

Counting The Number of Vertex Labelled Connected Graphs of Order Five with Minimum Five Edges and Maximum Ten Parallel Edges without Loops

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If given a graph $G(V,E)$ with n vertices and m edges many graphs can be constructed. The graphs constructed maybe connected graphs (there exists at least one path connecting every pair of vertices in the graph) or disconnected; either simple (contains loop or parallel edges) or not simple. In this paper we will discuss the formula for counting the number of connected vertex labelled graph of order five ($n=5$) without loops, with minimum five edges and may contain maximum ten parallel edges. **Keywords:** connected vertex labelled graph, order five, parallel edges.

1. Introduction

Graph theory is a branch of mathematics which discusses about discrete objects and the relationship among those objects. A point in a graph represents an object and an edge (a line) connecting a pair of vertices in graph represents the relationship between that pair of vertices. Points in a graph can represent cities, stations, houses, etc , while edges can represent roads, train track, fresh water pipeline, and so on. There is no a single correct way to draw a graph. A line can be drawn as a straight line, a curve , etc. Therefore, because of its flexibility, graph theoretical concepts are used to represent many real-life problems.

Not only can represent the real-life problem, graph theoretical concept also can be used to solve it. A comprehensive study about the application of graph theory in networks design is given in [1], while in [2] some applications in operations research problem are given. Enumerative or counting graph is pioneered by Cayley in 1874 who interested to count the isomer of hydrocarbon C_nH_{2n+2} and found that counting problem is related with counting rooted tree in graph. Inspired by the work of Cayley, Slomenski investigated the additive structural properties of hydrocarbon using the concept of graph [3]. However, theoretically enumerative graph is given in [4 – 6].

2. Literature Review

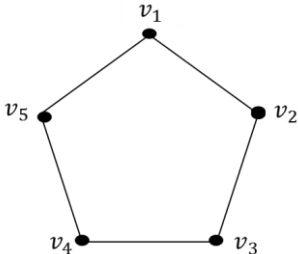
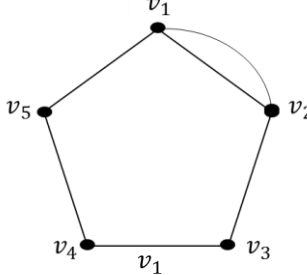
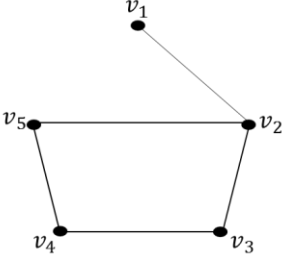
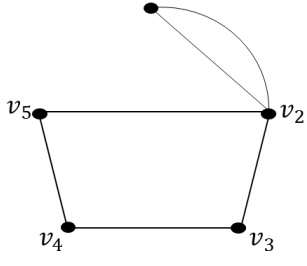
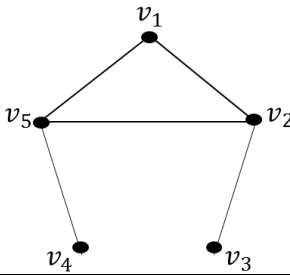
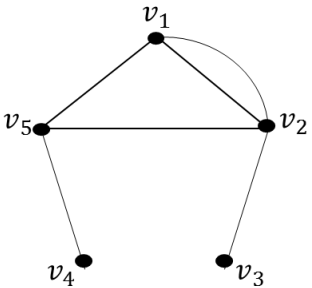
If given graph $G(V,E)$ with n number of vertices and m edges, there are a lot of graphs that can be formed, either simple graph or not simple. A simple graph is a graph which not contains loops nor paralel edges. Moeover, a graph constructed maybe connected or disconnected. To distinguished one graph with another graph with the same form or structure we can use labelling. If the label is given on every vertices, it is called vertex labelling, if the label is given on every edge than is is called as edge labelling, and if the label is given on every vertex and edge then is is called as total labelling The formula for counting vertex labelled graphs is given in [7], but in that formula there are no distinction if the graphs are connected or disconnected. In that formula, all graphs are included except loops which are not allowed to be in the graph. Based on [7] , in 2017 the number of disconnected vertex labelled graphs of order gour is investigated [8], and the number of disonconnected vertex labelled graphs of order five without parallel edges is investigated in [9]. The formula for counting disconnected vertex label graphs of order

