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On the Comparison of the Methods of Parameter Estimation for Pareto Distribution

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Abstract. The main purposes of this study is to asses on comparison of parameter estimation methods of the Pareto distribution. The estimation methods include moment, maximum likelihood estimation, probabilityweightedmoment, and generalizedmoment methods. Based on unbiasedness, variance, and consistency properties, the results demonstrate that in estimating parameters of the Pareto distribution, the maximum likelihood method is the one of the best estimation methods.

1. Introduction

The Pareto distribution is one of a continue probability distribution with scale parameter β and shift parameter k where $\beta > 0$ and $k > 0$ [1]. Pareto distribution introduced by an Italian economist Vilfredo Pareto, he was found that 80% land in Italia owned by no more than 20% of population. Based on those fact, then the Pareto’s law come up which stated that 20% effort will earn as much as 80%, this law also known as 20/80 or *law of the few* [2]. Probability density function of Pareto distribution is as follows [3]:

$$f(x; \beta, k) = \frac{k\beta^k}{x^{k+1}}; x \geq \beta, \beta > 0, k > \tag{1}$$

And the cumulative distribution of Pareto [3], is as follows:

$$F(x; \beta, k) = 1 - \left(\frac{\beta}{x}\right)^k; x \geq \beta, \beta > 0, k > \tag{2}$$

The estimation of parameters is a process by using a sample to estimate the unknown parameters of a population [4]. There are some methods that can be used to estimates the parameters, namely: methods of moment, method of maximum likelihood, method of probability weighted moment and method of generalized moment. An estimation of a parameter has attained the properties of unbiased, minimum variance, consistency, sufficient statistics and completeness.



In this study the comparison of those methods will be discussed to find the best method that can be used to estimate the parameters based on the criteria: of unbiased, minimum variance, consistency, sufficient statistics and completeness.

2. Materials and Methods

2.1 Method

The steps of the method conducted in this study:

1. Creating the curve of probability density function of Pareto distribution with parameter (β , κ) using software R
2. Estimating Pareto parameter (β , κ) using Method of Moments, Maximum Likelihood Estimation Method, Probability Weight Moment Method, and Generalized Method of Moments.
3. Examining the characteristics of unbiased from estimator of each parameter β and κ of the Method of Moments, Maximum Likelihood Estimation Method, Probability Weight Moment Method, and Generalized Method of Moments.
4. Examining the characteristics of consistency from estimator of each parameter β and κ of the Method of Moments, Maximum Likelihood Estimation Method, Probability Weighted Moment Method, and Generalized Method of Moments.
5. Examining the characteristics of minimum variance of Pareto distribution.
6. Examining the characteristics of sufficient statistic of Pareto distribution.
7. Examining the characteristics of and completeness of Pareto distribution.
8. Simulating using Software R for the Method of Moments, Maximum Likelihood Estimation Method, Probability Weight Moment Method, and Generalized Method of Moments.

3. Result and Discussion

3.1 The Curve of Probability of Pareto Distribution (β , κ)

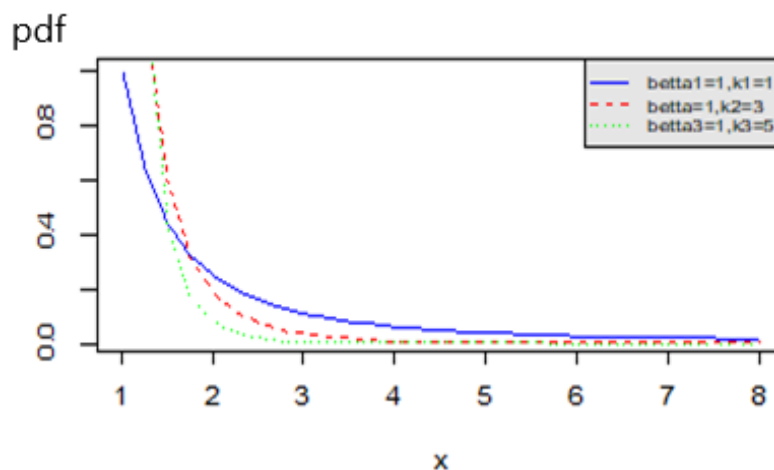


Figure 1. Pareto Distribution Probability Density Function

Pareto distribution probability density function is conducted using different parameter k which are $k = 1$, $k = 3$, $k = 5$ and the value of parameter $\beta = 1$ which is shown by the Figure 1. Parameter k is a shape parameter which numerical parameter to point the shape of curve. Meanwhile, parameter β is a scale parameter. It can be seen that the three graphs in Figure 1 has similar data variation which is caused by the same value of scale parameters.

