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Theoretical study of interaction between matter and curvature fluid in the theory of $f(R)$-gravity: Diffusion and friction

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The relativistic diffusion process and friction have been studied, especially in the framework of $f(R)$-gravity theory. The study of relativistic diffusion and friction processes based on $f(R)$-gravity is an alternative solution to solve the incompatibility problem emerging in the attempt to couple between the Fokker–Planck equation [FPE] to the Einstein field equation [EFE] encountered by Calogero. The energy–momentum tensor of the cosmological scalar field as proposed by Calogero is replaced by the presence of additional terms in the field equation of $f(R)$-gravity. The additional energy–momentum tensor in the field equation of $f(R)$-gravity in this context is regarded to compensate for the presence of the diffusion and another process like friction. The additional energy–momentum tensor is also regarded as due to the so-called curvature fluid or background fluid. Here we assume the presence of interaction between matter and the background fluid in the form of physical processes like diffusion, friction, etc. We also assume that there is 'interplay' between diffusion process and friction. In other words, the diffusion process and friction are not independent. As examples, we consider some viable models of $f(R)$ that satisfy both cosmological and local gravity constraints, i.e. $f(R) = R + \Lambda R^2, f(R) = R - \frac{\mathcal{M}^4}{R}$, and $f(R) = R - \mathcal{E} r_c \ln(1 + \frac{R}{r_c})$. Furthermore, we apply it to explain the diffusion and friction processes in the expanding universe by considering the Friedmann–Lemaitre–Robertson–Walker (FLRW) model.

Keywords: Curvature fluid; relativistic diffusion; diffusion process; friction.

Mathematics Subject Classification 2010: 83C40, 83C05, 83C10, 58J65

1. Introduction

The understanding of diffusion phenomena both non-relativistically and relativistically has developed quite rapidly. It can be said that diffusion is one of the physical phenomena that is still being studied. The study of diffusion processes date back to the observation by Brown and to the mathematical formulation of Einstein, Wiener, Smoluchowski, etc. in the last. In line with such non-relativistic formulation, the discussion of diffusion in the relativistic region has also been carried out by many authors. Currently, the study of the special relativistic as well as general relativistic diffusion processes are still being conducted by many researchers [1–20].

In the present work, we will focus on studying the diffusion process and friction in the GR theory related to the work of Calogero [17] and Alcantara and Calogero [18] in which they have constructed the Fokker–Planck equation (FPE) associated to diffusion process and friction in a curved spacetime. Furthermore, Calogero [17] has tried to study the coupling between FPE and Einstein field equation (EFE). However, he saw an incompatibility between the Bianchi identity and modified continuity equation derived from FPE due to the presence of diffusion process. To solve the problem of incompatibility, Calogero proposed two alternative solutions: (1) by adding additional material tensors to the right-hand side of the EFE or (2) by adding a cosmological scalar field on the left-hand side of the EFE. However, in his study [17] he has focused on adding the cosmological scalar field to the left-hand side of EFE.

The incompatibility of Bianchi identity and continuity equation of energy–momentum tensor also appears in the case of $f(R)$-gravity in which the divergence of left-hand side does not vanish and the divergence of right-hand side has to vanish due to the continuity equation of energy–momentum tensor. Koivisto [21] by making use of the so-called generalized Bianchi identity has shown that the energy–momentum conservation continuous to hold based on a Noether-law for gravitating matter.

Here, we would like to propose an alternative possible solution to the above problem of incompatibility. We try to study the possibility to couple the FPE obtained by Calogero with the field equation of $f(R)$-gravity. The alternative solution is expected to overcome the above problem of incompatibilities. If we take the divergence of field equation of $f(R)$-gravity (see Eq. (18)), then we obtain that the divergence of total energy–momentum tensor (the summation of the so-called effective energy–momentum tensor and matter energy–momentum tensor) is still zero. However, the divergence of the matter energy–momentum tensor may not be zero. As a consequence, the energy–momentum tensor of the cosmological scalar field as proposed by Calogero can be replaced by the presence of effective energy–momentum tensor, i.e. the additional terms in the field equation of $f(R)$-gravity itself. The additional energy–momentum tensor in the field equation of $f(R)$-gravity in this context can be regarded to compensate for the presence of the diffusion and another process like friction. The additional energy–momentum tensor can also be regarded as due to the so-called *effective stress–energy tensor* or *curvature fluid* energy–momentum tensor or *background fluid* energy–momentum tensor [22–25]. Related to the above-mentioned curvature fluid energy–momentum tensor, Capozziello *et al.* [25] have considered the cosmological perfect-fluids in $f(R)$-gravity in which they have shown that the presence of two terms in curvature fluid energy–momentum tensor prevents the presence of perfect fluid. To preserve the material energy–momentum tensor of the $f(R)$-gravity field equation still able to describe the perfect fluid, Cappozziello have proposed a generalized Robertson–Walker spacetime. They have shown that an n-dimensional generalized Robertson–Walker spacetime with divergence-free conformal curvature tensor exhibits a perfect fluid stress–energy tensor. They have also proved that a conformally flat generalized Robertson–Walker spacetime is still a perfect fluid in both $f(R)$ and quadratic gravity where other curvature invariants are considered. Different from the direction of Capozziello's work, here we use the presence of these additional terms to overcome the incompatibility problem between FPEs and EFEs, which means the curvature fluid is still regarded as unperfect fluid so that some physical processes (diffusion, friction, etc.) are present. Here, we assume the presence of interaction between matter and the curvature fluid in the form of physical processes like diffusion, friction, etc. We also assume that there is 'interplay' between diffusion process and friction. In other words, the diffusion process and friction are not independent. As examples, we will consider some viable models of $f(R)$ that satisfy both cosmological and local gravity constraints, i.e. the Starobinsky model $f(R) = R + \Lambda R^2, f(R) = R - \frac{\mathcal{M}^4}{R}$, and the model proposed by Miranda *et al.* $f(R) = R - \mathcal{E} r_c \ln(1 + \frac{R}{r_c})$. Furthermore, we will then apply it to explain the diffusion and friction phenomenon in the expanding universe by considering the Friedmann–Lemaitre–Robertson–Walker (FLRW) model.

In Sec. 2, we will review the general theory of $f(R)$-gravity by referring Capozziello, etc. In Sec. 3, we apply the general theory of $f(R)$-gravity to describe general relativistic diffusion processes, especially to give one of the possible solution of the incompatibility problem mentioned above. We also apply our idea to the special form of $f(R)$-gravity in Sec. 3. Furthermore, in Sec. 4 we apply the results obtained in Sec. 3 to explain the diffusion process in the expanding universe.

2. $f(R)$ Theory of Gravitation

In this section, we will review the theory of $f(R)$-gravity in the metric formalism [22]. The theory of $f(R)$-gravity is one type of modified gravity theory that generalizes Einstein's GR theory. This theory was first proposed by Buchdahl. Furthermore, this theory was developed by Starobinsky on cosmic inflation. The theory of $f(R)$-gravity is a set of theories that are characterized by various functions of Ricci scalar. In special cases, for $f(R) = R$, it will return to Einstein GR theory. As a consequence of the introduction of any function, it allows us the freedom to explain the expansion of the accelerated universe and the structure formation of the universe without the addition of dark energy or dark matter.

The field equation in $f(R)$-gravity is derived from the following action:

$$S = \frac{1}{2\kappa}\int d^4x\sqrt{-g}f(R) + S^{(m)}, \tag{1}$$

where κ is gravitational constants[a] and $S^{(m)}$ is matter action. In the case of vacuum, $S^{(m)} = 0$, the variational principle applied to the action (1) gives

$$\begin{aligned}\delta\int d^4x\sqrt{-g}f(R) &= \int d^4x[f(R)\delta\sqrt{-g} + \sqrt{-g}\delta f(R)] \\ &= \int d^4x\sqrt{-g}\left[f'(R)R_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu}\right]\delta g^{\mu\nu} \\ &\quad + \int d^4x\sqrt{-g}f'(R)g^{\mu\nu}\delta R_{\mu\nu}.\end{aligned} \tag{2}$$

Since in local inertial frame, we have the relation

$$\delta R_{\mu\nu}(0) = \frac{\partial}{\partial x^\alpha}(\delta\Gamma^\alpha_{\mu\nu}) - \frac{\partial}{\partial x^\nu}(\delta\Gamma^\alpha_{\mu\alpha}), \tag{3}$$

then

$$g^{\mu\nu}\delta R_{\mu\nu} = \partial_\sigma(g^{\mu\nu}\delta\Gamma^\sigma_{\mu\nu}) - \partial_\sigma(g^{\mu\sigma}\delta\Gamma^\nu_{\mu\nu}) \equiv \partial_\sigma W^\sigma, \tag{4}$$

where

$$W^\sigma \equiv g^{\mu\nu}\delta\Gamma^\sigma_{\mu\nu} - g^{\mu\sigma}\delta\Gamma^\nu_{\mu\nu}. \tag{5}$$

The second integral in the right-hand side of Eq. (2) can be written as follows:

$$\int d^4x\sqrt{-g}f'(R)g^{\mu\nu}\delta R_{\mu\nu} = \int d^4x\partial_\sigma[\sqrt{-g}f'(R)W^\sigma] - \int d^4x\partial_\sigma[\sqrt{-g}f'(R)]W^\sigma. \tag{6}$$

Meanwhile, the first integral in the right-hand side of Eq. (6) vanishes due to the fact that it is a surface integral, so Eq. (6) becomes

$$\int d^4x\sqrt{-g}f'(R)g^{\mu\nu}\delta R_{\mu\nu} = -\int d^4x\partial_\sigma[\sqrt{-g}f'(R)]W^\sigma. \tag{7}$$

By simple mathematical manipulation to the factor W^σ and some appropriate properties, we get

$$\int d^4x\sqrt{-g}f'(R)g^{\mu\nu}\delta R_{\mu\nu} = \int d^4x\partial_\sigma[\sqrt{-g}f'(R)][\partial^\mu(g_{\mu\nu}\delta g^{\sigma\nu}) - \partial^\sigma(g_{\mu\nu}\delta g^{\mu\nu})]. \tag{8}$$

[a]From now on we set $\kappa = 1$.

