A20 - Characteristics of Hazard Rate Functions of Log-Normal Distributions

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Characteristics of Hazard Rate Functions of Log-Normal Distributions

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Abstract. Distribution of Survival Analysis categorized in three functions those are: survival function, probability density function, and hazard rate function. Hazard rate function is used to analyze extreme value from a probability model of a distribution. One of the interesting distributions is log-normal distribution which is used for modeling of maintenance of a system. To analyze characteristics of hazard rate function of log-normal distribution, the Glaser method approach is used. The results are log-normal distribution have three hazard rate patterns those are increasing, decreasing or upside-down bathtub (∩).

1. Introduction

Survival analysis can be used to analyze data such as for the case of public health: For example, the incidence of an illness, recurrence of illness, healing and death [1]. One of the point that is interesting to be analyzed is *hazard rate*, namely the ratio of probability density function (pdf) and survival function (S(t)). The graph of *hazard rate* has the form as: *increasing* (I), *decreasing* (D), *bathtub* (∪), *upside-down bathtub* (∩) and constant.

The *log-normal* distribution is a probability from a continue random variable which was transformed from a normal distribution [2]. The *log-normal* distribution can be applied in many fields of studies, for instant in hydrology that can be used to analyze extreme values of daily, monthly or yearly rainfall. Besides, the log normal distribution also can be used for modeling of maintenance of a system.

The aims of this study are to discuss survival function, hazard function and the characteristics of *hazard rate from log-normal* by using Glaser method [3]. Besides, the behavior of the graph also will be presented by using software R.

2. Materials and Methods

2.1 Log-Normal Distribution

The *log-normal* distribution is defined as follows: consider a random variable T with region of $Rt = \{t|0 < t < \infty\}$ and $Y = \ln T$ has a normal distribution with mean μ and variance σ^2.

The probability distribution function of the random variable *log-normal* with a parameter $\mu > 0$ and $\sigma > 0$, is as follows:

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$$f(t) = \begin{cases} \frac{1}{\sqrt{2\pi}\sigma t} exp\left[-\frac{1}{2}\left(\frac{\ln t-\mu}{\sigma}\right)^2\right], & for\ t>0 \\ 0, & otherwise \end{cases} \tag{1}$$

The mean and variance are:

1. $E(t) = \mu_t = \exp\left(t + \frac{\sigma^2}{2}\right)$
2. $Var(t) = \left(e^{(2\mu+\sigma^2)}\right)\left(e^{\sigma^2} - 1\right)$

The rth moment of log-normal distribution [5] is:

$$\mu_t(r) = E[T^r] = \exp(r\mu + \frac{1}{2}r^2\sigma^2 \tag{2}$$

and the cumulative distribution function of log-normal [6] is:

$$F(t) = \varphi\left[\frac{\ln t - \mu}{\sigma}\right], \qquad t\,\epsilon\,(0,\infty) \tag{3}$$

where φ is a cumulative distribution function from normal distribution.

The survival function is defined [7]:

$$S(t) = 1 - F(t) \tag{4}$$

So that, the survival function of log-normal distribution is:

$$S(t) = 1 - \varphi\left[\frac{\ln t - \mu}{\sigma}\right] \tag{5}$$

and the hazard function of log-normal distribution is as follows:

$$h(t) = \frac{f(t)}{S(t)} \tag{7}$$

$$h(t) = \frac{\frac{1}{\sqrt{2\pi}\,\sigma t} e^{-\frac{1}{2}\left(\frac{\ln t-\mu}{\sigma}\right)^2}}{1 - \varphi\left[\frac{\ln t-\mu}{\sigma}\right]} \tag{8}$$

2.2 The first derivative of probability density function (pdf) of log-normal distribution

The first derivative of pdf can be used to find the value of $\eta(t)$. The pdf of log-normal distribution is:

$$f(t) = \frac{1}{\sqrt{2\pi}\,\sigma t} e^{-\frac{1}{2}\left(\frac{\ln t-\mu}{\sigma}\right)^2} \tag{9}$$