3.2 Estimation of Parameter of Pareto Distribution

3.2.1 Method of Moment

The method of moments is a method of estimation of parameters. First, it starts with deriving equations which relate the population moments (namely, the expected values of powers of the random variable under consideration) to the parameters of interest. Second, a sample is drawn and the population moments are estimated from the sample. The equations are then solved for the parameters of interest, using the sample moments in place of the (unknown) population moments. This results in estimates of those parameters [5]. The results are as follows:

$$\hat{k} = \frac{m_1^2}{m_1^2 - m_2} \text{ and } \hat{\beta} = \left(\frac{m_2}{m_1}\right) \tag{3}$$

3.2.2 Maximum Likelihood Estimation Method

The likelihood function of Pareto distribution is as follows:

$$L(\beta, k) = k\beta^k \prod_{i=1}^n \frac{1}{x_i^{k+1}} \tag{4}$$

Next, we take logarithm on both sides, we have:

$$\ln L(a, b) = n \ln k + n - (k + 1) \sum_{i=1}^n \ln x_i \tag{5}$$

The estimation of the parameters can be attained by the derivative with respect to the parameters β and k and then set equal to zero [6]. So that we found the estimation as follows:

$$\hat{k} = \frac{n}{\sum_{i=1}^n \ln x_i - n \ln \beta} \text{ and } \hat{\beta} = \min x_i \tag{6}$$

3.2.3 Probability Weighted Moment

To estimate the parameters of Pareto distribution by probability weighted moment, first we look for the inverse of its cumulative distribution function [7], and we have:

$$]x = \frac{\beta}{(1-F(x))^{1/k}} \tag{7}$$

Next, we look for the t -th moment by the following formula:

$$M_t = M_{1,0,t} = \int_0^1 (X(F)) (1 - F(x))^t dF = \int_0^1 \left(\frac{\beta}{(1-F(x))^{1/k}}\right) (1 - F(x))^t dF \tag{8}$$

$$M_t = M_{1,0,t} = -\frac{\beta k}{tk+k-1} \tag{9}$$

By substitution $t = 0$ and $t = 1$ we have:

$$M_0 = -\frac{\beta k}{k-1} \text{ and } M_1 = -\frac{\beta k}{2k-1} \tag{10}$$

Next, the estimation of parameter k is: $\hat{k} = \frac{M_0 - M_1}{M_0 - 2M_1}$ and the estimation of parameter β is: $\hat{\beta} = \frac{M_1 M_0}{M_0 - M_1}$

3.2.4. Generalized Method of Moment

The estimation of parameter Pareto distribution by using generalized method of moment [8] is found by the following formula:

$$M_{l,r} = \int_0^1 x^l [F(x)]^r = \int_0^1 \left[\frac{\beta}{(1-F(x))^{1/k}} \right]^l [F(x)]^r \tag{11}$$

If the value of $r = 0$ and l is arbitrary, we have:

$$M_l = \frac{k\beta^l}{(l-k)} \tag{12}$$

If the values of $l = l_1, l_2 (l_1 \neq l_2)$ we have:

$$M_{l_1} = \frac{k\beta^{l_1}}{(l_1-k)} \quad \text{and} \quad M_{l_2} = \frac{k\beta^{l_2}}{(l_2-k)} \tag{13}$$

So that we have:

$$\hat{\beta} = \left(\frac{M_{l_1} l_1 - M_{l_1} k}{k} \right)^{\frac{1}{l_1}} \quad \text{and} \quad \hat{k} = \frac{M_{l_2} l_2}{(\beta^{l_2} + M_{l_2})} \tag{14}$$