Then, Eq. (8) can be written as

$$\begin{aligned}&\int d^4x\sqrt{-g}f'(R)g^{\mu\nu}\delta R_{\mu\nu}\\&\quad=\int d^4x\partial^\mu[\partial_\sigma(\sqrt{-g}f'(R))(g_{\mu\nu}\delta g^{\sigma\nu})]-\int d^4x\partial^\mu\partial_\sigma(\sqrt{-g}f'(R))g_{\mu\nu}\delta g^{\sigma\nu}\\&\qquad-\int d^4x\partial^\sigma[\partial_\sigma(\sqrt{-g}f'(R))(g_{\mu\nu}\delta g^{\mu\nu})]+\int d^4x\partial^\sigma\partial_\sigma(\sqrt{-g}f'(R))(g_{\mu\nu}\delta g^{\mu\nu}).\end{aligned}\tag{9}$$

The first and third integrals of the right-hand side of Eq. (9) cancel each other. Therefore, Eq. (9) can be expressed in a simpler form as follows:

$$\begin{aligned}\int d^4x\sqrt{-g}f'(R)g^{\mu\nu}\delta R_{\mu\nu}&=\int d^4x g_{\mu\nu}\partial^\sigma\partial_\sigma[\sqrt{-g}f'(R)]\delta g^{\mu\nu}\\&\quad-\int d^4x g_{\mu\nu}\partial^\mu\partial_\sigma[\sqrt{-g}f'(R)]\delta g^{\sigma\nu}.\end{aligned}\tag{10}$$

Furthermore, Eq. (2) can be written as follows:

$$\begin{aligned}\delta\int d^4x\sqrt{-g}f(R)&=\int d^4x[g_{\mu\nu}\partial^\sigma\partial_\sigma(\sqrt{-g}f'(R))-g_{\sigma\nu}\partial^\mu\partial_\sigma(\sqrt{-g}f'(R))]\delta g^{\mu\nu}\\&\quad+\int d^4x\sqrt{-g}\left[f'(R)R_{\mu\nu}-\frac{1}{2}f(R)g_{\mu\nu}\right]\delta g^{\mu\nu}.\end{aligned}\tag{11}$$

If the total variation $\delta\int d^4x\sqrt{-g}f(R)$ of the action must vanish, we will have the fourth-order vacuum field equations

$$\nabla_\mu\nabla_\nu f'(R)-g_{\mu\nu}\Box f'(R)-f'(R)R_{\mu\nu}+\frac{f(R)}{2}g_{\mu\nu}=0.\tag{12}$$

Equations (12) can be rearranged in the Einstein-like form, that is

$$\begin{aligned}\nabla_\mu\nabla_\nu f'(R)-g_{\mu\nu}\Box f'(R)&=f'(R)R_{\mu\nu}-\frac{f(R)}{2}g_{\mu\nu}\\&=f'(R)R_{\mu\nu}-\frac{f'(R)}{2}g_{\mu\nu}R+\frac{f'(R)}{2}g_{\mu\nu}R-\frac{f(R)}{2}g_{\mu\nu}\\&=f'(R)\left[R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R\right]+\left[\frac{f'(R)R-f(R)}{2}\right]g_{\mu\nu}\\&=f'(R)G_{\mu\nu}+\left[\frac{f'(R)R-f(R)}{2}\right]g_{\mu\nu}.\end{aligned}\tag{13}$$

Equation (13) can be expressed as

$$G_{\mu\nu}=\frac{1}{f'(R)}\left\{\nabla_\mu\nabla_\nu f'(R)-g_{\mu\nu}\Box f'(R)+g_{\mu\nu}\frac{[f(R)-f'(R)R]}{2}\right\},\tag{14}$$

or

$$G_{\mu\nu} = T^{\text{eff}}_{\mu\nu}, \tag{15}$$

where

$$T^{\text{eff}}_{\mu\nu} = \frac{1}{f'(R)}\left\{\nabla_\mu\nabla_\nu f'(R) - g_{\mu\nu}\Box f'(R) + g_{\mu\nu}\frac{[f(R) - f'(R)R]}{2}\right\}, \tag{16}$$

is then regarded as an effective energy–momentum tensor. After Capozziello we will call it *the curvature fluid energy–momentum tensor* $T^{(\text{curv})}_{\mu\nu}$ sourcing the effective Einstein equations [22–25]. Capozziello *et al.* have shown that the first two terms of the right-hand side of Eq. (16) prevent the involved energy–momentum tensor from describing the perfect fluid [25].

In the presence of matter and energy, Eq. (12) or Eq. (16) can be generalized, respectively, as

$$\nabla_\mu\nabla_\nu f'(R) - g_{\mu\nu}\Box f'(R) - f'(R)R_{\mu\nu} + \frac{f(R)}{2}g_{\mu\nu} = T_{\mu\nu}, \tag{17}$$

or

$$G_{\mu\nu} = T_{\mu\nu} + T^{\text{eff}}_{\mu\nu}, \tag{18}$$

where $T_{\mu\nu}$ is the energy–momentum tensor of matter. In this paper, we assume the interaction between matter and curvature fluid or background fluid in the form of diffusion and friction. We will also assume the possibility of the presence of interplay between diffusion and friction.

3. Relativistic Diffusion Process in $f(R)$-Gravity

In this section, we apply the relativistic diffusion and friction found in the works of Calogero and Alcantara to the context of $f(R)$-gravity. At first, we will review the diffusion and friction in curved spacetime by referring to [17, 18] shortly. Kinetic diffusion and friction on curved spacetimes are considered in Lorentzian, time-oriented manifold (M, g), where M denotes the time-oriented manifold and g is the metric tensor on the manifold. Let x denote an arbitrary point of M and x^μ is a local coordinate system on an open set $U \subset M, x \in U$, with $x^0 \equiv t$ being timelike. The vectors ∂_{x^μ} form a basis of the tangent space T_xM and the components of $p \in T_xM$ in this basis will be denoted by p^μ. The pair (x^μ, p^ν) provides a coordinates system on $TU \subset TM$, where TM denotes the tangent bundle of M. The (future) mass-shell is the 7-dimensional submanifold of the tangent bundle defined as

$$\Pi M = \{(x, p) \in TM : g(x)(p, p) = -1, p \text{ future directed}\}. \tag{19}$$

On the subset $\Pi U = \{(x, p) \in \Pi M : x \in U\}$ of the mass-shell, the condition $g(x)(p, p) = -1$ is equivalent to $g_{\mu\nu}p^\mu p^\nu = -1$, which can be used to express p^0 in terms p^1, p^2, p^3, precisely;

$$p^0 = -\frac{1}{g_{00}}[g_{0j}p^j + \sqrt{(g_{0j}p^j)^2 - g_{00}(1 + g_{ij}p^ip^j)}]. \tag{20}$$

Through further calculations, the frictionless FPE associated to diffusion process in curved spacetime obtained by Calogero [17] is given by

$$\partial_t f + \hat{p}\cdot\nabla_x f = D\partial_{p^i}\left(\frac{\delta^{ij} + p^ip^j}{\sqrt{1+|p|^2}}\partial_{p^j} f\right), \tag{21}$$

where $D > 0$ is the diffusion constant and $\hat{p} = \frac{p}{p^0}$ is the particles 4-momentum.

The main physical assumption behind Eq. (21) is that the particles are moving in a background fluid in thermal equilibrium [17]. The molecules of the fluid are assumed to be much lighter than the particles, and the particles make up a sufficiently dilute system, the total force acting on the particles can be macroscopically approximated by dominant contributions from diffusion. The diffusion due to thermal fluctuations is associated to random collisions with the molecules of the fluid.

In the presence of friction, as described by Alcantara and Calogero, the general relativistic FPE is given by

$$\partial_t f + \hat{p}\cdot\nabla_x f = \partial_{p^i}\left(D\frac{\delta^{ij} + p^ip^j}{\sqrt{1+|p|^2}}\partial_{p^j} f + \mathcal{F}p^i f\right), \tag{22}$$

where $\mathcal{F}$ denotes friction constant. The friction takes into account the deterministic grazing collisions mainly between particles and fluid molecules and among the particles themselves. Equation (22) describes the process in which the diffusion process and the friction process are independent. It means that there is no interplay between the diffusion process and the friction process. In the case of the presence of interplay between diffusion process and friction process, the general relativistic FPE is not as simple as Eq. (22). In contrast to the work of Calogero and Alcantara, in this paper, we consider also the process in which the interplay between diffusion and friction is present.

The relativistic current density vector J^μ associated to diffusion process and matter energy–momentum tensor $T^{\mu\nu}$ are defined by

$$J^\mu(x) = \sqrt{|g|}\int_{\Pi_x U} f\frac{p^\mu}{-p^0}dp^{123}, \tag{23}$$

$$T^{\mu\nu}(x) = \sqrt{|g|}\int_{\Pi_x U} f\frac{p^\mu p^\nu}{-p^0}dp^{123}, \tag{24}$$

where dp^{123} is element volume of momentum in 3-spaces.

Based on Eqs. (22) and (23), the covariance derivative of the matter energy–momentum tensor in the absence of interplay between diffusion and friction is given by

$$\nabla_\mu T^{\mu\nu} = 3DJ^\nu + \mathcal{F}I^\nu, \tag{25}$$

where

$$I^\nu := \int p^2 \partial_{p^i}(fp^i)dp^{123}. \tag{26}$$

It is clear that

$$\nabla_\mu J^\mu = 0; \quad \nabla_\mu I^\mu \neq 0.$$

When the interplay between diffusion and friction presents, the covariant derivative of the matter energy–momentum tensor cannot be expressed by Eq. (25). In this case, however, we have

$$\nabla_\mu T^{\mu\nu} = \mathcal{C}K^\nu, \tag{27}$$

where K^ν is an appropriate vector field representing some kind of current density associated to the total process and $\mathcal{C}$ a constant.

Calogero tried to couple the above FPE in the absence of friction (Eq. (21)) with the standard EFE,

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = T_{\mu\nu}. \tag{28}$$

However, when we take the divergence of both sides of Eq. (28), we face a serious problem concerning the presence of current density vector field. The Bianchi identity implies the vanishing of the left-hand side. In turn, the right-hand side of Eq. (28) gives $\nabla^\mu T_{\mu\nu} = 0$. If we compare it to the covariance derivatives of energy–momentum tensor (Eq. (25)) obtained from the FPE (Eq. (21)), we have a contradiction. Calogero then proposed two alternative solutions to remove the incompatibility: (1) by adding additional matter tensors to the right-hand side of the EFE and (2) by adding a cosmological scalar field on the left-hand side of the EFE.

