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Determining the Number of Connected Vertices Labelled Graph of Order Five with Maximum Number of Parallel Edges is Five and Containing No Loops

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Abstract. A connected graph is a graph where there exists at least a path joining every pair of the vertices in the graph, and a graph is called simple if that graph containing no loops nor parallel edges. Given a graph G (V, E) with n vertices and m edges, there are a lot of graph that can be formed, either connected or disconnected, or simple or not simple. In this paper we will discuss the number of connected vertex labelled graph of order five (n=5) and $4 \leq m \leq 10$, with maximum number of parallel edges that connecting different pairs of vertices is five (the parallel edges that connecting the same pair of vertices are counted as one).

1. Introduction

Emerging as a new concept in mathematics after the representation and solution given by Euler in 1736 about the Konigsberg's problem, nowadays graph theory plays an important role especially in representing many real-life problems into the concept of graph. The two terminologies in graph that usually used to represent the real-life problem are vertices and edges. The vertices can be used to represent cities, stations, depots, warehouses, airports, and so on; while the edges can be used to represent roads, train tracks, airline routes, and so on. The nonstructural information that usually assigned to the edges can represent weight/capacity/cost/distance/time, etc. To draw a graph, there is no single correct way [1], and therefore that rule makes it flexible to represent any real-world problem. Because of its flexibility to accommodate diverse applications in daily-life, graph theory grows as one important area in mathematics. A comprehensive applications of graph theory in diverse areas in operations research and optimization including internet congestion control, data structures, algorithms, scheduling and resource allocation, and some combinatorial problems are exposed and discussed in [2]. Some application related with interconnection networks were given comprehensively in [3].

Graph enumeration problem was led by Cayley who interested in counting the isomer of hydrocarbon and found that the problem is related or similar with counting tree problem [4]. Some methods to enumerate and label graphs were given in [5-7]. The formula for counting simple graphs were given in [4,8], and the formula for counting simple labelled graph were given as $\binom{\frac{n(n-1)}{2}}{e}$, where n is the graph order and e is the number of edges [4], and of course, there are possibilities that the graphs obtained were disconnected or connected, and if the graph were connected then it must be tree, or if disconnected then it must contain circuit, because in that formula e = n-1.

Given graph G(V,E) of order n and the number of edges m, then there are many possible graphs can be constructed. That is possible that those graphs are simple which means those graphs not containing any loops nor parallel edges; or it also possible that those graphs are connected, or disconnected. For graph with order maximal four, the number of disconnected vertices labelled graphs is investigated [9]. For graph with order five, the number of disconnected vertices labelled graphs without parallel edges is investigated in [10]. This paper is organized as follow: Section 1 is Introduction. Graph Construction, Observation, and Patterns Obtained will be given in Section 2. In Section 3 will be given The Results and Discussion, followed by Conclusion in Section 4.

2. Graph Construction, Observation and Patterns Obtained

Given a graph G(V,E) with $|V| = n = 5$, and Let : $|E| = m$, $4 \leq m \leq 10$. Let $g = p_0$ = the number of non parallel edges, p_i = i-parallel edges ; $i \geq 2$, and j_i = the number of i-parallel edges, then $m = p_0 + \sum_{i=2} j_i \cdot p_i$

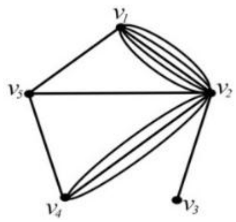


Figure 1. Example of a graph with 3-parallel and 5-parallel edges.