3.3 Unbiasedness

The estimation of parameters Pareto distribution by using those methods, then we check the unbiasedness as follows:

3.3.1 Method of Moment

The estimation parameter $k (\hat{k})$

$$E(\hat{k}) = E \left[\frac{m_1^2}{m_1^2 - m_2} \right] = \frac{\frac{1}{n} \sum_{i=1}^n X_i^1}{\frac{1}{n} \sum_{i=1}^n X_i^1 - \beta} = k \tag{15}$$

The estimation parameter $\beta (\hat{\beta})$

$$E(\hat{\beta}) = E \left[\frac{m_2}{m_1} \right] = \frac{\frac{1}{n} \sum_{i=1}^n X_i^2}{\frac{1}{n} \sum_{i=1}^n X_i^1} = \beta \tag{16}$$

3.3.2 Probability weighted moment

The estimation parameter $k (\hat{k})$

$$E(\hat{k}) = E \left[\frac{M_0 - M_1}{M_0 - 2M_1} \right] = \frac{\frac{-\beta k}{k-1} \left(\frac{-\beta k}{2k-1} \right)}{\frac{-\beta k}{k-1} \left(2 \left(\frac{-\beta k}{2k-1} \right) \right)} = k \tag{17}$$

The estimation parameter $\beta (\hat{\beta})$

$$E(\hat{\beta}) = E \left[\frac{M_1 M_0}{M_0 - M_1} \right] = \frac{\left(\frac{-\beta k}{2k-1} \right) \left(\frac{-\beta k}{k-1} \right)}{\left(\frac{-\beta k}{k-1} \right) - \left(\frac{-\beta k}{2k-1} \right)} = \beta \tag{18}$$

3.3.3. Generalized Method of Moment

The estimation parameter $k (\hat{k})$

$$E(\hat{k}) = E \left[\frac{M_{l_2} l_2}{(\beta^{l_2} + M_{l_2})} \right] = \frac{k\beta^{l_2} l_2}{\beta^{l_2} l_2} = k \tag{19}$$

The estimation parameter β ($\hat{\beta}$)

$$E(\hat{\beta}) = E\left[\left(\frac{M_{l_1} l_1 - M_{l_1} k}{k}\right)^{\frac{1}{l_1}}\right] = \left(\frac{k \beta^{l_1} (l_1 - k)}{k}\right)^{\frac{1}{l_1}} = \beta \tag{20}$$

So that the estimation $\hat{\beta}$ and \hat{k} are unbiased for β and k .

3.4 Consistency

To check the consistency, the estimation of parameter Pareto distribution by using Chebyshev theorem,

$$P(|\hat{k} - k| \geq \varepsilon) \leq \frac{var(\hat{k})}{\varepsilon^2} \text{ for } \forall \varepsilon > 0 \text{ [6],} \tag{21}$$

as follows:

3.4.1 Method of Moment

The estimation parameter k (\hat{k})

$$P(|\hat{k} - k| \geq \varepsilon) \leq \frac{var\left(\frac{m_1^2}{m_1^2 - m_2}\right)}{\varepsilon^2} \tag{22}$$

$$\lim_{n \rightarrow \infty} \left\{ \left[\left(\frac{m_1^2}{m_1^2 - m_2} \right) - k \right] \right\} \leq 0 \tag{23}$$

The estimation parameter β ($\hat{\beta}$)

$$P(|\hat{\beta} - \beta| \geq \varepsilon) \leq \frac{var\left(\frac{m_2}{m_1}\right)}{\varepsilon^2} \tag{24}$$

$$\lim_{n \rightarrow \infty} \left\{ \left[\left(\frac{m_2}{m_1} \right) - \beta \right] \right\} \leq 0 \tag{25}$$

3.4.2 Probability Weighted Moment

The estimation parameter k (\hat{k})

$$P(|\hat{k} - k| \geq \varepsilon) \leq \frac{var\left(\frac{M_0 - M_1}{M_0 - 2M_1}\right)}{\varepsilon^2} \tag{26}$$

$$\lim_{n \rightarrow \infty} \left\{ \left[\left(\frac{M_0 - M_1}{M_0 - 2M_1} \right) - k \right] \right\} \leq 0 \tag{27}$$

