)	0.0407	0.0403
	300	1.0112 (0.1153)	1.0079 (0.1151)	0.0134	0.0133
	500	0.9986 (0.0942)	0.9966 (0.0941)	0.0089	0.0089
	1000	0.9998 (0.0641)	0.9988 (0.0641)	0.0041	0.0041
4	5	2.6283 (5.0058)	1.0721 (2.7753)	27.7093	7.7075
	10	1.7362 (2.2949)	1.0422 (1.6267)	5.8085	2.6480
	20	1.3276 (1.1713)	1.0073 (0.9588)	1.4793	0.9193
	30	1.2274 (0.8892)	1.0167 (0.7750)	0.8424	0.6008
	60	1.1111 (0.5643)	1.0085 (0.5250)	0.3308	0.2757
	100	1.0647 (0.4448)	1.0038 (0.4260)	0.2021	0.1815
	300	1.0245 (0.2389)	1.0043 (0.2354)	0.0577	0.0554
	500	1.0092 (0.1889)	0.9972 (0.1872)	0.0358	0.0351
	1000	1.0013 (0.1272)	0.9953 (0.1267)	0.0162	0.0161
6	7	4.5153 (12.8404)	0.9378 (4.0255)	177.2319	16.2084
	10	3.6865 (8.1473)	1.1642 (3.3992)	73.5952	11.5816
	20	1.9674 (2.9034)	1.0227 (1.7467)	9.3656	3.0514
	30	1.5605 (1.8825)	0.9901 (1.3133)	3.8580	1.7250
	60	1.2954 (1.0360)	1.0220 (0.8541)	1.1606	0.7300
	100	1.2084 (0.7824)	1.0462 (0.6957)	0.6556	0.4861
	300	1.0621 (0.3793)	1.0109 (0.3641)	0.1477	0.1327
	500	1.0294 (0.2778)	0.9992 (0.2710)	0.0780	0.0734
	1000	1.0185 (0.1939)	1.0034 (0.1915)	0.0379	0.0367



(a) k=2 (b) k=4 (c) k=6

Figure 3: Bargraphs of the mean square errors of $T_{n;k,t}$ and $U_{n;k,t}$ for normal Poisson with $\mu_1 = 1$, $n \in \{10, 20, 30, 60, 100, 300, 500, 1000\}$ and $k \in \{2, 4, 6\}$.

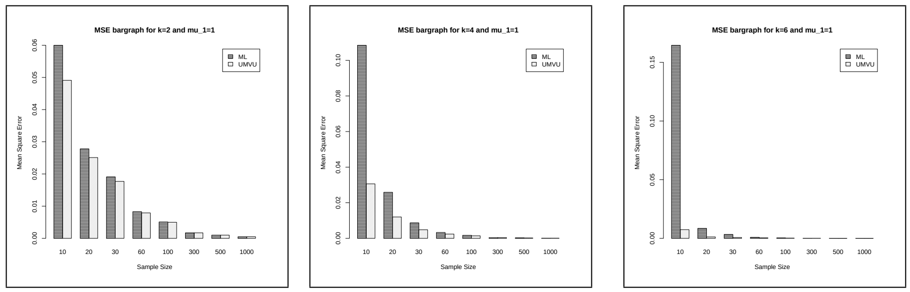
Table 5: The expected values (with standard errors) and MSE of $T_{n;k,t}$ and $U_{n;k,t}$ for normal Poisson with 1000 replications for given target value $\mu_1^k = 0.5^k$ and $k \in \{2, 4, 6\}$.