From Figure 1 n=5 ;m=12, $p_0 = g = 4$, $p_3=1$, $p_5=1$, $j_3=1$, $j_5=1$

$m = p_0 + \sum_{i=2} j_i \cdot p_i$

m= p_0 + j3. $p_3+j_5.\ p_5+p_0$

m= 4 + (1×3)+(1×5)=12

Table 1. The number of graphs for m = 4, g = 4

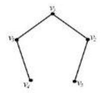
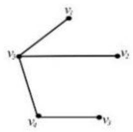
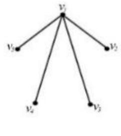
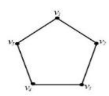
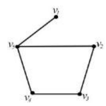
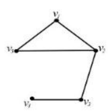
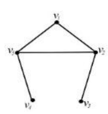
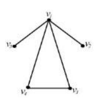
Pattern	The number of graphs
	60
	60
	5
Total	125

Table 2. The number of graphs for m = 5, g = 5

Pattern	The number of graphs
	12
	60
	60
	60
	30
Total	222

The following tables 1 and table 2 show some graph patterns and the number of graphs obtained. However, due to the space limitation, not all patterns will be given.

There are many more connected graphs constructed, however, because of the space limitation, the patterns are not exposed here. The following table shows the results:

Table 3. The number of graphs for n=5, 4 ≤ m ≤ 10

No	The number of i-parallel edges	The numbers of connected graphs						
		m						
		4	5	6	7	8	9	10
1	p_0	125	222	205	110	45	10	1
2	p_2=1		500	1110	1230	770	360	90
3	p_2=2			750	2220	3075	2310	1260
4	p_3=1			500	1110	1230	770	360
5	p_2=3				500	2220	4100	3850
6	p_2=1, p_3=1				1500	4440	6150	4620
7	p_4=1				500	1110	1230	770
8	p_2=4					125	1110	3075
9	p_3=2					750	2220	3075
10	p_2=2, p_3=1					1500	6660	12300
11	p_2=1, p_4=1					1500	4440	6150
12	p_5=1					500	1110	1230
13	p_1=1, p_3=2						1500	6660
14	p_2=1, p_5=1						1500	4440
15	p_2=2, p_4=1						1500	6660
16	p_2=3, p_3=1						500	4440
17	p_3=1, p_4=1						1500	4440
18	p_6=1						500	1110
19	p_2=5							222
20	p_2=1, p_3=1, p_4=1							3000
21	p_2=2, p_3=2							750
22	p_2=2, p_5=1							1500
23	p_2=3, p_4=1							500
24	p_1=1, p_5=1							1500
25	p_3=3							500
26	p_4=2							750
27	p_6=1, p_2=1							1500
28	p_7=1							500

After grouping the graphs on Table 3 using g, the number of non parallel edges, and m, the number of edges in the graphs, the number of graphs on Table 3 above can also be represented as follow:

Table 4. The number of graphs for n=5, $4 \leq m \leq 10$, and $4 \leq g \leq 10$

No.	*m*	The number of graphs						
		g						
		4	5	6	7	8	9	10
1	4	125						
2	5	500	222					
3	6	1250	1110	205				
4	7	2500	3330	1230	110			
5	8	4375	7770	4305	770	45		
6	9	7000	15540	11480	3080	360	10	
7	10	10500	27972	25830	9240	1620	90	1

By looking at the value of every column in Table 4, we can derive Table 5 as an alternative for presenting Table 4 as follows:

Table 5. The alternative representation of Table 4

No	m	The number of connected graphs						
		g						
		4	5	6	7	8	9	10
1	4	**1**x125						
2	5	**4**x125	**1**x222					
3	6	**10**x125	**5**x222	**1**x205				
4	7	**20**x125	**15**x222	**6**x205	**1**x110			
5	8	**35**x125	**35**x222	**21**x205	**7**x110	**1**x45		
6	9	**56**x125	**70**x222	**56**x205	**28**x110	**8**x45	**1**x10	
7	10	**84**x125	**126**x222	**126**x205	**84**x110	**36**x45	**9**x10	**1**x1

3. Results and Discussion

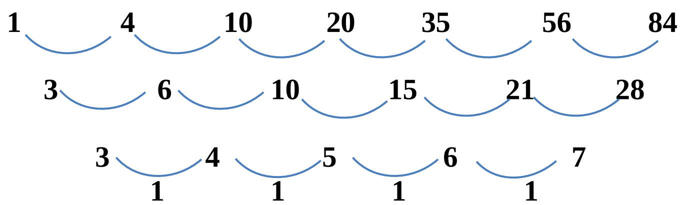
Notate the $G_{5,m,g}$ as the connected vertex labelled graph without loops of order 5 with m edges, g non parallel edges, and maximum parallel edges is five, $4 \leq m \leq 10$.