The estimation parameter β ($\hat{\beta}$)

$$P(|\hat{\beta} - \beta| \geq \varepsilon) \leq \frac{var\left(\frac{M_1 M_0}{M_0 - M_1}\right)}{\varepsilon^2} \tag{28}$$

$$\lim_{n \rightarrow \infty} \left\{ \left[\left(\frac{M_1 M_0}{M_0 - M_1} \right) - \beta \right] \right\} \leq 0 \tag{29}$$

3.4.3 Method of Generalized Moment

The estimation parameter k (\hat{k})

$$P(|\hat{k} - k| \geq \varepsilon) \leq \frac{var\left(\frac{M_{l_2} l_2}{(\beta^{l_2} + M_{l_2})}\right)}{\varepsilon^2} \tag{30}$$

$$\lim_{n \rightarrow \infty} \left\{ \left[\left(\frac{M_{l_2} l_2}{(\beta^{l_2} + M_{l_2})} \right) - k \right] \right\} \leq 0 \tag{31}$$

The estimation parameter β ($\hat{\beta}$)

$$P(|\hat{\beta} - \beta| \geq \varepsilon) \leq \frac{\text{var} \left(\left(\frac{M_{l_1} l_1 - M_{l_1} k}{k} \right)^{\frac{1}{l_1}} \right)}{\varepsilon^2} \tag{32}$$

$$\lim_{n \rightarrow \infty} \left\{ \left[\left(\frac{M_{l_1} l_1 - M_{l_1} k}{k} \right)^{\frac{1}{l_1}} - \beta \right] \right\} \leq 0 \tag{33}$$

So that the estimation $\hat{\beta}$ and \hat{k} are consistent estimation for β and k .

3.5 Check for Minimum Variance

To check the minimum variance of Pareto distribution, first we find the Fisher information matrix [6], as follows:

$$l_n(\beta, k) = - \begin{bmatrix} E \left[\frac{\partial}{\partial \beta} \left(\frac{\partial \ln L}{\partial \beta} \right) \right] & E \left[\frac{\partial}{\partial \beta} \left(\frac{\partial \ln L}{\partial k} \right) \right] \\ E \left[\frac{\partial}{\partial k} \left(\frac{\partial \ln L}{\partial \beta} \right) \right] & E \left[\frac{\partial}{\partial k} \left(\frac{\partial \ln L}{\partial k} \right) \right] \end{bmatrix} \tag{34}$$

where

$$\ln L(\beta, k) = n \ln k + nk \ln \beta - (k + 1) \sum_{i=1}^n \ln x \tag{35}$$

So that we have the Fisher information as follows:

$$l_n^{-1}(\beta, k) = \frac{\beta^2 k^2}{(n^2 k - n^2 k^2)} \begin{bmatrix} \frac{n}{k^2} & -\frac{n}{\beta} \\ -\frac{n}{\beta} & \frac{nk}{\beta^2} \end{bmatrix} \tag{36}$$

Then we calculate the Cramer-Rao inequality [9] as follows:

$$\text{Var}(\hat{\theta}) \geq \frac{1}{E \left[\left(\frac{\partial \ln f(x; \theta)}{\partial \theta} \right)^2 \right]} = \frac{1}{I(\theta)} = I^{-1}(\theta) \tag{37}$$

$$\text{Var}(\hat{\beta}, \hat{k}) \geq l_n^{-1}(\beta, k) = \frac{\beta^2 k^2}{(n^2 k - n^2 k^2)} \begin{bmatrix} \frac{n}{k^2} & -\frac{n}{\beta} \\ -\frac{n}{\beta} & \frac{nk}{\beta^2} \end{bmatrix} \tag{38}$$

The estimation parameter $(\hat{\beta}, \hat{k})$ is an efficient estimation since the variance is attains the minimum Cramer-Rao inequality.