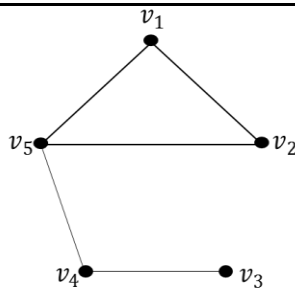
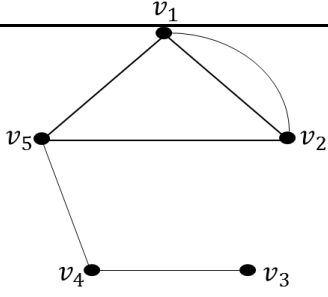
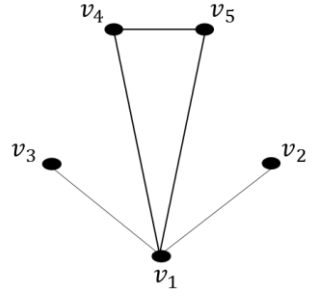
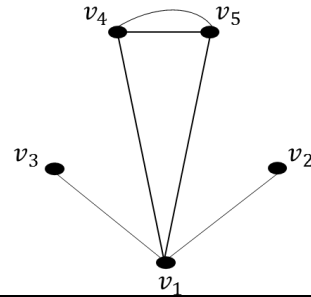
five with maximum six 3-parallel edges and containing no loops is investigated in [10], and the formula for counting the number of connected vertex labelled graph of order five with maximum 5 parallel edges is investigated in [11].

3. Graphs construction and the patterns obtained

Given $G(V,E)$ with the number of vertices is $5(n = 5)$ and the number of edges is m , $5 \leq m \leq 20$, some patterns obtained are as shown in Table 1, where t is the number of edges that connect different pairs of vertices (parallel edges are counted as one). **Note that isomorphism graphs are counted as one.** There are many graphs can be obtained, but due to space limitation, we only provide some patterns and the number of graphs that can be constructed using that pattern.

Table 1. Some connected vertex labelled graphs of order five constructed

Graph info	Patterns	The number of graphs	Graph info	Patterns	The number of graphs
$n = 5$ $t = 5$ $m = 5$		12	$n = 5$ $t = 5$ $m = 6$		60
		60			300
		60			300

	60		300
	30		150
Total	222	Total	1110

Notate t as the number of edges that connect different pairs of vertices (two or more edges that connect the same pair of vertices are counted as one), p_i is the number of i -parallel edges, and g is the number of non parallel edges, then $t = \sum_{i=1} p_i + g$, and $m = \sum_{i=1} j \cdot p_i + g$; $j \in \mathbb{N}$, where n is the order of graph and m is the number of edges in the graph. From Figure 1 we can see that $n = 5$, $p_2 = 2$, $g = 5$, $t = p_2 + g = 2 + 5 = 7$ and $m = 2 \cdot p_2 + g = 2 \cdot 2 + 5 = 9$.

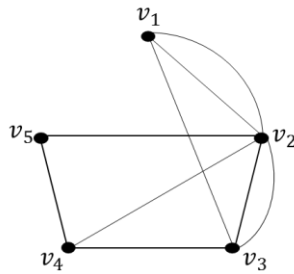


Figure 1. An example of graph with 2-parallel edges

4. Results and Discussion.

From construction and observation we get Table 2 as follows:

Table 2. The number of connected vertex labelled graphs of order five with minimum edges is five and maximum ten parallel edges.

m	The number of connected vertex labelled graphs of order five with minimum edges is five and maximum ten parallel edges					
	t					
	5	6	7	8	9	10
5	222					
6	1110	205				
7	3330	1230	110			
8	7770	4305	770	45		
9	15540	11480	3080	360	10	
10	27972	25830	9240	1620	90	1
11	46620	51660	23100	5400	450	10
12		94710	50820	14850	1650	55
13		162360	101640	35640	4950	220
14			188760	77220	12870	715
15			330330	154440	30030	2002
16				289575	64350	5005
17				514800	128700	11440
18					243100	24310
19					437580	48620
20						92378

The following table is derived from Table 2 by observing the patterns obtained on the column of the table.