Here, we try to couple the FPE describing relativistic diffusion and friction in the presence of interplay between both processes with the field equation of $f(R)$-gravity. From the above observation, we realize that the incompatibility is due mainly to the geometric part of EFE. Therefore, the modification should be made on that part. In Sec 2, we have derived the general form of the field equations of $f(R)$ gravity in the presence of matter as

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + T^{\text{eff}}_{\mu\nu} = T_{\mu\nu}, \tag{29}$$

where we assume the existence of the so-called curvature fluid represented by effective energy–momentum tensor $T^{\text{eff}}_{\mu\nu}$. We will assume here that the curvature fluid (Eq. (16)) plays the role of background fluid in the relativistic diffusion and friction formulated by Calogero and Alcantara.

The consequence of Bianchi's identity, $\nabla^\mu(R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R)=0$ to the field Eq. (29) is

$$\nabla^\mu T^{\text{eff}}_{\mu\nu} = \nabla^\mu T_{\mu\nu}. \tag{30}$$

From the FPE on curved spacetime, we have then $\nabla^\mu T_{\mu\nu} = \mathcal{C}K_\nu$ (see [17]), so the relation between the curvature fluid or background fluid energy–momentum tensor $T^{\text{eff}}_{\mu\nu}$ and current density K_ν in $f(R)$-gravity theory can be written as

$$\begin{aligned}\nabla^\mu T^{\text{eff}}_{\mu\nu} &= \nabla^\mu\left[\frac{1}{f'(R)}\nabla_\mu\nabla_\nu f'(R) - g_{\mu\nu}\frac{1}{2f'(R)}[2\Box f'(R)+f(R)-f'(R)R]\right]\\ &= \mathcal{C}K_\nu.\end{aligned} \tag{31}$$

Equation (31) describes the relation between the effective momentum–energy tensor and current density for any $f(R)$-function. This equation is also assumed to overcome the problem emerging in the attempt to couple the FPE with the EFE. In this case, we assume that the energy–momentum tensor of the additional matter fields added by Calogero on the right-hand side of the EFE has been represented by the effective momentum–energy tensor of the field equation of $f(R)$-gravity.

The current density K_ν expressed by Eq. (31) certainly provides different information which are different from that of the current density associated to diffusion process only as proposed by Calogero. In this context, we interpret that the current density described by Eq. (31) is an effective current density which is the accumulation of the current density of diffusion and friction in the presence of interplay between both processes. If we assume that diffusion is the only process which presents in the situation, then

$$\mathcal{C}K_\nu = 3DJ_\nu. \tag{32}$$

Taking the divergence of both side of Eq. (32) gives $\mathcal{C}\nabla^\nu K_\nu = 0$. It means

$$\nabla^\nu\left[\nabla^\mu\left[\frac{1}{f'(R)}\nabla_\mu\nabla_\nu f'(R) - g_{\mu\nu}\frac{1}{2f'(R)}[2\Box f'(R)+f(R)-f'(R)R]\right]\right] = 0. \tag{33}$$

Now, we can also write Eq. (31) in the following form:

$$\nabla^\mu\left[\frac{1}{f'(R)}\nabla_\mu\nabla_\nu f'(R)\right] - \nabla^\mu\left[g_{\mu\nu}\frac{1}{2f'(R)}[2\Box f'(R)+f(R)-f'(R)R]\right] = \mathcal{C}K_\nu. \tag{34}$$

Furthermore, we define a scalar field $\mathcal{D}$ as

$$\mathcal{D} =: \frac{1}{2f'(R)}[2\Box f'(R)+f(R)-f'(R)R], \tag{35}$$

and tensor field $\mathcal{O}_{\mu\nu}$ as

$$\mathcal{O}_{\mu\nu} =: \frac{1}{f'(R)}\nabla_\mu\nabla_\nu f'(R). \tag{36}$$

Therefore, Eq. (31) can be written as

$$\nabla^\mu \mathcal{O}_{\mu\nu} - \nabla^\mu (g_{\mu\nu}\mathcal{D}) = \mathcal{C}K_\nu, \tag{37}$$

and Eq. (29) as

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \mathcal{D}g_{\mu\nu} - \mathcal{O}_{\mu\nu} = T_{\mu\nu}. \tag{38}$$

Compare Eq. (38) and the gravitational field equation proposed by Calogero in [17]. If we identify the term $\mathcal{D}g_{\mu\nu}$ with the term related to diffusion process, the term $\mathcal{O}_{\mu\nu}$ with the term related to friction, and we assume the absence of interplay between both processes, then Eq. (31) can be written as

$$\mathcal{C}K_\nu =: \mathcal{F}I_\nu + 3DJ_\nu. \tag{39}$$

If we take the divergence of Eq. (39), then we get $\mathcal{C}\nabla^\nu K_\nu = \mathcal{F}\nabla^\nu I_\nu$. It means

$$\nabla^\nu\nabla^\mu \left[\frac{1}{f'(R)}\nabla_\mu\nabla_\nu f'(R)\right] = \mathcal{F}\nabla^\nu I_\nu \tag{40}$$

$$\nabla^\nu\nabla^\mu \left[g_{\mu\nu}\frac{1}{2f'(R)}[2\Box f'(R) + f(R) - f'(R)R]\right] = 0. \tag{41}$$

3.1. *Special case I*: $f(R) = R + \Lambda R^2$

Here, we will consider the simplest model that was first introduced by Starobinsky, i.e. $f(R) = R + \Lambda R^2$, where Λ is constant [26]. This model is usually related to inflation. This model does not strongly affect large-scale cosmological behavior. At first, we will construct the field equations for $f(R) = R + \Lambda R^2$. By substituting $f(R) = R + \Lambda R^2$ and $f'(R) = 1 + 2\Lambda R$ into Eq. (12), we get the field equation for the vacuum case, i.e. [22, 24]

$$G_{\mu\nu} + 2\Lambda g_{\mu\nu}\Box R - \frac{1}{2}\Lambda g_{\mu\nu}R^2 + 2\Lambda R R_{\mu\nu} - 2\Lambda\nabla_\mu\nabla_\nu R = 0. \tag{42}$$

While in the presence of matter, the field equation for the function $f(R) = R+\Lambda R^2$ is given by [22, 24]

$$G_{\mu\nu} + 2\Lambda g_{\mu\nu}\Box R - \frac{1}{2}\Lambda g_{\mu\nu}R^2 + 2\Lambda R R_{\mu\nu} - 2\Lambda\nabla_\mu\nabla_\nu R = T^{(m)}_{\mu\nu}. \tag{43}$$

Thus, the form of the effective momentum–energy tensor for the function $f(R) = R + \Lambda R^2$ is the additional term in Eq. (43), i.e.

$$T^{\text{eff}}_{\mu\nu} = g_{\mu\nu}\left[2\Lambda\Box R - \frac{1}{2}\Lambda R^2\right] + 2\Lambda R R_{\mu\nu} - 2\Lambda\nabla_\mu\nabla_\nu R. \tag{44}$$

Furthermore, from Eq. (31), we obtain an equation that describes the relation between the effective current density to the matter energy–momentum tensor as the impact of interaction of matter with curvature fluid, that is

$$\nabla^\mu \left[g_{\mu\nu} \left[2\Lambda\Box R - \frac{1}{2}\Lambda R^2 \right] + 2\Lambda R R_{\mu\nu} - 2\Lambda\nabla_\mu\nabla_\nu R \right] =: \mathcal{C}K_\nu. \tag{45}$$

If we assumed that diffusion is the only process involved in the interaction between matter and curvature fluid, then we should have

$$\nabla^\nu \left[\nabla^\mu \left[g_{\mu\nu} \left[2\Lambda\Box R - \frac{1}{2}\Lambda R^2 \right] + 2\Lambda R R_{\mu\nu} - 2\Lambda\nabla_\mu\nabla_\nu R \right]\right] = 0. \tag{46}$$

However, a tedious calculation on the left-hand side of Eq. (46) yields

$$\Lambda\nabla_\mu R\nabla^\mu R + 2\Lambda(\nabla^\nu\nabla^\mu R)R_{\mu\nu}, \tag{47}$$

which in general may not vanish. Therefore, we face a contradiction to Eq. (46). It means, diffusion is not the only process involved in the interaction between matter and curvature fluid. Thus, the curvature fluid energy–momentum tensor $T^{\rm eff}_{\mu\nu}$ may cover both the contribution of diffusion as well as of friction. Now, if the interplay between diffusion and friction *didn't* present, then we can apply the identification given by Eqs. (34)–(36) and (39). From the identification we have the equation of current density of both the diffusion and of the friction, respectively, that are

$$\nabla^\mu \mathcal{D} = \nabla^\mu \left[g_{\mu\nu} \left[2\Lambda\Box R - \frac{1}{2}\Lambda R^2 \right]\right] = 3DJ^\nu, \tag{48}$$

$$\nabla^\mu \mathcal{O}_{\mu\nu} = \nabla^\mu \left[2\Lambda R R_{\mu\nu} - 2\Lambda\nabla_\mu\nabla_\nu R \right] = \mathcal{F}I^\nu. \tag{49}$$

If we take the divergence on the left-hand side of Eqs. (48) and (49), we get

$$\nabla^\nu \left[\nabla^\mu \left[g_{\mu\nu} \left[2\Lambda\Box R - \frac{1}{2}\Lambda R^2 \right]\right]\right] = 3D\nabla^\nu J_\nu = 0, \tag{50}$$

$$\nabla^\nu \left[\nabla^\mu \left[2\Lambda R R_{\mu\nu} - 2\Lambda\nabla_\mu\nabla_\nu R \right]\right] = \mathcal{F}\nabla^\nu I^\nu \neq 0. \tag{51}$$

If we calculate the left-hand side of Eq. (50), then we have

$$\nabla^\nu \left[\nabla^\mu \left[g_{\mu\nu} \left[2\Lambda\Box R - \frac{1}{2}\Lambda R^2 \right]\right]\right] = 2\Lambda\Box\Box R - \Lambda R\Box R - \Lambda\nabla_\nu R\nabla^\nu R \neq 0. \tag{52}$$

It contradicts Eq. (50). It means that in this case there must be in general an interplay between diffusion and friction.

3.2. *Special case II*: $f(R) = R - \frac{\mathcal{M}^4}{R}$

Since the acceleration of expansion universe was well established from observations, additional models were introduced in order to account for the acceleration. One of the first choices attempting to explain cosmological acceleration was $f(R)$ of the form $f(R) = R - \frac{\mathcal{M}^4}{R}$, where $\mathcal{M}$ is a constant [27]. This model also turns out to experience instability as expressed by [28]. It can be shown that the first-order correction of the field equation has a very large coefficient. The characteristic time of the instability is about $\sim 10^{-26}$ sec. It means that the instability grows very fast and reaches high values.

