REMARKS ON MOMENT PROPERTIES OF GENERALIZED DISTRIBUTIONS

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Abstract. Because of their flexibility, several authors propose to use generalized distributions. In particular, in fitting the size distribution of income the generalized beta of the second kind (GB2) is outlined by McDonald (1987). In modeling data of lung and other cancers, the generalized log-logistic (GLL) distribution is suggested by Singh (1989). In fitting air pollutant concentrations, the generalized gamma (GG) model is proposed by Marani, Lavagni, and Buttazoni (1986). Since in those papers there is no discussion about moment properties of those particular distributions, this paper derives moment generating functions (MGFs) of GLL, GB2, and GG distributions. Moreover, based on their moment generating functions this paper proves that relation among generalized distributions In deriving MGFs of generalized distributions and their relation, this paper utilizes MacLaurin series and Stirling's approximation formula.

Keywords: Generalized beta of the second kind, Generalized log-logistic distribution, Generalized gamma distribution, Moment generating function, MacLaurin series, and Stirling approximation.

1. Introduction

Because of their flexibility, several authors propose to use generalized distributions. In particular, in fitting the size distribution of income the generalized beta of the second kind (GB2) is outlined by McDonald (1987). In modeling data of lung and other cancers, the generalized log-logistic (GLL) distribution is suggested by Singh (1989). In fitting air pollutant concentrations, the generalized gamma (GG) model is proposed by Marani, Lavagni, and Buttazoni (1986). Based on the probability density function, McDonald (1987) states that the GG distribution is a special case of the GB2 distribution. Moreover, the GLL distribution has relation to the GB2 distribution and includes the generalized gamma (GG) distribution (Warsono, Usman, and Nusyirwan, 2000). However, in all of these papers there is no discussion about relationship of generalized distributions in term of moment properties.

Warsono (2009) derives moments of the GG and GB2 distributions. Based on a moment property of the GB2 distribution, he proves that there is relationship between the GG distribution and the GB2 distribution. This paper then extends the results obtained by Warsono (2009). In this paper the relationship of the generalized distributions is outlined based on a moment generating function (MGF) of the GLL distribution. The main objective of this paper is to provide a discussion of moment properties of the generalized distributions. In order to achieve this purpose, in the Section 2, this paper states and proves a moment generating function of the GLL distribution. In the section 3, this paper provides reparameterization of the moment of the GLL to the moment of the GB2 distribution. Section 4 contains a description of limiting behavior of the GLL's moment as a general case of the GG's moment. Finally, this paper provides conclusion in the Section 5.

2. The Moment of the GLL Distribution

Let a random variable X have four-parameter generalized log-logistic distribution with shape parameters m_1 and m_2, denoted by $X \sim \mathrm{GLL}(\alpha,\beta,m_1,m_2)$ or $\mathrm{GLL}(m_1,m_2)$, and the corresponding probability density function (PDF) of the $\mathrm{GLL}(\alpha,\beta,m_1,m_2)$ can be written in the form (Singh, Warsono, Bartolucci (1997):

$$g(x) = \frac{\alpha}{xB(m_1,m_2)}[F(x)]^{m_1}[1-F(x)]^{m_2}$$

where $B(m_1,m_2)$ represents the complete beta function, and

$$F(x) = [1+e^{-[\beta+\alpha\ln(x)]}]^{-1}$$

is the log-logistic distribution function. In this section the MGF of the GLL distribution is stated and proved.

Theorem 2.1 Let X be a random variable of the $\mathrm{GLL}(\alpha,\beta,m_1,m_2)$ distribution, then moment generating function (MGF) of X is given by

$$M_x(t) = \sum_{n=0}^{\infty}\frac{\left(te^{-\left(\beta/\alpha\right)}\right)^n}{n!}\cdot\frac{\Gamma\left(m_1+\frac{n}{\alpha}\right)\cdot\Gamma\left(m_2-\frac{n}{\alpha}\right)}{\Gamma(m_1)\Gamma(m_2)}$$

Proof:

$$M_X(t) = E(e^{tX}) = \int_{-\infty}^{\infty} e^{tx}\cdot g(x)\,dx$$

$$= \int_0^{\infty} e^{tx}\left(\frac{\alpha}{xB(m_1,m_2)}\right)[F(x)]^{m1}[1-F(x)]^{m2}\,dx$$

$$= \frac{\alpha}{B(m_1,m_2)}\int_0^{\infty}\frac{e^{tx}}{x}\left[\frac{1}{1+e^{-(\beta+\alpha\ln(x))}}\right]^{m1}\left[1-\frac{1}{1+e^{-(\beta+\alpha\ln(x))}}\right]^{m2}dx$$

$$= \frac{\alpha}{B(m_1,m_2)}\int_0^{\infty}\frac{e^{tx}}{x}\left[\frac{1}{1+\frac{1}{e^{(\beta+\varepsilon\ln(x))}}}\right]^{m_1}\left[1-\frac{1}{1+\frac{1}{e^{(\beta+\alpha\ln(x))}}}\right]^{m_2}dx$$