Table 3. The patterns obtained in every column of the table.

m	The number of connected vertex labelled graphs of order five with minimum edges is five and maximum ten parallel edges					
	t					
	5	6	7	8	9	10
5	1 x 222					
6	5 x 222	1 x 205				
7	15 x 222	6 x 205	1 x 110			
8	35 x 222	21 x 205	7 x 110	1 x 45		
9	70 x 222	56 x 205	28 x 110	8 x 45	1 x 10	
10	126 x 222	126 x 205	84 x 110	36 x 45	9 x 10	1 x 1
11	210 x 222	252 x 205	210 x 110	120 x 45	45 x 10	10 x 1
12		462 x 205	462 x 110	330 x 45	165 x 10	55 x 1
13		792 x 205	924 x 110	792 x 45	495 x 10	220 x 1
14			1716 x 110	1716 x 45	1287 x 10	715 x 1
15			3003 x 110	3432 x 45	3003 x 10	2002 x 1
16				6435 x 45	6435 x 10	5005 x 1
17				11440 x 45	12870 x 10	11440 x 1
18					24310 x 10	24310 x 1
19					43758 x 10	48620 x 1
20						92378 x 1

By grouping the graphs that can be constructed in terms of m and t , we found that the number in every column of the table formed patterns. In every column the numbers can be rewrite as a product of fixed constant with a number, and the numbers in every column makes a sequences.

For $t=5$: 1, 5, 15, 35, 70, 126, 210

1	5	15	35	70	126	210
4	10	20	35	56	84	
	6	10	15	21	28	
		4	5	6	7	
			1	1	1	

The fixed difference occurs on the fourth level, therefore the polynomial that can represent that sequence is polynomial of order four: $P_4(m) = A_4m^4 + A_3m^3 + A_2m^2 + A_1m + A_0$

Result 1 : Given $G(V,E)$, $|V| = 5$, $|E| = m$, $5 \leq m \leq 20$, $t = 5$, t is the number of edges that connect different pair of vertices (parallel edges are counted as one), then the number of connected vertex labelled graphs of order five with minimum five edges and maximum ten parallel edges is

$$N(G_{5,m,t}) = 222 \binom{m-1}{4}.$$

Proof :

Substituting the value of the number of graph obtained in the first column we get:

$$222 = 625 A_4 + 125 A_3 + 25 A_2 + 5 A_1 + A_0$$

$$1110 = 1296 A_4 + 216 A_3 + 36 A_2 + 6 A_1 + A_0$$

$$3330 = 2401 A_4 + 343 A_3 + 49 A_2 + 7 A_1 + A_0$$

$$7770 = 4096 A_4 + 512 A_3 + 64 A_2 + 8 A_1 + A_0$$

$$15540 = 6561 A_4 + 729 A_3 + 81 A_2 + 9 A_1 + A_0$$

Solving the above system of equations we get $A_4 = \frac{2664}{288}$, $A_3 = \frac{-26640}{288}$, $A_2 = \frac{93240}{288}$, $A_1 = \frac{-133200}{288}$, and $A_0 = \frac{63936}{288}$.

$$\begin{aligned} \text{Therefore : } P_4(m) &= \frac{2664}{288} m^4 - \frac{26640}{288} m^3 + \frac{93240}{288} m^2 - \frac{133200}{288} m + \frac{63936}{288} \\ &= \frac{222}{24} (m^4 - 10m^3 + 35m^2 - 50m + 24) \\ &= \frac{222}{24} (m-1)(m-2)(m-3)(m-4) \\ &= 222 \binom{m-1}{4}. \end{aligned}$$

Therefore $N(G_{5,m,t})$ for $t = 5$ is $222 \binom{m-1}{4}$.

For $t=6$, the sequence of number is 1, 6, 21, 56, 126, 252, 462, 792.

1	6	21	56	126	252	462	792
5	15	35	70	126	210	330	
	10	20	35	56	84	120	
		10	15	21	28	36	
			5	6	7	8	
				1	1	1	

The fixed difference occurs on the fourth level, therefore the polynomial that can represent that sequence is polynomial of order five: $P_5(m) = A_5m^5 + A_4m^4 + A_3m^3 + A_2m^2 + A_1m + A_0$

Result 2 : Given $G (V,E)$, $|V| = 5$, $|E| = m$, $5 \leq m \leq 20$, $t = 6$, t is the number of edges that connect different pair of vertices (parallel edges are counted as one), then the number of connected vertex labelled graphs of order five with minimum five edges and maximum ten parallel edges is

$$N(G_{5,m,t}) = 205 \binom{m-1}{5}.$$

Proof :