Here, we will consider $f(R)$-function of the form $f(R) = R - \frac{\mathcal{M}^4}{R}$. At first, we will construct the field equations for $f(R)$ of this form. By substituting $f(R) = R - \frac{\mathcal{M}^4}{R}$ and $f'(R) = 1 + \frac{\mathcal{M}^4}{R^2}$ into Eq. (12), we get the field equation for the vacuum case, i.e.

$$G_{\mu\nu} + \mathcal{M}^4 R^{-2} R_{\mu\nu} + \frac{1}{2}\mathcal{M}^4 g_{\mu\nu} R^{-1} - \mathcal{M}^4 \nabla_\mu \nabla_\nu R^{-2} + \mathcal{M}^4 g_{\mu\nu} \Box R^{-2} = 0. \tag{53}$$

While in the presence of matter, the field equation for this $f(R)$ is given by

$$G_{\mu\nu} + \mathcal{M}^4 R^{-2} R_{\mu\nu} + \frac{1}{2}\mathcal{M}^4 g_{\mu\nu} R^{-1} - \mathcal{M}^4 \nabla_\mu \nabla_\nu R^{-2} + \mathcal{M}^4 g_{\mu\nu} \Box R^{-2} = T^{(m)}_{\mu\nu}. \tag{54}$$

Thus, the form of the effective momentum–energy tensor for this $f(R)$ is the additional term in Eq. (54), i.e.

$$T^{\text{eff}}_{\mu\nu} = g_{\mu\nu}\left[\mathcal{M}^4 \Box R^{-2} + \frac{1}{2}\mathcal{M}^4 R^{-1}\right] + \mathcal{M}^4 R^{-2} R_{\mu\nu} - \mathcal{M}^4 \nabla_\mu \nabla_\nu R^{-2}. \tag{55}$$

Furthermore, from Eq. (31), we obtain an equation that describes the relation between the effective current density and the matter energy–momentum tensor as the impact of interaction of matter with curvature fluid, that is

$$\nabla^\mu \left[g_{\mu\nu}\left[\mathcal{M}^4 \Box R^{-2} + \frac{1}{2}\mathcal{M}^4 R^{-1}\right] + \mathcal{M}^4 R^{-2} R_{\mu\nu} - \mathcal{M}^4 \nabla_\mu \nabla_\nu R^{-2}\right] =: \mathcal{C} K_\nu. \tag{56}$$

If we assumed that diffusion is the only process involved in the interaction between matter and curvature fluid, then we should have

$$\nabla^\nu \left[\nabla^\mu \left[g_{\mu\nu}\left[\mathcal{M}^4 \Box R^{-2} + \frac{1}{2}\mathcal{M}^4 R^{-1}\right] + \mathcal{M}^4 R^{-2} R_{\mu\nu} - \mathcal{M}^4 \nabla_\mu \nabla_\nu R^{-2}\right]\right] = 0. \tag{57}$$

However, a tedious calculation on the left-hand side of Eq. (57) yields that the left-hand side of Eq. (57) doesn't vanish, i.e.

$$-2\mathcal{M}^4 R^{-3} R_{\mu\nu} \nabla^\nu \nabla^\mu R + 6\mathcal{M}^4 R^{-4} R_{\mu\nu} \nabla^\nu R \nabla^\mu R - \mathcal{M}^4 R^{-3} \nabla_\mu R \nabla^\mu R. \tag{58}$$

Therefore, we face a contradiction to Eq. (57). It means, diffusion is not the only process involved in the interaction between matter and curvature fluid. Thus, the curvature fluid energy–momentum tensor $T^{\text{eff}}_{\mu\nu}$ may cover both the contribution of diffusion as well as of friction. Now, if we assume that the interplay between diffusion and friction *didn't* present, then we can apply the identification given by Eqs. (34)–(36) and (39). From the identification, we have the equation of current density both of diffusion and friction, respectively, i.e.

$$\nabla^{\mu}\mathcal{D} = \nabla^{\mu}\left[g_{\mu\nu}\left[\mathcal{M}^4\Box R^{-2} + \frac{1}{2}\mathcal{M}^4 R^{-1}\right]\right] = 3DJ^{\nu}, \tag{59}$$

$$\nabla^{\mu}\mathcal{O}_{\mu\nu} = \nabla^{\mu}[\mathcal{M}^4 R^{-2} R_{\mu\nu} - \mathcal{M}^4\nabla_{\mu}\nabla_{\nu}R^{-2}] = \mathcal{F}I^{\nu}. \tag{60}$$

If we take the divergence on the left-hand side of Eqs. (59) and (60), then we get

$$\nabla^{\nu}\left[\nabla^{\mu}\left[g_{\mu\nu}\left[\mathcal{M}^4\Box R^{-2} + \frac{1}{2}\mathcal{M}^4 R^{-1}\right]\right]\right] = 3D\nabla^{\nu}J_{\nu} = 0, \tag{61}$$

$$\nabla^{\nu}\left[\nabla^{\mu}\left[\mathcal{M}^4 R^{-2} R_{\mu\nu} - \mathcal{M}^4\nabla_{\mu}\nabla_{\nu}R^{-2}\right]\right] = \mathcal{F}\nabla^{\nu}I^{\nu} \neq 0. \tag{62}$$

Using a tedious calculation, the left-hand side of Eq. (61) in general doesn't vanish, i.e.

$$\begin{aligned}&\nabla^{\nu}\left[\nabla^{\mu}\left[g_{\mu\nu}\left[\mathcal{M}^4\Box R^{-2} + \frac{1}{2}\mathcal{M}^4 R^{-1}\right]\right]\right]\\&\quad = \mathcal{M}^4\Box\Box R^{-2} - \frac{1}{2}\mathcal{M}^4 R^{-2}\Box R + \mathcal{M}^4 R^{-3}\nabla^{\nu}R\nabla_{\nu}R \neq 0.\end{aligned} \tag{63}$$

This contradiction to Eq. (61) means that there must be also an interplay between diffusion and friction.

3.3. *Special case III*: $f(R) = R - \mathcal{E}r_c\ln(1+\frac{R}{r_c})$

A viable $f(R)$ theory must comply cosmological observations and local gravity constraints. One of the existing viable models that satisfy both cosmological and local gravity constraints, i.e the model initiated by [29], $f(R) = R - \mathcal{E}r_c\ln(1+\frac{R}{r_c})$, where Λ is a constant and r_c is a critical radius. The logarithmic correction here plays an important role in cosmology and gravitational waves [30, 31]. The cosmological background of gravitational waves can be tuned by the higher-order correction to the gravitational Lagrangian. In [31], Capozziello *et al.* have shown that by assuming $R^{1+\epsilon}$, where ϵ denotes a generic correction to the Hilbert–Einstein action in the Ricci scalar R, gives a parametric approach to control the evolution and the production mechanism of gravitational waves in the early universe. Here, we will study the logarithmic correction of $f(R)$ to investigate the diffusion process and friction in the curvature fluid related to this function. At first, we will derive the field equations for $f(R)$ of this form. By substituting $f(R) = R - \mathcal{E}r_c\ln(1+\frac{R}{r_c})$ and $f'(R) = 1-\mathcal{E}(1+\frac{R}{r_c})^{-1}$ into Eq. (12), we get the field equation for the vacuum case, i.e.

$$\begin{aligned}&G_{\mu\nu} - \mathcal{E}\left(1+\frac{R}{r_c}\right)^{-1}R_{\mu\nu} - \frac{1}{2}\mathcal{E}r_c g_{\mu\nu}\ln\left(1+\frac{R}{r_c}\right)\\&\quad - \mathcal{E}\nabla_{\mu}\nabla_{\nu}\left(1+\frac{R}{r_c}\right)^{-1} + \mathcal{E}g_{\mu\nu}\Box\left(1+\frac{R}{r_c}\right)^{-1} = 0.\end{aligned} \tag{64}$$

While in the presence of matter, the field equation for this $f(R)$ is given by

$$G_{\mu\nu} - \mathcal{E}\left(1+\frac{R}{r_c}\right)^{-1} R_{\mu\nu} - \frac{1}{2}\mathcal{E} r_c g_{\mu\nu} \ln\left(1+\frac{R}{r_c}\right) - \mathcal{E}\nabla_\mu\nabla_\nu\left(1+\frac{R}{r_c}\right)^{-1} + \mathcal{E} g_{\mu\nu}\Box\left(1+\frac{R}{r_c}\right)^{-1} = T^{(m)}_{\mu\nu}. \tag{65}$$

Thus, the form of the effective momentum–energy tensor for the function $f(R) = R - \mathcal{E} r_c \ln(1+\frac{R}{r_c})$ is the additional term in Eq. (65), i.e.

$$T^{\text{eff}}_{\mu\nu} = g_{\mu\nu}\left[\mathcal{E}\Box\left(1+\frac{R}{r_c}\right)^{-1} - \frac{1}{2}\mathcal{E} r_c \ln\left(1+\frac{R}{r_c}\right)\right] - \mathcal{E}\left(1+\frac{R}{r_c}\right)^{-1} R_{\mu\nu} - \mathcal{E}\nabla_\mu\nabla_\nu\left(1+\frac{R}{r_c}\right)^{-1}. \tag{66}$$

Furthermore, from Eq. (31), we obtain an equation that describes the relation between the effective current density and the matter energy–momentum tensor as the impact of the interaction between matter and curvature fluid, namely

$$\nabla^\mu\left[g_{\mu\nu}\left[\mathcal{E}\Box\left(1+\frac{R}{r_c}\right)^{-1} - \frac{1}{2}\mathcal{E} r_c \ln\left(1+\frac{R}{r_c}\right)\right] - \mathcal{E}\left(1+\frac{R}{r_c}\right)^{-1} R_{\mu\nu} - \mathcal{E}\nabla_\mu\nabla_\nu\left(1+\frac{R}{r_c}\right)^{-1}\right] =: \mathcal{C}K_\nu. \tag{67}$$

If we assumed that diffusion is the only process involved in the interaction between matter and curvature fluid, then we should have

$$\nabla^\nu\left[\nabla^\mu\left[g_{\mu\nu}\left[\mathcal{E}\Box\left(1+\frac{R}{r_c}\right)^{-1} - \frac{1}{2}\mathcal{E} r_c \ln\left(1+\frac{R}{r_c}\right)\right] - \mathcal{E}\left(1+\frac{R}{r_c}\right)^{-1} R_{\mu\nu} - \mathcal{E}\nabla_\mu\nabla_\nu\left(1+\frac{R}{r_c}\right)^{-1}\right]\right] = 0. \tag{68}$$