$N(G_{5,m,g}) = |G_{5,m,g}|$ = the number of $G_{5,m,g}$.

Result 1: For $4 \leq m \leq 10$ and g = 4, $N\left(G_{5,m,4}\right) = 125 \times C_3^{(m-1)}$

Proof:

From the first column of Table 5 with g = 4 we found a sequence of numbers occurs which are 1, 4, 10, 20, 35, 56, and 84.



Since the fixed difference occurs on the third level, then that sequence related with the arithmetic polynomial of order three $a_m = \alpha_1 m^3 + \alpha_2 m^2 + \alpha_3 m + \alpha_4$, where $\alpha_1, \alpha_2, \alpha_3$, and α_4 are the constants need to be determined. By setting m = 1, 2,3 and 4 we get the following system of equations:

$$125 = 64\alpha_1 + 16\alpha_2 + 4\alpha_3 + \alpha_4 \quad (1)$$
$$500 = 125\alpha_1 + 25\alpha_2 + 5\alpha_3 + \alpha_4 \quad (2)$$
$$1250 = 216\alpha_1 + 36\alpha_2 + 6\alpha_3 + \alpha_4 \quad (3)$$
$$2500 = 343\alpha_1 + 49\alpha_2 + 7\alpha_3 + \alpha_4 \quad (4)$$

Solving this system of equations we get $a_m = \frac{125}{6}m^3 - \frac{750}{6}m^2 + \frac{1375}{6}m - \frac{750}{6}$ and by doing some mathematical calculation we get $a_m = 125 \times C_3^{(m-1)}$. Therefore $N(G_{5,m,4}) = 125 \times C_3^{(m-1)}$

Result 2: For $4 \le m \le 10$ and g = 5, $N(G_{5,m,5}) = 222 \times C_4^{(m-1)}$
Proof:
From Table 5, we can see that the numbers in column with g=5 formed a sequence of number which are 1, 5, 15, 35, 70, 126. This sequence is related with arithmetic polynomial of the fourth order.

1 5 15 35 70 126
4 10 20 35 56
6 10 15 21
4 5 6
1 1

$$1222 = 625\alpha_1 + 125\alpha_2 + 25\alpha_3 + 5\alpha_4 + \alpha_5 \quad (5)$$
$$1110 = 1296\alpha_1 + 216\alpha_2 + 36\alpha_3 + 6\alpha_4 + \alpha_5 \quad (6)$$
$$3330 = 2401\alpha_1 + 343\alpha_2 + 49\alpha_3 + 7\alpha_4 + \alpha_5 \quad (7)$$
$$7770 = 4096\alpha_1 + 512\alpha_2 + 64\alpha_3 + 8\alpha_4 + \alpha_5 \quad (8)$$
$$15540 = 6561\alpha_1 + 729\alpha_2 + 81\alpha_3 + 9\alpha_4 + \alpha_5 \quad (9)$$

Solving this system of equations we get $a_m = \frac{222}{24}m^4 - \frac{2220}{24}m^3 + \frac{7770}{24}m^2 - \frac{11100}{24}m + \frac{5328}{24}$ and by doing some mathematical calculation we get $a_m = 222 \times C_4^{(m-1)}$.
Therefore $N(G_{5,m,5}) = 222 \times C_4^{(m-1)}$

Result 3 : For $4 \le m \le 10$ and g = 6, $N(G_{5,m,6}) = 205 \times C_5^{(m-1)}$
Proof:
From Table 5, we can see that the numbers in column with g=6 formed a sequence of numbers which are 1, 6, 21, 56, 126, 252. This sequence is related with arithmetic polynomial of fifth order
$a_m = \alpha_1 m^5 + \alpha_2 m^4 + \alpha_3 m^3 + \alpha_4 m^2 + \alpha_5 m + \alpha_6$