Substituting the value of the number of graph obtained in the first column we get:

$$205 = 7776 A_5 + 1296A_4 + 216 A_3+ 36A_2 + 6A_1 + A_0$$

$$1230 = 16807A_5 + 4012 A_4 + 343 A_3+ 49A_2 + 7A_1 + A_0$$

$$4305 = 32768A_5 + 4096A_4 + 512A_3 + 64A_2 + 8A_1 + A_0$$

$$11480 = 59049A_5 + 6561A_4 + 729 A_3+ 81A_2 + 9A_1 + A_0$$

$$25850 = 100000A_5 + 10000A_4 + 1000A_3 + 100A_2 + 10A_1 + A_0$$

$$51660 = 161051A_5 + 114641A_4 + 1331A_3 + 121A_2 + 11A_1 + A_0$$

$$\text{Solving the above system of equations we get } A_5 = \frac{59040}{34560}, A_4 = \frac{-885600}{34560}, A_3 = \frac{5016400}{34560}, A_2 = \frac{-13284000}{34560}$$

$$, A_1 = \frac{16176960}{34560}, \text{ and } A_0 = \frac{-7084800}{34560}.$$

$$\begin{aligned} \text{Therefore : } P_5(m) &= \frac{59040}{34560} m^5 - \frac{885600}{34560} m^4 + \frac{5016400}{34560} m^3 - \frac{13284000}{34560} m^2 + \frac{16176960}{34560} m - \frac{7084800}{34560}. \\ &= \frac{205}{120} (m^5 - 15m^4 + 85m^3 - 225m^2 + 274m - 120) \\ &= \frac{205}{120} (m-1)(m-2)(m-3)(m-4)(m-5) \\ &= 205 \binom{m-1}{5}. \end{aligned}$$

Therefore $N(G_{5,m,t})$ for $t = 6$ is $205 \binom{m-1}{5}$.

Based on Results 1 and 2, we get the following results:

Result 3 : Given $G (V,E)$, $|V| = 5$, $|E| = m$, $5 \leq m \leq 20$, $t = 7$, t is the number of edges that connect different pair of vertices (parallel edges are counted as one), then the number of connected vertex labelled graphs of order five with minimum five edges and maximum ten parallel edges is

$$N(G_{5,m,t}) = 110 \binom{m-1}{6}.$$

Result 4 : Given $G (V,E)$, $|V| = 5$, $|E| = m$, $5 \leq m \leq 20$, $t = 8$, t is the number of edges that connect different pair of vertices (parallel edges are counted as one), then the number of connected vertex labelled graphs of order five with minimum five edges and maximum ten parallel edges is

$$N(G_{5,m,t}) = 45 \binom{m-1}{7}.$$

Result 5 : Given $G (V,E)$, $|V| = 5$, $|E| = m$, $5 \leq m \leq 20$, $t = 9$, t is the number of edges that connect different pair of vertices (parallel edges are counted as one), then the number of connected vertex labelled graphs of order five with minimum five edges and maximum ten parallel edges is

$$N(G_{5,m,t}) = 10 \binom{m-1}{8}.$$

Result 6 : Given $G (V,E)$, $|V| = 5$, $|E| = m$, $5 \leq m \leq 20$, $t = 10$, t is the number of edges that connect different pair of vertices (parallel edges are counted as one), then the number of connected vertex labelled graphs of order five with minimum five edges and maximum ten parallel edges is

$$N(G_{5,m,t}) = \binom{m-1}{9}.$$

5. Conclusion

Based on the above discussion we can conclude that given $G (V,E)$, $|V| = 5$, $|E| = m$, $5 \leq m \leq 20$, and t , t is the number of edges that connect different pair vertices (parallel edges are counted as one), then

the number of connected vertex labelled graphs of order five with minimum five edges and maximum ten parallel edges $N(G_{5,m,t})$ is :

1. For $t = 5$, $N(G_{5,m,t}) = 222 \binom{m-1}{4}$,
2. For $t = 6$, $N(G_{5,m,t}) = 205 \binom{m-1}{5}$,
3. For $t = 7$, $N(G_{5,m,t}) = 110 \binom{m-1}{6}$,
4. For $t = 8$, $N(G_{5,m,t}) = 45 \binom{m-1}{7}$,
5. For $t = 9$, $N(G_{5,m,t}) = 10 \binom{m-1}{8}$,
6. For $t = 10$, $N(G_{5,m,t}) = \binom{m-1}{9}$.

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Acknowledgement

This research is fund by Universitas Lampung under contract number 2843/UN26.21/PN/2019, and the authors want to thanks for the contribution of the research.