By algebra manipulation we may find the following equation

$$= \frac{\alpha}{B(m_1, m_2)} \int_0^\infty \left(\frac{e^{tx}}{x} \right) \frac{\left[e^{\beta+\alpha \ln(x)} \right]^{m_1}}{\left[1 + e^{\beta+\alpha \ln(x)} \right]^{m_1+m_2}} \, dx \qquad \text{.............. (1)}$$

By letting $y = e^{\beta+\alpha \ln(x)}$ we may rerwite the equation (1) in the following form

$$M_X(t) = \frac{\alpha}{B(m_1, m_2)} \int_0^\infty \left(\frac{e^{tx}}{\alpha\, y} \right) \frac{y^{m_1}}{(1+y)^{m_1+m_2}} \, dy$$

$$= \frac{1}{B(m_1, m_2)} \int_0^\infty e^{tx} \frac{y^{m_1-1}}{(1+y)^{m_1+m_2}} \, dy \qquad \text{..........(2)}$$

Making use of a well-known property of MacLaurin series of the e^{tx} function, then equation (2) can be written as

$$M_X(t) = \frac{1}{B(m_1, m_2)} \int_0^\infty \left(\sum_{n=0}^\infty \frac{(tx)^n}{n!} \right) \frac{y^{m_1}}{(1+y)^{m_1+m_2}} \, dy$$

$$= \frac{1}{B(m_1, m_2)} \int_0^\infty \left(1 + tx + \frac{(tx)^2}{2!} + \frac{(tx)^3}{3!} + \dots \right) \frac{y^{m_1}}{(1+y)^{m_1+m_2}} \, dy$$

$$= \frac{1}{B(m_1, m_2)} \int_0^\infty \left(\frac{y^{m_1}}{(1+y)^{m_1+m_2}} + tx \frac{y^{m_1}}{(1+y)^{m_1+m_2}} + \frac{(tx)^2}{2!} \frac{y^{m_1}}{(1+y)^{m_1+m_2}} + \frac{(tx)^3}{3!} \frac{y^{m_1}}{(1+y)^{m_1+m_2}} + \dots \right) dy$$

$$= \frac{1}{B(m_1, m_2)} \left[\int_0^\infty \frac{y^{m_1-1}}{(1+y)^{m_1+m_2}} \, dy + \int_0^\infty tx \frac{y^{m_1-1}}{(1+y)^{m_1+m_2}} \, dy + \int_0^\infty \frac{(tx)^2}{2!} \frac{y^{m_1-1}}{(1+y)^{m_1+m_2}} \, dy + \right.$$

$$\left. \int_0^\infty \frac{(tx)^3}{3!} \frac{y^{m_1-1}}{(1+y)^{m_1+m_2}} \, dy + \dots \right]$$

$$= \frac{1}{B(m_1, m_2)} \left[\int_0^\infty \frac{y^{m_1-1}}{(1+y)^{m_1+m_2}} \, dy + t e^{-\left(\beta/\alpha\right)} \int_0^\infty y^{\frac{1}{\alpha}} \cdot \frac{y^{m_1-1}}{(1+y)^{m_1+m_2}} \, dy + \right.$$

$$\left.\frac{\left(te^{-\left(\beta/\alpha\right)}\right)^2}{2!}\int_0^\infty y^{\frac{2}{\alpha}}\cdot\frac{y^{m_1-1}}{(1+y)^{m_1+m_2}}\,dy+\frac{\left(te^{-\left(\beta/\alpha\right)}\right)^3}{3!}\int_0^\infty y^{\frac{3}{\alpha}}\cdot\frac{y^{m_1-1}}{(1+y)^{m_1+m_2}}\,dy+\ldots\ldots\right]$$

$$=\frac{1}{B(m_1,m_2)}\left[\int_0^\infty\frac{y^{m_1-1}}{(1+y)^{m_1+m_2}}\,dy+te^{-\left(\beta/\alpha\right)}\int_0^\infty\frac{y^{m_1+\frac{1}{\alpha}-1}}{(1+y)^{m_1+\frac{1}{\alpha}+m_2-\frac{1}{\alpha}}}\,dy+\right.$$