		Expected values and Standard errors		MSE	
k	n	$T_{n;k,t}$	$U_{n;k,t}$	$T_{n;k,t}$	$U_{n;k,t}$
2	3	0.3930 (0.5426)	0.2320 (0.4223)	0.3148	0.1787
	10	0.2868 (0.2421)	0.2378 (0.2212)	0.0600	0.0491
	20	0.2652 (0.1660)	0.2407 (0.1583)	0.0278	0.0251
	30	0.2642 (0.1374)	0.2476 (0.1332)	0.0191	0.0177
	60	0.2598 (0.0903)	0.2514 (0.0888)	0.0083	0.0079
	100	0.2534 (0.0712)	0.2484 (0.0705)	0.0051	0.0050
	300	0.2495 (0.0418)	0.2478 (0.0417)	0.0017	0.0017
	500	0.2491 (0.0313)	0.2482 (0.0313)	0.0010	0.0010
	1000	0.2495 (0.0221)	0.2490 (0.0221)	0.0005	0.0005
4	5	0.2999 (0.8462)	0.0685 (0.3474)	0.7724	0.1207
	10	0.1696 (0.3115)	0.0689 (0.1750)	0.1085	0.0306
	20	0.1089 (0.1541)	0.0658 (0.1097)	0.0259	0.0120
	30	0.0886 (0.0894)	0.0617 (0.0689)	0.0087	0.0048
	60	0.0774 (0.0559)	0.0642 (0.0487)	0.0033	0.0024
	100	0.0704 (0.0403)	0.0627 (0.0370)	0.0017	0.0014
	300	0.0643 (0.0207)	0.0618 (0.0201)	0.0004	0.0004
	500	0.0635 (0.0158)	0.0620 (0.0156)	0.0003	0.0002
	1000	0.0631 (0.0115)	0.0624 (0.0114)	0.0001	0.0001
6	7	0.2792 (1.2521)	0.0268 (0.2274)	1.6371	0.0519
	10	0.1212 (0.3918)	0.0165 (0.0858)	0.1646	0.0074
	20	0.0427 (0.0883)	0.0124 (0.0345)	0.0085	0.0012
	30	0.0356 (0.0539)	0.0151 (0.0271)	0.0033	0.0007
	60	0.0236 (0.0281)	0.0149 (0.0196)	0.0009	0.0004
	100	0.0211 (0.0183)	0.0159 (0.0145)	0.0004	0.0002
	300	0.0173 (0.0089)	0.0157 (0.0082)	0.0001	0.0001
	500	0.0166 (0.0068)	0.0157 (0.0064)	0.0000	0.0000
	1000	0.0164 (0.0044)	0.0159 (0.0043)	0.0000	0.0000

for the three models show that UMVU produces better estimation than ML for small sample sizes. However, the two methods are consistent and they become more similar when the sample size increases.

References

[1] K. K. Aase, Representative agent pricing of financial assets based on Lévy processes with normal inverse Gaussian marginals, *Annals of Operations Research*, **114** (2002), 15-31. https://doi.org/10.1023/a:1021093615674



(a) k=2 (b) k=4 (c) k=6

Figure 4: Bargraphs of the mean square errors of $T_{n;k,t}$ and $U_{n;k,t}$ for normal Poisson with $\mu_1 = 0.5$, $n \in \{10, 20, 30, 60, 100, 300, 500, 1000\}$ and $k \in \{2, 4, 6\}$.

[2] J. Andersson, On the normal inverse Gaussian stochastic volatility model, *Journal of Business & Economic Statistics*, **19** (2001), no. 1, 44-54. https://doi.org/10.1198/07350010152472607

[3] Y. Boubacar Maïnassara and C. C. Kokonendji, On normal stable Tweedie models and power-generalized variance functions of only one component, *TEST,* **23** (2014), 585-606. https://doi.org/10.1007/s11749-014-0363-9

[4] O. E. Barndorff-Nielsen, J. Kent and M. Sørensen, Normal variance-mean mixtures and z distributions, *International Statistical Review,* **50** (1982), 145-159. https://doi.org/10.2307/1402598

[5] J. M. Bernardo and A. F. M. Smith, *Bayesian Theory,* Wiley, New York, 1993. https://doi.org/10.1002/9780470316870

[6] M. Casalis, The $2d + 4$ simple quadratic natural exponential families on $\mathbb{R}^d$, *The Annals of Statistics*, **24** (1996), 1828-1854. https://doi.org/10.1214/aos/1032298298

[7] W. Feller, *An Introduction to Probability Theory and its Applications Vol. II*, Second edition, Wiley, New York, 1971.

[8] C. C. Kokonendji and A. Masmoudi, On the Monge-Ampère equation for characterizing gamma-Gaussian model, *Statistics and Probability Letters*, **83** (2013), 1692-1698. https://doi.org/10.1016/j.spl.2013.03.023

[9] K. Nisa, C.C. Kokonendji and A. Saefuddin, Characterizations of multivariate normal Poisson model, *Journal of Iranian Statistical Society*, **14** (2015), 37-52.

[10] M. C. K. Tweedie, An index which distinguishes between some important exponential families, In: *Statistics: Applications and New Directions. Proceedings of the Indian Statistical Golden Jubilee International Conference*,

Eds. J. K. Ghosh and J. Roy, Indian Statistical Institute, Calcutta, 1984, 579–604.

Received: September 16, 2016; Published: November 14, 2016