To find the derivative of the pdf of log-normal distribution, we can used the multiplicative formula:

$$f'(t) = u'v + uv' \tag{10}$$

so we have,

$$f'^{(t)} = \frac{1}{\sqrt{2\pi}\,\sigma t} e^{-\frac{1}{2}\left(\frac{\ln t-\mu}{\sigma}\right)^2}\left[-\frac{1}{t} - \frac{1}{\sigma^2 t}(\ln t - \mu)\right]. \tag{11}$$

2.3 The value of $\eta(t)$ and the first derivative of $(\eta(t))$

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The value $\eta(t)$ from the log normal distribution is as follows:

$$\eta(t) = -\frac{f'(t)}{f(t)} \tag{12}$$

$$\eta(t) = -\frac{\frac{1}{\sqrt{2\pi}\,\sigma t} e^{-\frac{1}{2}\left(\frac{\ln t-\mu}{\sigma}\right)^2}\left[-\frac{1}{t}-\frac{1}{\sigma^2 t}(\ln t-\mu)\right]}{\frac{1}{\sqrt{2\pi}\,\sigma t} e^{-\frac{1}{2}\left(\frac{\ln t-\mu}{\sigma}\right)^2}} \tag{13}$$

$$\eta(t) = \frac{1}{t} + \frac{1}{\sigma^2 t}(\ln t - \mu) \tag{14}$$

After we have $\eta(t)$, then we can find the derivative of $\eta(t)$ as follows:

$$\eta'(t) = \frac{d}{dt}\eta(t) \tag{15}$$

$$\eta'(t) = \frac{d}{dt}\left(\frac{1}{t} + \frac{1}{\sigma^2 t}(\ln t - \mu)\right) \tag{16}$$

$$\eta'(t) = \frac{-\sigma^2 + 1 - \ln t + \mu}{\sigma^2 t^2} \tag{17}$$

2.4 Methods

The method that is used to analyze the characteristics function of hazard rate of log-normal distribution we use Glaser approach [3] as follows: (a) If $\eta'(t) > 0$ for all $t > 0$, then it is *Increasing (I)*. (b) If $\eta'(t) < 0$ for all $t > 0$, then it is *Decreasing (D)*. (c) Let $t_0 > 0$ so that $\eta'(t) < 0$ for all $t \in (0, t_0)$, $\eta'(t_0) = 0$, $\eta'(t) > 0$ for all $t > t_0$ and

If $\lim_{t\to 0} pdf(t) = 0$, then it is *Increasing (I)*.

If $\lim_{t\to 0} pdf(t) \to \infty$, then it is *Bathtub*($\cup$).

(d) Let $t_0 > 0$ so that $\eta'(t) > 0$ for all $t \in (0, t_0)$, $\eta'(t_0) = 0$, $\eta'(t) < 0$ for all $t > t_0$ and

If $\lim_{t\to 0} pdf(t) = 0$, then it is Upside-*down bathtub* ($\cap$).

If $\lim_{t\to 0} pdf(t) \to \infty$, then it is *Decreasing* (D).

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3. Results and Discussion

3.1 The Pattern of Hazard Rate

The pattern of *hazard rate* [8] can be estimated by $\eta'(t) = 0$ and the sign of its coefficients. From the equation:

$$\eta'(t) = \frac{-\sigma^2 + 1 - \ln t + \mu}{\sigma^2 t^2} \tag{18}$$

We find the critical points by setting the equation $\eta'(t) = 0$, so that we can find the pattern of hazard rate function:

$$-\sigma^2 + 1 - \ln t + \mu = 0 \tag{19}$$

Based on the equation above, then the pattern of *hazard rate* is as follows:

a. There is no quadratic coefficient in the equation.

b. Coefficient of linear:

$$-\ln t \quad ; \quad t > 0$$

c. Coefficient of constant:

$$-\sigma^2 + 1 + \mu = 0$$
$$-\sigma^2 + \mu = -1$$
$$\sigma^2 - \mu = 1$$

for $\sigma^2 - \mu > 1$ coefficient is positive and $\sigma^2 - \mu < 1$ coefficient is negative.