3.6 Check for sufficiency

To find the sufficient statistic of Pareto distribution, we used Fisher-Neymann factorization [10, 11], as follows:

$$f(\underline{X}; \beta, k) = \prod_{i=1}^n \frac{k \beta^k}{x_i^{k+1}} \tag{39}$$

$$f(\underline{X}; \beta, k) = k \beta^k \underbrace{x_i^{-kn}}_{k_1(U_i(\underline{X}); \beta, k)} \underbrace{x_i^{-n}}_{k_2(\underline{X})}$$

Since $k_2(\underline{X})$ is independent of β, k , so X_i is sufficient statistic.

3.7 Check for completeness

It will be shown that X_i is complete statistic if $E(g(x)) = 0$ and $P(g(x) = 0) = 1$ [6].

$$E(g(x)) = \sum g(x) \cdot f(X; \beta, k) = \sum g(x) \cdot k\beta^k x_i^{-kn} x_i^{-n} \tag{40}$$

If $E(g(x)) = 0$ then $\sum g(x) \cdot k\beta^k x_i^{-kn} x_i^{-n} = 0$. This means that $k\beta^k x_i^{-kn} x_i^{-n}$ is not equal to 0. If $x = 0$ then $g(0) = 0$, and if $x = 1$ then $g(1) = 0$ and so on, if we take $x = x$ then $g(x) = 0$. So that $P(g(x) = 0) = 1$ and so it is a complete statistic.

3.8 Simulation for the estimation of parameters β and k

Simulation of the estimation of parameters of Pareto distribution β and k by using software R version 3.3.2 based on the form of the curve of probability density function, for the value of parameters β and k , where $\beta = 1$ and $k = 1, 3, 5$. The sample size are $n = 10, 20, 40, 80$ and 100 . In this simulation the values of mean and Mean Square Error (MSE) are as follows:

Table 1. Estimation values of Parameter $\beta = 1$ and $k = 1$

n		$\beta=1$		$k=1$					
		MM ($\hat{\beta}$)	MLE ($\hat{\beta}$)	MM ($\hat{\beta}$)	MLE ($\hat{\beta}$)	MM ($\hat{\beta}$)	MLE ($\hat{\beta}$)	MM ($\hat{\beta}$)	MLE ($\hat{\beta}$)
10	Mea	100.815	0.03265	0.37977	66.6671	0.00147	0.18064	0.42400	0.05236
	n	3		6	2	8	5	9	5
	MS E	353755. 8	1.06058 7	1.27014	0.43768	1.01088	1.08623 2	1.39387	0.82726
20	Mea	4.84897	0.00676	0.23841	0.97815	0.00021	0.06713	0.22529	0.03572
	n		6		6	6	2	5	3
	MS E	356.483 6	1.02412 3	1.20852	0.42578	1.00521	1.07413	1.30743	0.85548
40	Mea	3.08625	0.00112	0.06983	0.56128	5.23E- 05	0.03988	0.12574	0.02469
	n		3	9	4		6	5	5
	MS E	109.199 7	1.01393 8	1.18725	0.38859	1.00166	1.02416 9	1.24523	0.90124 2
80	Mea	1.62732	0.00040	0.04998	0.61811	5E-06	0.01512	0.07966	0.02390
	n		1		5		9	3	2
	MS E	9.76338 9	1.00924 6	1.17962	0.43796	1.00165	1.23930 4	1.24462	0.87375
100	Mea	1.37139	0.00017	0.04311	0.47771	6.99E- 06	0.00817	0.07943	0.01845
	n		1	5	1			4	2
	MS E	16.035	0.03265	0.37977	66.6671	0.00147	0.18064	0.42400	0.05236
				6	2	8	5	9	5

Table 2. Estimation values of parameters $\beta=1$ and $k=3$

n		$\beta=1$		$k=3$					
		MM ($\hat{\beta}$)	MLE ($\hat{\beta}$)	MM ($\hat{\beta}$)	MLE ($\hat{\beta}$)	MM ($\hat{\beta}$)	MLE ($\hat{\beta}$)	MM ($\hat{\beta}$)	MLE ($\hat{\beta}$)
10	Mea	1.95926	1.03774	1.01945	1.20592	1.01238	3.60136	3.77759	3.59327
	n	8		6	7	6	7	7	6
	MS E	1.79660 9	0.00317	0.00526 5	0.06710 4	3.95061 9	1.54717 1	2.85488 4	0.38410 8
20	Mea	1.74417	1.01738	0.99741	1.20794	1.00293	3.35290	3.38725	3.59821
	n		5		4		5		6