However, a tedious calculation on the left-hand side of Eq. (68) yields

$$2\mathcal{E}\Box\Box\left(1+\frac{R}{r_c}\right)^{-1} + \frac{\mathcal{E}}{r_c}\left(1+\frac{R}{r_c}\right)^{-2} R_{\mu\nu}\nabla^\nu\nabla^\mu R + \frac{\mathcal{E}}{2r_c}\left(1+\frac{R}{r_c}\right)^{-2}\nabla_\mu R\nabla^\mu R - \frac{2\mathcal{E}}{r_c}\left(1+\frac{R}{r_c}\right)^{-3} R_{\mu\nu}\nabla^\nu R\nabla^\mu R, \tag{69}$$

which in general does not vanish. This contradiction to Eq. (68) means that diffusion is not the only process involved in the interaction between matter and curvature fluid. Thus, the curvature fluid energy–momentum tensor $T^{\text{eff}}_{\mu\nu}$ may cover both the contribution of diffusion as well as of friction. Now, if the interplay between diffusion and friction *didn't* present, then we can apply the identification given by Eqs. (34)–(36) and (39). From the identification, we have respectively the equation of current density of both diffusion and friction, namely

$$\nabla^\mu \mathcal{D} = \nabla^\mu \left[g_{\mu\nu} \left[\mathcal{E}\Box \left(1 + \frac{R}{r_c}\right)^{-1} - \frac{1}{2}\mathcal{E} r_c \ln\left(1 + \frac{R}{r_c}\right)\right]\right] = 3DJ^\nu, \quad (70)$$

$$\nabla^\mu \mathcal{O}_{\mu\nu} = \nabla^\mu \left[-\mathcal{E}\left(1 + \frac{R}{r_c}\right)^{-1} R_{\mu\nu} - \mathcal{E}\nabla_\mu \nabla_\nu \left(1 + \frac{R}{r_c}\right)^{-1}\right] = \mathcal{F} I^\nu. \quad (71)$$

If we take the divergence of the left-hand side of Eqs. (70) and (71), then we get

$$\nabla^\nu \left[\nabla^\mu \left[g_{\mu\nu} \left[\mathcal{E}\Box \left(1 + \frac{R}{r_c}\right)^{-1} - \frac{1}{2}\mathcal{E} r_c \ln\left(1 + \frac{R}{r_c}\right)\right]\right]\right] = 3D\nabla^\nu J_\nu = 0, \quad (72)$$

$$\nabla^\nu \left[\nabla^\mu \left[-\mathcal{E}\left(1 + \frac{R}{r_c}\right)^{-1} R_{\mu\nu} - \mathcal{E}\nabla_\mu \nabla_\nu \left(1 + \frac{R}{r_c}\right)^{-1}\right]\right] = \mathcal{F}\nabla^\nu I^\nu \neq 0. \quad (73)$$

If we calculate the term of left-hand side of Eq. (72), then we have

$$\begin{aligned} &\nabla^\nu \left[\nabla^\mu \left[g_{\mu\nu} \left[\mathcal{E}\Box \left(1 + \frac{R}{r_c}\right)^{-1} - \frac{1}{2}\mathcal{E} r_c \ln\left(1 + \frac{R}{r_c}\right)\right]\right]\right] \\ &= \mathcal{E}\Box\Box\left(1 + \frac{R}{r_c}\right)^{-1} + \frac{1}{2}\mathcal{E}\left(1 + \frac{R}{r_c}\right)^{-1}\Box R - \frac{1}{2r_c}\mathcal{E}\left(1 + \frac{R}{r_c}\right)^{-2}\nabla_\nu R \nabla^\nu R \\ &\neq 0. \end{aligned} \quad (74)$$

It contradicts Eq. (72). It means that there must be an interplay between diffusion and friction. More specifically, Eq. (45) can be used to explain the diffusion process in expanding universe, for example, FLRW model.

4. Diffusion Process in Expanding Universe

In this section, we will consider the special case of diffusion process and friction in expanding universe. The model of expanding universe that will be considered here is FLRW model. The FLRW model describes a homogeneous and isotropic expanding universe. The form of the FLRW metric is

$$ds^2 = -dt^2 + \frac{A(t)^2}{1 - Kr^2}dr^2 + A(t)^2 r^2 d\theta^2 + A(t)^2 r^2 \sin^2\theta d\phi^2, \quad (75)$$

where $A(t)$ is scale factor in the parameter of t.[b] The nonzero Ricci tensor components are given by

$$R_{tt} = -\frac{3\ddot{A}}{A}, \quad (76a)$$

[b]To make simplification, we write $A(t) = A, \dot{A}(t) = \dot{A}, \ddot{A}(t) = \ddot{A}$, etc.

$$R_{rr} = \frac{A\ddot{A} + 2\dot{A}^2 + 2K}{1 - Kr^2}, \tag{76b}$$

$$R_{\theta\theta} = r^2[A\ddot{A} + 2\dot{A}^2 + 2K], \tag{76c}$$

$$R_{\phi\phi} = r^2 \sin^2\theta[A\ddot{A} + 2\dot{A}^2 + 2K]. \tag{76d}$$

From Eqs. (76a)–(76d), the Ricci scalar for FLRW metric is

$$R = \frac{6[A\ddot{A} + \dot{A}^2 + K]}{A^2}. \tag{77}$$

Now, we will consider the above-mentioned cases of $f(R)$ to describe diffusion and friction processes in the expanding universe.

4.1. *Special case I*: $f(R) = R + \Lambda R^2$

At first, we will consider the quadratic form, $f(R) = R + \Lambda R^2$. By substituting both values of R and $R_{\mu\nu}$, respectively, in Eq. (44), we obtain the effective energy–momentum tensor or the curvature fluid tensor, $T^{\text{eff}}_{\mu\nu}$, that generates the diffusion process, friction, and interplay between both processes. The nonzero effective energy–momentum tensor components are

$$T^{\text{eff}}_{tt} = \frac{18\Lambda}{A^4}[2A^2\dot{A}\dddot{A} - A^2\ddot{A}^2 + 2A\dot{A}^2\ddot{A} - 3\dot{A}^4 - 2K\dot{A}^2 + K^2], \tag{78}$$

$$\begin{aligned} T^{\text{eff}}_{\theta\theta} = {}& -\frac{6\Lambda r^2}{A^2}[2A^3\ddddot{A} + 4A^2\dot{A}\dddot{A} + 3A^2\ddot{A}^2 - 12A\dot{A}^2\ddot{A} \\ & - 4KA\ddot{A} + 3\dot{A}^4 + 2K\dot{A}^2 - K^2], \end{aligned} \tag{79}$$

$$\begin{aligned} T^{\text{eff}}_{rr} = {}& \frac{6\Lambda}{A^2(Kr^2 - 1)}[2A^3\ddddot{A} + 4A^2\dot{A}\dddot{A} + 3A^2\ddot{A}^2 - 12A\dot{A}^2\ddot{A} \\ & - 4KA\ddot{A} + 3\dot{A}^4 + 2K\dot{A}^2 - K^2], \end{aligned} \tag{80}$$

$$\begin{aligned} T^{\text{eff}}_{\phi\phi} = {}& -\frac{6\Lambda r^2 \sin^2\theta}{A^2}[2A^3\ddddot{A} + 4A^2\dot{A}\dddot{A} + 3A^2\ddot{A}^2 - 12A\dot{A}^2\ddot{A} \\ & - 4KA\ddot{A} + 2K\dot{A}^2 + 3\dot{A}^4 - K^2]. \end{aligned} \tag{81}$$

By substituting each component of the effective energy–momentum tensor to Eq. (45), the total effective current density components, K_ν,[c] induced by the reaction of spacetime to various physical processes in the expanding universe are

$$\begin{aligned} K_t = {}& \frac{12\Lambda}{A^4}\Bigg[A^3\overset{\cdot\cdot\cdot\cdot\cdot}{A} + 2A^2\dot{A}\ddddot{A} + (-2KA + 5A^2\ddot{A} - 11A\dot{A}^2)\dddot{A} \\ & + 6\dot{A}\ddot{A}\left(K - 2A\ddot{A} + \frac{5}{2}\dot{A}^2\right)\Bigg], \end{aligned} \tag{82}$$

$$K_r = K_\theta = K_\phi = 0.$$

[c]where $\mu = t, r, \theta, \phi$ are moving index.

Equation (82) shows that the processes in the expanding universe propagate to the time coordinate only. Now, if we take the divergence of the left-hand side of Eq. (82), then based on Eq. (47), we obtain

$$\begin{aligned}\nabla^\nu K_\nu = -\frac{12\Lambda}{A^5}\Big[&A^4\overset{\cdot\cdot\cdot\cdot\cdot\cdot}{A} + A^3\dot{A}\overset{\cdot\cdot\cdot\cdot\cdot}{A} - 2KA^2\ddddot{A} + 7A^3\ddot{A}\ddddot{A} - 15A^2\dot{A}^2\ddddot{A}\\ &+ 5A^3\dddot{A}^2 + 12KA\dot{A}\dddot{A} - 56A^2\dot{A}\ddot{A}\dddot{A} + 48A\dot{A}^3\dddot{A}\\ &- 12A^2\ddot{A}^3 + 6KA\ddot{A}^2 + 81A\dot{A}^2\ddot{A}^2 - 4K\dot{A}^2 - 10\dot{A}^4\Big]. \end{aligned} \tag{83}$$

Since $\nabla^\nu K_\nu \neq 0$, diffusion is not the only process which presents in expanding universe.