$$\left.\frac{\left(te^{-\left(\beta/\alpha\right)}\right)^2}{2!}\int_0^\infty\frac{y^{m_1+\frac{2}{\alpha}-1}}{(1+y)^{m_1+\frac{2}{\alpha}+m_2-\frac{2}{\alpha}}}\,dy+\frac{\left(te^{-\left(\beta/\alpha\right)}\right)^3}{3!}\int_0^\infty\frac{y^{m_1+\frac{3}{\alpha}-1}}{(1+y)^{m_1+\frac{3}{\alpha}+m_2-\frac{3}{\alpha}}}\,dy+\ldots\ldots\right]$$

$$=\frac{1}{B(m_1,m_2)}\left[B(m_1,m_2)+te^{-\left(\beta/\alpha\right)}\cdot B\left(m_1+\frac{1}{\alpha},m_2-\frac{1}{\alpha}\right)+\right.$$

$$\left.\frac{\left(te^{-\left(\beta/\alpha\right)}\right)^2}{2!}\cdot B\left(m_1+\frac{2}{\alpha},m_2-\frac{2}{\alpha}\right)+\frac{\left(te^{-\left(\beta/\alpha\right)}\right)^3}{3!}\cdot B\left(m_1+\frac{3}{\alpha},m_2-\frac{3}{\alpha}\right)+\ldots\ldots\right]$$

$$=1+te^{-\left(\beta/\alpha\right)}\cdot\frac{B\left(m_1+\frac{1}{\alpha},m_2-\frac{1}{\alpha}\right)}{B(m_1,m_2)}+\frac{\left(te^{-\left(\beta/\alpha\right)}\right)^2}{2!}\cdot\frac{B\left(m_1+\frac{2}{\alpha},m_2-\frac{2}{\alpha}\right)}{B(m_1,m_2)}+$$

$$\frac{\left(te^{-\left(\beta/\alpha\right)}\right)^3}{3!}\cdot\frac{B\left(m_1+\frac{3}{\alpha},m_2-\frac{3}{\alpha}\right)}{B(m_1,m_2)}+\ldots$$

$$= \sum_{n=0}^{\infty} \frac{\left(te^{-\left(\beta/\alpha\right)}\right)^n}{n!} \cdot \frac{B\left(m_1 + \frac{n}{\alpha}, m_2 - \frac{n}{\alpha}\right)}{B(m_1, m_2)}$$

$$= \sum_{n=0}^{\infty} \frac{\left(te^{-\left(\beta/\alpha\right)}\right)^n}{n!} \cdot \frac{\dfrac{\Gamma\left(m_1 + \frac{n}{\alpha}\right)\cdot\Gamma\left(m_2 - \frac{n}{\alpha}\right)}{\Gamma\left(m_1 + \frac{n}{\alpha} + m_2 - \frac{n}{\alpha}\right)}}{\dfrac{\Gamma(m_1)\Gamma(m_2)}{\Gamma(m_1 + m_2)}}$$

Therefore, the MGF of the GLL distribution is

$$M_x(t) = \sum_{n=0}^{\infty} \frac{\left(te^{-\left(\beta/\alpha\right)}\right)^n}{n!} \cdot \frac{\Gamma\left(m_1 + \frac{n}{\alpha}\right)\cdot\Gamma\left(m_2 - \frac{n}{\alpha}\right)}{\Gamma(m_1)\Gamma(m_2)}$$

3. The Moment of the GB2 Distribution

Recently, by using MGF's definition Warsono (2009) has mathemattically derived the MGF of the GB2 distribution. Based on reparameterization of the MGF of the GLL distribution, in this section the author provides the MGF of the GB2 distribution. The reparameterization proposition is stated and proved.

Proposition 3.1 Let X be a random variable having the GLL (α, β, m_1, m_2) moment and

$\alpha = a$ and $\beta = -a\ln(b)$, then GB2(a, b, m_1, m_2) moment.