n		$\beta=1$		k=3					
		MM ($\hat{\beta}$)	MLE ($\hat{\beta}$)	MM ($\hat{\beta}$)	MLE ($\hat{\beta}$)	MM ($\hat{\beta}$)	MLE ($\hat{\beta}$)	MM ($\hat{\beta}$)	MLE ($\hat{\beta}$)
40	MS	0.67605	0.00057	0.00343	0.06417	3.98827	0.69908	1.31292	0.38328
	E	9	2	3	6		4	7	2
	Mea n	1.82381	1.00873	1.0012	1.22336	1.00076	3.05988	3.31467	3.61807
80	MS	0.87026	0.00015	0.00055	0.06890	3.99695	0.22793	0.47885	0.40283
	E	5	2	5	8	4	7	8	
	Mea n	1.88181	1.00452	1.0043	1.20021	1.00019	3.04348	3.13774	3.59701
100	MS	1.00665	0.00004	0.00025	0.04573	3.99921	0.10988	0.21747	0.36404
	E	6	23	8	6	3	3	8	5
	Mea n	1.95285	1.00299	0.99914	1.19891	1.00013	3.08188	3.07644	3.59505
	MS	1.25291	0.00001	0.00021	0.04639	3.99948	0.10955	0.20594	0.36213
	E	7	55			2	4	4	5

Table 3. The estimation values of parameters $\beta = 1$ and $k = 5$

n		$\beta=1$		k=					
		MM ($\hat{\beta}$)	MLE ($\hat{\beta}$)	MM ($\hat{\beta}$)	MLE ($\hat{\beta}$)	MM ($\hat{\beta}$)	MLE ($\hat{\beta}$)	MM ($\hat{\beta}$)	MLE ($\hat{\beta}$)
10	Mea n	1.33803	1.019809	1.00143	0.99571	1.01065	6.30227	6.1625	5.5385
	MS	0.18685	0.000687	0.00205	0.00487	15.9148	6.70947	8.0690	0.3185
	E	5		5	2	9	1	3	4
20	Mea n	1.32645	1.009209	0.99742	0.99932	1.00265	5.58986	5.3153	5.5494
	MS	0.12605	0.000193	0.00078	0.00332	15.9788	2.07402	2.8441	0.3214
	E	3		5	5	2	6	6	2
40	Mea n	1.31870	1.005312	1.0012	0.99303	1.00066	5.26881	5.3595	5.5361
	MS	0.10849	0.000061	0.00055	0.00159	15.9947	0.69983	1.4252	0.2979
	E	4	9	5	6	1	2	6	4
80	Mea n	1.33677	1.0022	1.0043	1.00361	1.00016	5.09930	5.2197	5.5632
	MS	0.11844	0.000009	0.00025	0.00090	15.9986	0.42775	0.4723	0.3257
	E	4	7	8	8	7	5	6	8
100	Mea n	1.33424	1.002085	0.99914	1.00379	1.000107	5.11813	5.0933	5.5640
	MS	0.11698	0.000009	0.00021	0.00065	15.9991	0.29119	0.4082	0.3220
	E	6	56		9	5	4	1	5

Based on the tables above, it was found that the values of estimation for mean of the parameters β and k by using method of moment, methods of maximum likelihood, probability weighted moment, and method of generalized moment, the best method is the maximum likelihood method, since it is close to the real values of parameters (β, k) and with the minimum Mean Square Error (MSE).

4. Conclusion

Based on the result above, we conclude that:

1. The estimation of parameters(β, k) by using method of moment, method of maximum likelihood, method of probability weighted moment, and generalized moment are the estimation methods that can be use since all the method attains the properties of unbiasedness, minimum variance, consistency, sufficient statistic and completeness.
2. Based on the simulation result by using software R the values of *mean square error* for the method of maximum likelihood has the smallest values compared with the others methods. Therefore, we can conclude that the method of maximum likelihood is the best method to estimate the parameters of Pareto distribution if the sample size is large.

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