From the identification Eq. (48), we obtain $J_\nu = 0$ and $\nabla^\nu J_\nu = 0$, whereas the effective current of friction process is given by

$$\begin{aligned}I_t = \frac{12\Lambda}{A^5}\Bigg[&A^4\overset{\cdot\cdot\cdot\cdot\cdot}{A} + 2A^3\dot{A}\ddddot{A} + A^2\dddot{A}(K + 8A\ddot{A} - 8\dot{A}^2)\\ &- 6\dot{A}\left(\frac{3}{2}A^2\ddot{A}^2 - \frac{1}{2}KA\ddot{A} - 2A\dot{A}^2\ddot{A} + \dot{A}^4 + 2K\dot{A}^2 + K^2\right)\Bigg], \end{aligned} \tag{84}$$

$$I_r = I_\theta = I_\phi = 0. \tag{85}$$

From Eq. (51), we get the divergence of I_ν, i.e.

$$\begin{aligned}\nabla^\nu I_\nu = \frac{12\Lambda}{A^6}[&A^5\overset{\cdot\cdot\cdot\cdot\cdot\cdot}{A} + A^4\dot{A}\overset{\cdot\cdot\cdot\cdot\cdot}{A} + A^3\ddddot{A}(K + 10A\ddot{A} - 12\dot{A}^2) + 8A^4\dddot{A}^2\\ &- 50A^3\dot{A}\ddot{A}\dddot{A} + 36A^2\dot{A}^3\dddot{A} - 9A^3\ddot{A}^3 + 3KA^2\ddot{A}^2 + 63A^2\dot{A}^2\ddot{A}^2\\ &- 6A\ddot{A}(K^2 + 8K\dot{A}^2 + 13\dot{A}^4) + 30\dot{A}^2(K + \dot{A}^2)^2]. \end{aligned} \tag{86}$$

4.2. *Special case II*: $f(R) = R - \frac{\mathcal{M}^4}{R}$

Now, we consider the form $f(R) = R - \frac{\mathcal{M}^4}{R}$ to describe diffusion and friction processes as well as the interplay between diffusion and friction in the expanding universe. At first, we derive the expression of current density to describe all the processes. By substituting both values of R and $R_{\mu\nu}$, respectively, in Eq. (55), we obtain the effective energy–momentum tensor or the curvature fluid tensor, $T^{\text{eff}}_{\mu\nu}$, that generates the diffusion process, friction, and interplay between both processes. The nonzero effective energy–momentum tensor components are

$$\begin{aligned}T^{\text{eff}}_{tt} = -\frac{\mathcal{M}^4 A}{12(K + A\ddot{A} + \dot{A}^2)^3}[&2A^3\dot{A}\dddot{A} + 6A^2\ddot{A}^3 + 12A\dot{A}^2\ddot{A}^2 + A^3\ddot{A}^2\\ &+ 12KA\ddot{A}^2 + 6\dot{A}^4\ddot{A} + 4A^2\dot{A}^2\ddot{A} + 12K\dot{A}^2\ddot{A} + 2KA^2\ddot{A} + 6K^2\ddot{A}\\ &+ A(K + \dot{A}^2)(K - 3\dot{A}^2)], \end{aligned} \tag{87}$$

$$T_{rr}^{\text{eff}} = \frac{\mathcal{M}^4 A^2}{18(K + A\ddot{A} + \dot{A}^2)^4 (Kr^2 - 1)} \Bigg[A^5 \ddddot{A}(K + A\ddot{A} + \dot{A}^2) - 3A^6 \dddot{A}^2$$

$$+ 14A^4 \dot{A}\dddot{A}\left(K - \frac{2}{7}A\ddot{A} + \dot{A}^2\right) + 3A^4\ddot{A}^4 + 15A^3\ddot{A}^3\left(\dot{A}^2 + \frac{1}{6}A^2 + K\right)$$

$$+ 27A^2\ddot{A}^2\left(\dot{A}^4 - \frac{7}{54}A^2\dot{A}^2 + 2K\dot{A}^2 + \frac{7}{54}KA^2 + K^2\right)$$

$$+ 21A\ddot{A}\left(\dot{A}^4 + \frac{25}{42}A^2\dot{A}^2 + 2K\dot{A}^2 + \frac{5}{42}KA^2 + K^2\right)(K + \dot{A}^2)$$

$$+ 6\left(\dot{A}^4 - \frac{17}{12}A^2\dot{A}^2 + 2K\dot{A}^2 + \frac{1}{4}KA^2 + K^2\right)(K + \dot{A}^2)^2\Bigg], \tag{88}$$

$$T_{\theta\theta}^{\text{eff}} = \frac{\mathcal{M}^4 r^2 A^2}{18(K + A\ddot{A} + \dot{A}^2)^4} \Bigg[A^5 \ddddot{A}(K + A\ddot{A} + \dot{A}^2) - 3A^6 \dddot{A}^2$$

$$+ 14A^4 \dot{A}\dddot{A}\left(K - \frac{2}{7}A\ddot{A} + \dot{A}^2\right) + 3A^4\ddot{A}^4 + 15A^3\ddot{A}^3\left(\dot{A}^2 + \frac{1}{6}A^2 + K\right)$$

$$+ 27A^2\ddot{A}^2\left(\dot{A}^4 - \frac{7}{54}A^2\dot{A}^2 + 2K\dot{A}^2 + \frac{7}{54}KA^2 + K^2\right)$$

$$+ 21A\ddot{A}\left(\dot{A}^4 + \frac{25}{42}A^2\dot{A}^2 + 2K\dot{A}^2 + \frac{5}{42}KA^2 + K^2\right)(K + \dot{A}^2)$$

$$+ 6\left(\dot{A}^4 - \frac{17}{12}A^2\dot{A}^2 + 2K\dot{A}^2 + \frac{1}{4}KA^2 + K^2\right)(K + \dot{A}^2)^2\Bigg], \tag{89}$$

$$T_{\phi\phi}^{\text{eff}} = -\frac{\mathcal{M}^4 A^2 r^2 \sin^2\theta}{6(K + A\ddot{A} + \dot{A}^2)^4} \Bigg[\frac{1}{3}A^5 \ddddot{A}(K + A\ddot{A} + \dot{A}^2) - A^6 \dddot{A}^2$$

$$+ \frac{4}{3}A^4 \dot{A}\dddot{A}\left(-\frac{7}{2}K + A\ddot{A} - \frac{7}{2}\dot{A}^2\right) + A^4\ddot{A}^4 + 5A^3\ddot{A}^3\left(\dot{A}^2 + \frac{1}{6}A^2 + K\right)$$

$$+ 9A^2\ddot{A}^2\left(\dot{A}^4 - \frac{7}{54}A^2\dot{A}^2 + 2K\dot{A}^2 + \frac{7}{54}KA^2 + K^2\right)$$

$$+ 7A\ddot{A}\left(\dot{A}^4 + \frac{25}{42}A^2\dot{A}^2 + 2K\dot{A}^2 + \frac{5}{42}KA^2 + K^2\right)(K + \dot{A}^2)$$

$$+ 2\left(\dot{A}^4 - \frac{17}{12}A^2\dot{A}^2 + 2K\dot{A}^2 + \frac{1}{4}KA^2 + K^2\right)(K + \dot{A}^2)^2\Bigg]. \tag{90}$$

Using the similar procedure with quadratic case, we obtain the value of K_ν as follows:

$$K_\nu = -\frac{\mathcal{M}^4}{18A(K + A\ddot{A} + \dot{A}^2)^5}\Bigg[A^6 \overset{\cdot\cdot\cdot\cdot\cdot}{A}(K + A\ddot{A} + \dot{A}^2)^2 - 81A^6\dot{A}\dddot{A}^2\left(K - \frac{1}{3}A\ddot{A} + \dot{A}^2\right)$$

$$
\begin{aligned}
&+20A^5\ddddot{A}(K+A\ddot{A}+\dot{A}^2)\left(-\frac{9}{20}A^2\dddot{A}+K\dot{A}-\frac{7}{20}A\dot{A}\ddot{A}+\dot{A}^3\right)+12A^8\dddot{A}^3\\
&-9A^2\dddot{A}\left(3A^2\ddot{A}^2\left(\dot{A}^4-\frac{157}{54}A^2\dot{A}^2+2K\dot{A}^2-\frac{31}{54}KA^2+K^2\right)+3A\ddot{A}(K+\dot{A}^2)\right.\\
&\times\left(\dot{A}^4+\frac{247}{54}A^2\dot{A}^2+2K\dot{A}^2-\frac{41}{54}KA^2+K^2\right)+(K+\dot{A}^2)^2\\
&\left.\times\left(\dot{A}^4-\frac{233}{18}A^2\dot{A}^2+2K\dot{A}^2+\frac{1}{18}KA^2+K^2\right)+A^3\ddot{A}^3\left(\dot{A}^2+\frac{11}{8}A^2+K\right)\right)\\
&-18\dot{A}\left(\frac{1}{2}A^3\ddot{A}^3\left(\dot{A}^4-\frac{17}{2}A^2\dot{A}^2+2K\dot{A}^2-\frac{7}{6}KA^2+K^2\right)\right.\\
&-\frac{3}{2}A^2\ddot{A}^2(K+\dot{A}^2)\left(\dot{A}^4-\frac{29}{6}A^2\dot{A}^2+2K\dot{A}^2+\frac{1}{6}KA^2+K^2\right)\\
&-\frac{5}{2}A\ddot{A}(K+\dot{A}^2)^2\left(\dot{A}^4+\frac{3}{2}A^2\dot{A}^2+2K\dot{A}^2-\frac{7}{10}KA^2+K^2\right)\\
&-(K+\dot{A}^2)^3\left(\dot{A}^4-\frac{3}{2}A^2\dot{A}^2+2K\dot{A}^2\right)-\frac{1}{6}KA^2+K^2\\
&\left.\left.+\frac{1}{2}A^4\ddot{A}^4\left(\dot{A}^2+\frac{5}{2}A^2+K\right)\right)\right]. \qquad (91)
\end{aligned}
$$