Proof:

$$M_X\left(GLL(\alpha = a, \beta = -a\ln(b), m_1, m_2)\right) = \sum_{n=0}^{\infty} \frac{\left(te^{-\left(\frac{\beta}{\alpha}\right)}\right)^n}{n!} \frac{\Gamma\left(m_1 + \frac{n}{\alpha}\right)\cdot\Gamma\left(m_2 - \frac{n}{\alpha}\right)}{\Gamma(m_1)\Gamma(m_2)}$$

$$= \sum_{n=0}^{\infty} \frac{\left(te^{-\left(\frac{-a\ln(b)}{a}\right)}\right)^n}{n!} \frac{\Gamma\left(m_1 + \frac{n}{a}\right)\cdot\Gamma\left(m_2 - \frac{n}{a}\right)}{\Gamma(m_1)\Gamma(m_2)}$$

$$= \sum_{n=0}^{\infty} \frac{\left(e^{\ln(b)}\right)^n}{n!} \frac{\Gamma\left(m_1 + \frac{n}{a}\right) \cdot \Gamma\left(m_2 - \frac{n}{a}\right)}{\Gamma(m_1)\Gamma(m_2)}$$

$$= \sum_{n=0}^{\infty} \frac{(b)^n}{n!} \frac{\Gamma\left(m_1 + \frac{n}{a}\right) \cdot \Gamma\left(m_2 - \frac{n}{a}\right)}{\Gamma(m_1)\Gamma(m_2)}$$

This is the moment generating function of the GB2(a, b, m_1, m_2) stated by Warsono (2009).

4. The Moment of the GG as a Limiting Moment of the GLL Distributions

In this section, two propositions of limiting moment properties of the GLL distribution are stated and proved.

Proposition 4.1 The GLL(α,β,m_1,m_2) distribution converges to the GG distribution as m_2 tends to ∞ and $\alpha = a$, and $\beta = -a\ln\left(\gamma (m_2)^{-\frac{1}{a}}\right)$.

Proof:

$$\lim_{m_2\to\infty} M_X(t)_{GLL\left(\alpha = a, \beta = -a\ln\left(\gamma (m_2)^{-\frac{1}{a}}\right), m_1, m_2\right)}$$

$$= \lim_{m_2\to\infty} \sum_{n=0}^{\infty} \frac{\left(te^{-\frac{\beta}{\alpha}}\right)^n}{n!} \cdot \frac{\Gamma\left(m_1 + \frac{n}{\alpha}\right) \cdot \Gamma\left(m_2 - \frac{n}{\alpha}\right)}{\Gamma(m_1)\Gamma(m_2)} = \lim_{m_2\to\infty} \sum_{n=0}^{\infty} \frac{\left(te^{-\frac{\left(-a\ln\left(\gamma (m_2)^{-\frac{1}{a}}\right)\right)}{a}}\right)^n}{n!} \cdot \frac{\Gamma\left(m_1 + \frac{n}{a}\right) \cdot \Gamma\left(m_2 - \frac{n}{a}\right)}{\Gamma(m_1)\Gamma(m_2)}$$

$$= \lim_{m_2\to\infty} \sum_{n=0}^{\infty} \frac{\left(te^{\ln\left(\gamma m_2^{-1/a}\right)}\right)^n}{n!} \cdot \frac{\Gamma\left(m_1 + \frac{n}{a}\right) \cdot \Gamma\left(m_2 - \frac{n}{a}\right)}{\Gamma(m_1)\Gamma(m_2)} = \lim_{m_2\to\infty} \sum_{n=0}^{\infty} \frac{\left(t\gamma (m_2)^{-\frac{1}{a}}\right)^n}{n!} \cdot \frac{\Gamma\left(m_1 + \frac{n}{a}\right) \cdot \Gamma\left(m_2 - \frac{n}{a}\right)}{\Gamma(m_1)\Gamma(m_2)}$$

$$= \lim_{m_2\to\infty} 1 + \lim_{m_2\to\infty} \left(\gamma (m_2)^{-\frac{1}{a}}\right) \cdot \frac{\Gamma\left(m_1 + \frac{1}{a}\right) \cdot \Gamma\left(m_2 - \frac{1}{a}\right)}{\Gamma(m_1)\Gamma(m_2)} + \lim_{m_2\to\infty} \frac{\left(\gamma (m_2)^{-\frac{1}{a}}\right)^2}{2!} \cdot \frac{\Gamma\left(m_1 + \frac{2}{a}\right) \cdot \Gamma\left(m_2 - \frac{2}{a}\right)}{\Gamma(m_1)\Gamma(m_2)} +$$

$$\lim_{m_2\to\infty} \frac{\left(\gamma (m_2)^{-\frac{1}{a}}\right)^3}{3!} \cdot \frac{\Gamma\left(m_1 + \frac{3}{a}\right) \cdot \Gamma\left(m_2 - \frac{3}{a}\right)}{\Gamma(m_1)\Gamma(m_2)} + \ldots$$

$$
=\lim_{m_2\to\infty} 1+\left(\gamma\right)\cdot\frac{\Gamma\left(m_1+\frac{1}{a}\right)}{\Gamma\left(m_1\right)}\lim_{m_2\to\infty}\left(m_2\right)^{\frac{1}{a}}\cdot\frac{\Gamma\left(m_2-\frac{1}{a}\right)}{\Gamma\left(m_2\right)}+
$$