3.2 The analysis of pattern of Hazard functions by Glaser

Analysis of the pattern of hazard function according to Glaser is as follows: Let we define a number which satisfy $0 < \mu \leq 1$, $0 < \sigma \leq 1$ for $t > 0$ so that we have:

a. If $\mu < \sigma$, $\mu = 0{,}1$ and $\sigma = 0{,}5$ then $\eta'(t)$ will have a negative value at $t = 3$. If $\mu = 0{,}5$ and $\sigma = 1$ then $\eta'(t)$ will have negative value at $t = 2$.

b. If $\mu > \sigma$ and the value of μ and σ are 0.5 and 0.1 respectively, then $\eta'(t)$ will have a negative value at t=5.

If $\mu = 1$ and $\sigma = 0{,}5$, then $\eta'(t)$ will have a negative value at $t = 6$.

Therefore, the result we have for the values of $0 < \mu \leq 1$ and $0 < \sigma \leq 1$ with $t > 0$, we have $\eta'(t)$ positive $(\eta'(t) > 0)$ and $\eta'(t)$negative $(\eta'(t) < 0)$ then in those region values, they are *increasing* and *decreasing* at different t values and depend on values μ and σ.

2. Let us take some number which satisfy $\mu > 1$ and $\sigma > 1$ for $t > 0$ so that we have:
   a. If $\mu < \sigma$, $\mu = 2$ and $\sigma = 3$ then $\eta'(t)$ has a negative values up to t=n.
   b. If $\mu > \sigma$, $\mu = 7$ and $\sigma = 5$ then $\eta'(t)$ has negative values up to t=n.
   c. Therefore, from the results above for $\mu > 1$, $\sigma > 1$ and $t > 0$ we have the value $\eta'(t) < 0$ up to $t = n$ then in this region is *decreasing*.
3. Let $t_0 > 0$ such that $\eta'(t) < 0$ for every $t \in (0, t_0)$, $\eta'(t) = 0$, $\eta'(t) > 0$ for all $t > t_0$ and

$$\lim_{t \to 0} pdf(t) = \lim_{t \to 0} \frac{1}{t\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{\ln t - \mu}{\sigma}\right)^2} \tag{20}$$

$$\lim_{t \to 0} pdf(t) = 0 \tag{21}$$

Since $\lim_{t \to 0} pdf(t) \to 0$ so that *Increasing (I)*

4. Let $t_0 > 0$ such that $\eta'(t) > 0$ for all $t \in (0, t_0)$, $\eta'(t) = 0$, $\eta'(t) < 0$ for all $t > t_0$ and $\lim_{t \to 0} pdf(t) \to 0$ so that *upside-down bathtub* (∩).

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3.3 Graph of Hazard Function of Log-Normal Distribution

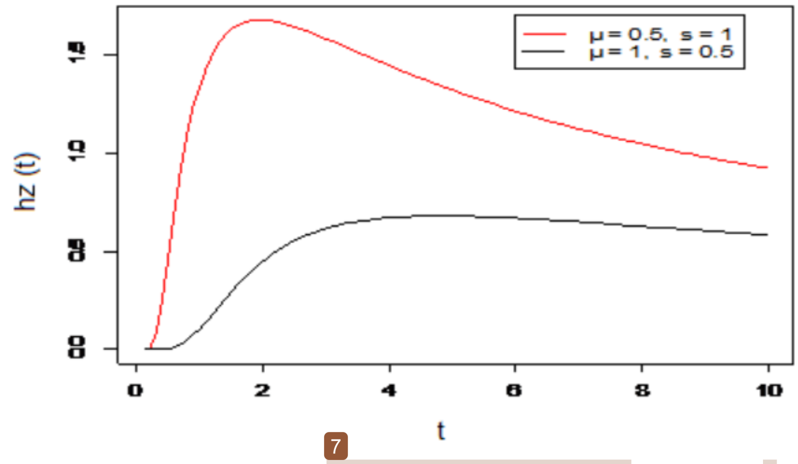

Figure 1. Graph of Hazard Function of *Log-Normal Distribution* with region $0 < \mu \leq 1$ and $0 < \sigma \leq 1$

From figure 1, it can be explained that the graph of hazard rate function from log-normal distribution at $\mu > \sigma$ with the values $\mu = 0.5$ and $\sigma = 1$ is increasing up to the maximum at $t = 1.9$ and then *decreasing*. But at $\mu < \sigma$ with the value $\mu = 1$ and $\sigma = 0.5$ the pattern of the graph almost the same as the graph $\mu > \sigma$, that is increasing up to the maximum point at $t = 4.8$ and then decreasing.