The divergence of Eq. (91) is given by

$$
\begin{aligned}
\nabla^\nu K_\nu = {}&\frac{25M^4}{18A^2(K+A\ddot{A}+\dot{A}^2)^6}\left[\frac{1}{25}A^7\overset{\cdot\cdot\cdot\cdot\cdot\cdot}{A}\left(K+A\ddot{A}+\dot{A}^2\right)^3\right.\\
&-\frac{9}{25}A^8\overset{\cdot\cdot\cdot\cdot\cdot}{A}{}^2(K+A\ddot{A}+\dot{A}^2)+A^6\overset{\cdot\cdot\cdot\cdot\cdot}{A}(K+A\ddot{A}+\dot{A}^2)^2\\
&\times\left(-\frac{12}{25}A^2\dddot{A}+K\dot{A}-\frac{11}{25}A\dot{A}\ddot{A}+\dot{A}^3\right)-\frac{9}{25}A^3\ddddot{A}\left(\frac{101}{3}A^4\dot{A}\dddot{A}\right.\\
&\times\left[K-\frac{43}{101}A\ddot{A}+\dot{A}^2\right]+A^3\ddot{A}^3\left[\dot{A}^2+\frac{25}{18}A^2+K\right]\\
&-8A^6\dddot{A}^2+3A^2\ddot{A}^2\left[\dot{A}^4-\frac{361}{54}A^2\dot{A}^2+2K\dot{A}^2-\frac{19}{18}KA^2+K^2\right]\\
&+3A\ddot{A}[K+\dot{A}^2]\left[\dot{A}^4+\frac{517}{54}A^2\dot{A}^2+2K\dot{A}^2-\frac{3}{2}KA^2+K^2\right]\\
&\left.+[K+\dot{A}^2]^2\left[\dot{A}^4-\frac{131}{6}A^2\dot{A}^2+2K\dot{A}^2+\frac{1}{18}KA^2+K^2\right]\right)(K+A\ddot{A}+\dot{A}^2)\\
&-\frac{12}{5}A^10\dddot{A}^4+\frac{516}{25}A^8\dot{A}\dddot{A}^3\left(K-\frac{17}{43}A\ddot{A}+\dot{A}^2\right)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{18}{25}A^4\dddot{A}^2\left(A^3\ddot{A}^3\left[\dot{A}^2+\frac{19}{9}A^2+K\right]+3A^2\ddot{A}^2\right.\\
&\times\left[\dot{A}^4-\frac{79}{6}A^2\dot{A}^2+2K\dot{A}^2-\frac{13}{6}KA^2+K^2\right]\\
&+3A\ddot{A}\left[\dot{A}^4+\frac{230}{9}A^2\dot{A}^2+2K\dot{A}^2-\frac{22}{9}KA^2+K^2\right][K+\dot{A}^2]\\
&\left.+\left[\dot{A}^4-\frac{1111}{18}A^2\dot{A}^2+2K\dot{A}^2+\frac{23}{18}KA^2+K^2\right][K+\dot{A}^2]^2\right)\\
&-\frac{54}{25}A^2\dot{A}\dddot{A}\left(\frac{1}{3}A^5\ddot{A}^5+A^4\ddot{A}^4\left[\dot{A}^2-\frac{229}{18}A^2+K\right]\right.\\
&+\frac{58}{9}A^5\ddot{A}^5\left[K+\frac{233}{58}\dot{A}^2\right]+\frac{4}{3}A^2\ddot{A}^2\\
&\times\left[\dot{A}^4-\frac{859}{24}A^2\dot{A}^2+2K\dot{A}^2+\frac{35}{24}KA^2+K^2\right][K+\dot{A}^2]\\
&+\frac{7}{3}A\ddot{A}\left[\dot{A}^4+\frac{862}{63}A^2\dot{A}^2+2K\dot{A}^2-\frac{227}{63}KA^2+K^2\right][K+\dot{A}^2]^2\\
&\left.+\left[\dot{A}^4-\frac{92}{9}A^2\dot{A}^2+2K\dot{A}^2+\frac{1}{3}KA^2+K^2\right][K+\dot{A}^2]^3\right)\\
&-\frac{9}{25}A^6\ddot{A}^6\left(3\dot{A}^2+\frac{5}{2}A^2+K\right)-\frac{18}{25}A^5\ddot{A}^5\\
&\times\left(-2\dot{A}^4-24A^2\dot{A}^2-K\dot{A}^2+\frac{2}{3}KA^2+K^2\right)+\frac{18}{25}A^4\ddot{A}^4\\
&\times\left(13\dot{A}^6-\frac{153}{2}A^2\dot{A}^4+27K\dot{A}^4-\frac{61}{3}KA^2\dot{A}^2+15K^2\dot{A}^2+\frac{5}{6}K^2A^2+K^3\right)\\
&+\frac{72}{25}A^3\ddot{A}^3\left(\frac{11}{4}\dot{A}^6+\frac{47}{2}A^2\dot{A}^4+\frac{13}{2}K\dot{A}^4-\frac{115}{24}KA^2\dot{A}^2+\frac{19}{4}K^2\dot{A}^2\right.\\
&\left.-\frac{3}{8}K^2A^2+K^3\right)(K+\dot{A}^2)+\frac{63}{25}A^2\ddot{A}^2\left(-\frac{5}{7}\dot{A}^6-\frac{231}{14}A^2\dot{A}^4-\frac{3}{7}K\dot{A}^4\right.\\
&\left.+\frac{122}{21}KA^2\dot{A}^2+\frac{9}{7}K^2\dot{A}^2-\frac{23}{42}K^2A^2+K^3\right)(K+\dot{A}^2)^2+\frac{18}{25}A\ddot{A}\\
&\times\left(-5\dot{A}^6+\frac{45}{3}A^2\dot{A}^4-9K\dot{A}^4-\frac{41}{6}KA^2\dot{A}^2-3K^2\dot{A}^2-\frac{1}{6}K^2A^2+K^3\right)\\
&\left.\times(K+\dot{A}^2)^3-\frac{18}{25}\dot{A}^2\left(\dot{A}^4+\frac{3}{2}A^2\dot{A}^2+2K\dot{A}^2+\frac{1}{6}KA^2+K^2\right)(K+\dot{A}^2)^4\right].
\end{aligned}
\tag{92}
$$

Equation (91) means that diffusion is not the only process which presents in the expanding universe. By making use of identification Eq. (59), we obtain the effective current density of diffusion process J_ν, i.e.

$$J_\nu = \frac{2K\mathcal{M}^4 A\dot{A} + 2\mathcal{M}^4 A\dot{A}^3 - \mathcal{M}^4 A^2\dot{A}\ddot{A} - \mathcal{M}^4 A^3\dddot{A}}{12(K + A\ddot{A} + \dot{A}^2)^2}, \tag{93}$$

and from Eq. (60), then we obtain the expression of current density of the friction, namely,

$$\begin{aligned} I_\nu = &-\frac{\mathcal{M}^4}{18A(K + A\ddot{A} + \dot{A}^2)^5}\left[A^6 \overset{\cdot\cdot\cdot\cdot\cdot}{A}(K + A\ddot{A} + \dot{A}^2)^2 - 81A^6\dot{A}\dddot{A}^2\left(K - \frac{1}{3}A\ddot{A} + \dot{A}^2\right)\right.\\ &+ 20A^5\ddddot{A}\left(-\frac{9}{20}A^2\dddot{A} + K\dot{A} - \frac{7}{20}A\dot{A}\ddot{A} + \dot{A}^3\right)(K + A\ddot{A} + \dot{A}^2)\\ &- 9A^2\dddot{A}\left(A^3\ddot{A}^3\left[\dot{A}^2 + \frac{7}{9}A^2 + K\right] + 3A^2\ddot{A}^2\right.\\ &\times\left[\dot{A}^4 - \frac{74}{27}A^2\dot{A}^2 + 2K\dot{A}^2 - \frac{11}{27}KA^2 + K^2\right] + 3A\ddot{A}[K + \dot{A}^2]\\ &\times\left[\dot{A}^4 + \frac{128}{27}A^2\dot{A}^2 + 2K\dot{A}^2 - \frac{16}{27}KA^2 + K^2\right]\\ &\left.+\left[\dot{A}^4 - \frac{115}{9}A^2\dot{A}^2 + 2K\dot{A}^2 + \frac{2}{9}KA^2 + K^2\right][K + \dot{A}^2]^2\right)\\ &- 9\dot{A}\left(A^4\ddot{A}^4\left[\dot{A}^2 + \frac{8}{3}A^2 + K\right] + A^3\ddot{A}^3\left[\dot{A}^4 - \frac{25}{3}A^2\dot{A}^2 + 2K\dot{A}^2 - KA^2 + K^2\right]\right.\\ &- 3A^2\ddot{A}^2[K + \dot{A}^2]\left[\dot{A}^4 - \frac{14}{3}A^2\dot{A}^2 + 2K\dot{A}^2 + \frac{1}{3}KA^2 + K^2\right]\\ &- 5A\ddot{A}[K + \dot{A}^2]^2\left[\dot{A}^4 + \frac{5}{3}A^2\dot{A}^2 + 2K\dot{A}^2 - \frac{8}{15}KA^2 + K^2\right]\\ &\left.\left.- 2[K + \dot{A}^2]^3\left[\dot{A}^4 - \frac{4}{3}A^2\dot{A}^2 + 2K\dot{A}^2 + K^2\right]\right)\right]. \end{aligned} \tag{94}$$

Furthermore, the divergence of J_ν is given by

$$\begin{aligned} \nabla^\nu J_\nu = \frac{\mathcal{M}^4}{6(K + A\ddot{A} + \dot{A}^2)^3}&\left[-\frac{1}{2}A^3\ddddot{A}(K + A\ddot{A} + \dot{A}^2) + A^4\dddot{A}^2\right.\\ &- 4A^2\dot{A}\dddot{A}\left(K - \frac{1}{2}A\ddot{A} + \dot{A}^2\right)^2 + \frac{1}{2}KA^2\ddot{A}^2 + \frac{9}{2}A^2\dot{A}^2\ddot{A}^2\\ &\left.- \frac{1}{2}A^3\ddot{A}^3 + A\ddot{A}(K - 3\dot{A}^2)(K + \dot{A}^2) + \dot{A}^2(K + \dot{A}^2)^2\right]. \end{aligned} \tag{95}$$

Equation (95) means that diffusion and friction are present in the expanding universe and there is interplay between diffusion and friction.