Therefore, we get:

$$205 = 7776\alpha_1 + 1296\alpha_2 + 216\alpha_3 + 36\alpha_4 + 6\alpha_5 + \alpha_6 \quad (10)$$
$$1230 = 16807\alpha_1 + 2401\alpha_2 + 343\alpha_3 + 49\alpha_4 + 7\alpha_5 + \alpha_6 \quad (11)$$
$$4305 = 32768\alpha_1 + 4096\alpha_2 + 512\alpha_3 + 64\alpha_4 + 8\alpha_5 + \alpha_6 \quad (12)$$
$$11480 = 59049\alpha_1 + 6561\alpha_2 + 729\alpha_3 + 81\alpha_4 + 9\alpha_5 + \alpha_6 \quad (13)$$
$$25830 = 100000\alpha_1 + 10000\alpha_2 + 1000\alpha_3 + 100\alpha_4 + 10\alpha_5 + \alpha_6 \quad (14)$$
$$516600 = 161051\alpha_1 + 14641\alpha_2 + 1331\alpha_3 + 121\alpha_4 + 11\alpha_5 + \alpha_6 \quad (15)$$

Solving this system of equations we get $a_m = \frac{205}{120}m^5 - \frac{3075}{120}m^4 + \frac{17425}{120}m^3 - \frac{46125}{120}m^2 + \frac{56170}{120}m - \frac{24600}{120}$, and by doing some mathematical calculation we get $a_m = 222 \times C_4^{(m-1)}$.
Therefore $N(G_{5,m,6}) = 222 \times C_5^{(m-1)}$.

From the results 1, 2, and 3 we can see that the formulas obtained make a pattern which is a multiplication of a constant with combinations of (m-1) and (g-1) : $125 \times C_3^{(m-1)}$, $222 \times C_4^{(m-1)}$, and $205 \times C_5^{(m-1)}$. Therefore, we can derive the following results:

Result 4: For $4 \leq m \leq 10$ and g = 7, $N(G_{5,m,7}) = 110 \times C_6^{(m-1)}$

Result 5: For $4 \leq m \leq 10$ and g = 8, $N(G_{5,m,8}) = 45 \times C_7^{(m-1)}$

Result 6 : For $4 \leq m \leq 10$ and g = 9, $N(G_{5,m,9}) = 10 \times C_8^{(m-1)}$

Result 7 : For $4 \leq m \leq 10$ and g = 10, $N(G_{5,m,10}) = 1 \times C_9^{(m-1)}$

4. Conclusion

From discussion above we can conclude that given n = 5, m edges, $4 \leq m \leq 10$, then the number of connected vertices labelled graphs with maximum number of parallel edges is five and containing no loops can be determined by its g, where g is the number of non parallel edges, which are:

a. For $4 \leq m \leq 10$ and g = 4, $N(G_{n,m,4}) = 125 \times C_3^{(m-1)}$
b. For $4 \leq m \leq 10$ and g = 5, $N(G_{n,m,5}) = 222 \times C_4^{(m-1)}$
c. For $4 \leq m \leq 10$ and g = 6 , $N(G_{n,m,6}) = 205 \times C_5^{(m-1)}$
d. For $4 \leq m \leq 10$ and g = 7, $N(G_{n,m,7}) = 110 \times C_6^{(m-1)}$
e. For $4 \leq m \leq 10$ and g = 8 , $N(G_{n,m,8}) = 45 \times C_7^{(m-1)}$
f. For $4 \leq m \leq 10$ and g = 9 , $N(G_{n,m,9}) = 10 \times C_8^{(m-1)}$
g. For $4 \leq m \leq 10$ and g = 10, $N(G_{n,m,10}) = C_9^{(m-1)}$

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