The pattern of the graph is resembles a ridge or can be said as *upside-down bathtub* (∩). Where at $\mu > \sigma$ and the value of t from $t = 1.2$ to $t = 3.6$, while at $\mu < \sigma$ and the value of t from $t = 3.9$ to $t = 6$.

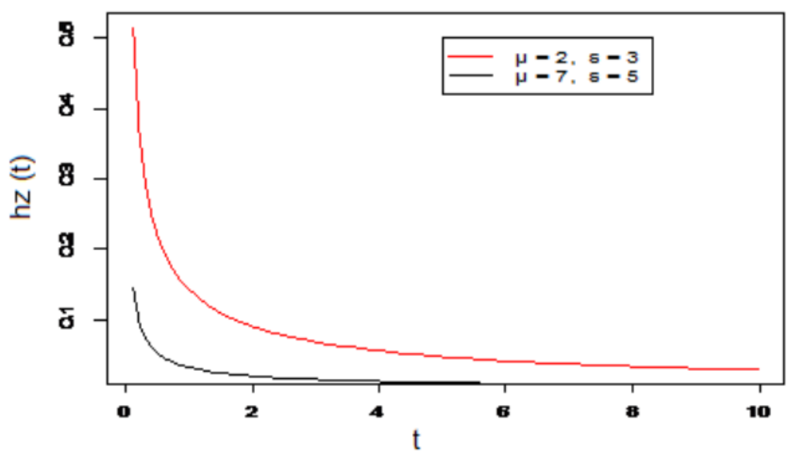

Figure 2. Hazard rate function from log-normal distribution at region $\mu > 1$ and $\sigma > 1$

Figure 2 explain that for the value $= 2$, $\sigma = 3$ and $\mu = 7$, $\sigma = 5$ the pattern of the graph is decreasing. This means that as the time increase of a system, and then the hazard rate will decrease.

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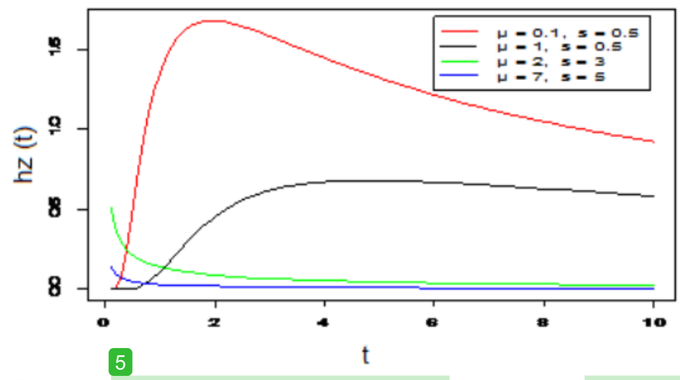

Figure 3. Graph of Hazard Rate Function of Log-Normal Distribution

Figure 3, graph of *hazard rate function of log-normal* distribution where x axis denotes time (t) and y axis denotes hazard function of log normal distribution (hz (t)) and we have three pattern of hazard rate, namely: *increasing* (I), *decreasing* (D) and *upside-down bathtub* (∩).

4. Conclusions

Based on the results of study we can conclude as follows:

1. The characteristic of hazard rate of log-normal distribution has the pattern: increasing, decreasing and upside-down bathtub (∩).
2. Hazard rate from log-normal distribution at region $0 < \mu \leq 1$ and $0 < \sigma \leq 1$ with $t > 0$ will increase and up to the maximum point (t) then will decrease depend on the value of μ and σ, either for the value $\mu > \sigma$ or $\mu < \sigma$.
3. Hazard rate from log-normal distribution at region $\mu > 1$ and $\sigma > 1$ with $t > 0$ has a pattern decreasing.
4. Log-normal distribution has the pattern upside-down bathtub (∩) when the value of $0 < \mu \leq 1$, $0 < \sigma \leq 1$, $t > 0$ and $\lim_{t \to 0} \text{pdf}(t) = 0$.
5. Graphically, the characteristic of hazard rate of log-normal distribution have the pattern increasing, decreasing, or upside-down bathtub (∩).

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