4.3. *Special case III*: $f(R) = R - \mathcal{E} r_c \ln(1 + \frac{R}{r_c})$

Finally, we consider the form $f(R) = R - \mathcal{E} r_c \ln(1 + \frac{R}{r_c})$ to describe diffusion and friction processes as well as the interplay between diffusion and friction in the expanding universe. At first, we derive the expression of current density to describe all the processes. By substituting both values of R and $R_{\mu\nu}$, respectively, in Eq. (66), we obtain the effective energy–momentum tensor or the curvature fluid tensor, $T^{\text{eff}}_{\mu\nu}$, i.e.

$$T^{\text{eff}}_{tt} = \frac{\mathcal{E} r_c}{2(A\ddot{A} + \dot{A}^2 + \frac{r_c}{6}A^2 + K)^2}\Bigg[- A^2\dot{A}\dddot{A} + A^2\ddot{A} + \frac{1}{6}r_c A^3\ddot{A} + KA\ddot{A} + 2K\dot{A}^2 + 2\dot{A}^4 + \left(A\ddot{A} + \dot{A}^2 + \frac{r_c}{6}A^2 + K\right)^2 \ln\left(\frac{r_c A^2 + 6K + 6A\ddot{A} + 6\dot{A}^2}{r_c A^2}\right)\Bigg], \tag{96}$$

$$\begin{aligned} T^{\text{eff}}_{rr} = {}& \frac{\mathcal{E} r_c A^2}{3(Kr^2 - 1)(A\ddot{A} + \dot{A}^2 + \frac{r_c}{6}A^2 + K)^3} \\ & \times \Bigg[\frac{3}{2}\left(A\ddot{A} + \dot{A}^2 + \frac{r_c}{6}A^2 + K\right)^3 \ln\left(\frac{r_c A^2 + 6K + 6A\ddot{A} + 6\dot{A}^2}{r_c A^2}\right) \\ & - \frac{1}{2}A^3\ddddot{A}\left(A\ddot{A} + \dot{A}^2 + \frac{r_c}{6}A^2 + K\right) + A^4\dddot{A}^2 \\ & - 5A^2\dot{A}\dddot{A}\left(-\frac{1}{5}A\ddot{A} + \dot{A}^2 + \frac{r_c}{30}A^2 + K\right) + \frac{5}{2}A^2\ddot{A}^2\left(\frac{11}{5}\dot{A}^2 + \frac{r_c}{30}A^2 + K\right) \\ & + \frac{1}{2}A\ddot{A}(\dot{A}^4 + 2r_c A^2\dot{A}^2 + 8K\dot{A}^2) + \frac{1}{2}A\ddot{A}\left(\frac{1}{6}r_c A^2 + K\right)\left(\frac{1}{6}r_c A^2 + 7K\right) \\ & + (K + \dot{A}^2)\left(4\dot{A}^4 + \frac{1}{6}r_c A^2\dot{A}^2 + 5K\dot{A}^2 + \left[\frac{1}{6}r_c A^2 + K\right]^2\right)\Bigg], \end{aligned} \tag{97}$$

$$\begin{aligned} T^{\text{eff}}_{\theta\theta} = {}& -\frac{\mathcal{E} r_c r^2 A^2}{3(A\ddot{A} + \dot{A}^2 + \frac{r_c}{6}A^2 + K)^3}\Bigg[- \frac{1}{2}A^3\ddddot{A}\left(A\ddot{A} + \dot{A}^2 + \frac{r_c}{6}A^2 + K\right) + A^4\dddot{A}^2 \\ & + \frac{3}{2}\left(A\ddot{A} + \dot{A}^2 + \frac{r_c}{6}A^2 + K\right)^3 \ln\left(\frac{r_c A^2 + 6K + 6A\ddot{A} + 6\dot{A}^2}{r_c A^2}\right) \\ & - 5A^2\dot{A}\dddot{A}\left(-\frac{1}{5}A\ddot{A} + \dot{A}^2 + \frac{r_c}{30}A^2 + K\right) + \frac{5}{2}A^2\ddot{A}^2\left(\frac{11}{5}\dot{A}^2 + \frac{r_c}{30}A^2 + K\right) \\ & + \frac{1}{2}A\ddot{A}(\dot{A}^4 + 2r_c A^2\dot{A}^2 + 8K\dot{A}^2) + \frac{1}{2}A\ddot{A}\left(\frac{1}{6}r_c A^2 + K\right)\left(\frac{1}{6}r_c A^2 + 7K\right) \\ & + (K + \dot{A}^2)\left(4\dot{A}^4 + \frac{1}{6}r_c A^2\dot{A}^2 + 5K\dot{A}^2 + \left[\frac{1}{6}r_c A^2 + K\right]^2\right)\Bigg], \end{aligned} \tag{98}$$

$$
\begin{aligned}
T^{\text{eff}}_{\phi\phi} = &-\frac{\mathcal{E} r_c r^2 \sin^2\theta A^2}{2(A\ddot{A}+\dot{A}^2+\frac{r_c}{6}A^2+K)^3}\\
&\times\Bigg[\left(A\ddot{A}+\dot{A}^2+\frac{r_c}{6}A^2+K\right)^3 \ln\left(\frac{r_cA^2+6K+6A\ddot{A}+6\dot{A}^2}{r_cA^2}\right)\\
&-\frac{1}{3}A^3\ddddot{A}\left(A\ddot{A}+\dot{A}^2+\frac{r_c}{6}A^2+K\right)+\frac{2}{3}A^4\dddot{A}^2\\
&-\frac{1}{3}A^2\dot{A}\dddot{A}\left(-2A\ddot{A}+10\dot{A}^2+\frac{r_c}{3}A^2+10K\right)+\frac{1}{3}A^2\ddot{A}^2\left(11\dot{A}^2+\frac{r_c}{6}A^2+5K\right)\\
&+\frac{1}{3}A\ddot{A}(\dot{A}^4+2r_cA^2\dot{A}^2+8K\dot{A}^2)+\frac{1}{3}A\ddot{A}\left(\frac{1}{6}r_cA^2+K\right)\left(\frac{1}{6}r_cA^2+7K\right)\\
&+\frac{1}{3}(K+\dot{A}^2)\left(8\dot{A}^4+\frac{1}{3}r_cA^2\dot{A}^2+10K\dot{A}^2+2\left[\frac{1}{6}r_cA^2+K\right]^2\right)\Bigg]. \qquad (99)
\end{aligned}
$$

Based on Eq. (67), we get the expression of current density K_ν that describes the diffusion, friction, and interplay between both process as follows:

$$
\begin{aligned}
K_t = &\frac{3024\mathcal{E}r_c}{A(r_cA^2+6K+6A\ddot{A}+6\dot{A}^2)^4}\Bigg[\frac{1}{14}A^4\overset{\cdot\cdot\cdot\cdot\cdot}{A}\left(A\ddot{A}+\dot{A}^2+\frac{r_c}{6}A^2+K\right)\\
&-A^3\ddddot{A}\left(\frac{3}{7}A^2\dddot{A}+\frac{2}{7}A\dot{A}\ddot{A}-\dot{A}^3-\frac{r_c}{42}A^2\dot{A}-K\dot{A}\right)\left(A\ddot{A}+\dot{A}^2+\frac{r_c}{6}A^2+K\right)\\
&+\frac{3}{7}A^6\dddot{A}^3-3A^4\dot{A}\dddot{A}^2\left(-\frac{2}{7}A\ddot{A}+\dot{A}^2+\frac{r_c}{42}A^2+K\right)\\
&+\frac{2}{7}A^2\dddot{A}\Bigg(-\frac{1}{4}A^3\ddot{A}^3+\frac{53}{4}A^2\dot{A}^2\ddot{A}^2+\frac{5r_c}{12}A^4\ddot{A}^2+5KA^2\ddot{A}^2+\frac{55}{4}\dot{A}^6\\
&+\frac{1}{4}A\ddot{A}\left[-53\dot{A}^4+\frac{56r_c}{7}A^2\dot{A}^2-28K\dot{A}^2+\left(\frac{17r_c}{6}A^2+25K\right)\left(\frac{r_c}{6}A^2+K\right)\right]\\
&+\frac{63}{4}\dot{A}^2\left[\frac{r_c}{324}A^4-\frac{1}{189}Kr_cA^2+K^2\right]+\left[\frac{1}{4}r_cA^2+K\right]\left[\frac{1}{6}r_cA^2+K\right]^2\\
&-\frac{2r_c}{3}A^2\dot{A}^4+\frac{57K}{2}\dot{A}^4\Bigg)+\frac{6}{7}\dot{A}\Bigg(\frac{1}{2}A^3\ddot{A}^3\left[\frac{17}{2}\dot{A}^2-\frac{2}{3}r_cA^2+K\right]\\
&-\frac{3}{2}A^4\ddot{A}^4-\frac{1}{1296}r_cA^6+\frac{1}{3}A^2\ddot{A}^2[-18\dot{A}^4+3r_cA^2\dot{A}^2-18K\dot{A}^2+Kr_cA^2]\\
&+3A\ddot{A}\Bigg[\frac{1}{4}\dot{A}^6-\frac{1}{2}r_cA^2\dot{A}^4-\frac{1}{2}K\dot{A}^4+\frac{7}{4}\dot{A}^2\left(\frac{1}{252}r_c^2A^4-\frac{25}{63}Kr_cA^2-K^2\right)
\end{aligned}
$$

$$- \frac{7}{36}K^2 r_c A^2 - K^3\Big] - [K + \dot{A}^2]\Big[\dot{A}^6 - \frac{1}{6}r_c A^2 \dot{A}^4 + 3K\dot{A}^4 + 3\dot{A}^2$$

$$\times \left(\frac{1}{36}r_c^2 A^4 + \frac{1}{9}K r_c A^2 + K^2\right) + \frac{1}{6}r_c A^2 + K\Big]\Big)\Big],$$

$$K_r = K_\theta = K_\phi = 0. \tag{100}$$

By tedious calculation, we obtain that $\nabla^\nu K_\nu \neq 0$. It means, diffusion is not the only process which presents in the expanding universe. It is also simple to check the divergence $\nabla^\nu J_\nu$ and $\nabla^\nu I_\nu$ of diffusion and friction current obtained from the identification. Equations (70) and (71) do not vanish. It means that the diffusion process and friction is not independent in this case. There is interplay between diffusion process and friction.

5. Conclusion

The general formulation of the total effective current density that describes diffusion, friction, and interplay between diffusion and friction for any $f(R)$ is shown by Eq. (31). From the identification, we obtain the equation of the effective current density related to diffusion process only and the equation of the effective current density related to friction is expressed by Eq. (39). To obtain a more phenomenological picture related to the diffusion and friction processes and interactions among them, we have studied several functional forms of $f(R)$ i.e. $f(R) = R + \Lambda R^2$, $f(R) = R - \frac{\mathcal{M}^4}{R}$, and $f(R) = R - \mathcal{E} r_c \ln(1 + \frac{R}{r_c})$. From these three functions, then we get the conclusion that Eq. (31) doesn't describe the diffusion process only, but also involves other processes such as friction. Moreover, diffusion and friction are not independent. It means that there is an interplay between diffusion and friction. Furthermore, the total effective current equation and the identified equation (diffusion and friction) obtained from the considered $f(R)$-function are applied to explain diffusion and friction processes in the expanding universe with the FLRW metric. In general, the consideration of these three functions in the expanding universe also provides the same information as previously disclosed that diffusion and friction present in the expanding universe and there is interplay between diffusion and friction. In addition, the total effective current equation obtained from the study of expanding universe (FLRW model) shows that the effective current density depends only on time, whereas the other current density components are zero. It means that the processes in the expanding universe propagate to the time coordinate only.

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