$$
\frac{\left(\gamma\right)^{2}}{2!}\cdot\frac{\Gamma\left(m_1+\frac{2}{a}\right)}{\Gamma\left(m_1\right)}\lim_{m_2\to\infty}\left(m_2\right)^{\frac{2}{a}}\cdot\frac{\Gamma\left(m_2-\frac{2}{a}\right)}{\Gamma\left(m_2\right)}+
$$

$$
\frac{\left(\gamma\right)^{3}}{3!}\cdot\frac{\Gamma\left(m_1+\frac{3}{a}\right)}{\Gamma\left(m_1\right)}\lim_{m_2\to\infty}\left(m_2\right)^{\frac{3}{a}}\cdot\frac{\Gamma\left(m_2-\frac{3}{a}\right)}{\Gamma\left(m_2\right)}+\ldots
$$

The Stirling's approximation formula of the gamma function (Spiegel, 1968) is

$$
\Gamma\left(az+b\right)\sim\sqrt{2\pi}\cdot e^{-az}\left(az\right)^{az+b-\frac{1}{2}}
$$

$$
\frac{\Gamma\left(m_2-\frac{1}{a}\right)}{\Gamma\left(m_2\right)}\sim\frac{\sqrt{2\pi}\cdot e^{-m_2}\left(m_2\right)^{m_2-\frac{1}{a}-\frac{1}{2}}}{\sqrt{2\pi}\cdot e^{-m_2}\left(m_2\right)^{m_2-\frac{1}{2}}}=\left(m_2\right)^{-\frac{1}{a}}=\frac{1}{\left(m_2\right)^{\frac{1}{a}}}
$$

and others.

Then the limiting moment property of the GLL(α,β,m_1,m_2) distribution can be written as:

$$
\lim_{m_2\to\infty} M_X\left(\right)GLL\left(\alpha=a,\beta=-a\ln\left(\gamma\left(m_2\right)^{\frac{1}{a}}\right),m_1,m_2\right)
$$

$$
=\lim_{m_2\to\infty} 1+\left(\gamma\right)\cdot\frac{\Gamma\left(m_1+\frac{1}{a}\right)}{\Gamma\left(m_1\right)}\lim_{m_2\to\infty}\left(m_2\right)^{\frac{1}{a}}\cdot\frac{1}{\left(m_2\right)^{\frac{1}{a}}}+\frac{\left(\gamma\right)^{2}}{2!}\cdot\frac{\Gamma\left(m_1+\frac{2}{a}\right)}{\Gamma\left(m_1\right)}\lim_{m_2\to\infty}\left(m_2\right)^{\frac{2}{a}}\cdot\frac{1}{\left(m_2\right)^{\frac{2}{a}}}+\ldots
$$

$$= 1 + (\gamma)\frac{\Gamma\left(m_1 + \frac{1}{a}\right)}{\Gamma(m_1)} + \frac{(\gamma)^2}{2!}\cdot\frac{\Gamma\left(m_1 + \frac{2}{a}\right)}{\Gamma(m_1)} + \cdots = \sum_{n=0}^{\infty}\frac{(\gamma)^n}{n!}\cdot\frac{\Gamma\left(m_1 + \frac{n}{a}\right)}{\Gamma(m_1)}$$

This is the moment generating function of the GG stated by Warsono (2009). The GLL distribution converges to the GG distribution as m_2 tends to ∞ and $\alpha = a$, and $\beta = -a\ln\left(\gamma (m_2)^{-\frac{1}{a}}\right)$.

Proposition 4.2 The GLL(α,β,m_1,m_2) distribution converges to the gamma distribution as m_2 tends to ∞ and $\alpha = 1$ and $\beta = -\ln(m_2)$.

Proof: Letting $\alpha = 1$ and using similar way to the proof of Proposition 4.1, the Proposition 4.2 can easily be proven.

Therefore, another moment limiting property of the GLL distribution is similar to the form of the gamma moment stated by Casella and Berger (1990).

5. Conclusion

The moment of the generalized beta of the second kind distribution is reparameterization of the generalized log-logistic (GLL) The moment of the generalized gamma (GG) distribution is the limiting moment of (GLL) distribution. Moreover, since the moments of the gamma and exponential distributions are special cases of the moment of the generalized gamma distribution (Warsono, 2009), the moments of both special distributions are also special cases of the moments of the moment of the GLL distribution as stated in the Proposition 4.2.

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