Bulletin of Mathematics
Vol. 04, No. 02 (2012), pp. 161–167.

PARTITION DIMENSION OF AMALGAMATION OF STARS

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Abstract.
Let G be a connected graph and $S_i \subseteq V(G)$ for all $i = 1, 2, ..., k$. For an ordered k-partition $\Pi = \{S_1, S_2, ..., S_k\}$ of $V(G)$ and a vertex v of G, the representation of v with respect to Π is defined as the k-vector $r(v|\Pi) = (d(v, S_1), d(v, S_2), ..., d(v, S_k))$. The partition is called a resolving partition if the k-vectors $r(v|\Pi), v \in V(G)$ are distinct. The partition dimension of G, denoted by $pd(G)$ is the smallest k such that G has a resolving k-partition of $V(G)$. In this paper, we determine the partition dimension of amalgamation of stars $S_{k,m}$. $S_{k,m}$ is obtained from k copies of star $K_{1,m}$ by identifying a leaf from each star.

1. INTRODUCTION

Let G be a finite, simple, and connected graph. The distance $d(u, v)$ between two vertices u and v is the length of a shortest $u-v$ path in G. For a subset S of $V(G)$ and a vertex v of G, the *distance $d(v, S)$ between v and S* is defined as $d(v, S) = \min\{d(v, x)|x \in S\}$. For an ordered k-partition $\Pi = S_1, S_2, ..., S_k$ of $V(G)$ and a vertex v of G, the *representation of v with respect to* Π is defined as the k-vector $r(v|\Pi) = (d(v, S_1), d(v, S_2), ..., d(v, S_k))$. The partition Π is called a *resolving partition* if the k-vectors $r(v|\Pi), v \in V(G)$ are distinct. The minimum k for which there is a resolving k-partition of $V(G)$ is the *partition dimension* $pd(G)$ of G.

Received 25-08-2011, Accepted 28-03-2012.
2000 Mathematics Subject Classification: 05C05, 05C12, 05C15
Key words and Phrases: Amalgamation of stars, Partition dimension, Graph.

We consider the amalgamation of stars, for $i = 2, 3, 4, ..., k$, let $G_i = K_{1,m}, m \geq 2$ are v_{oi} be any leaf of G_i. Then the amalgamation of these k isomorphic stars $K_{1,m}$ is denoted by $S_{k,m}$. The identified vertex as the center (denoted by x), the vertices of distance 1 from the center as the intermediate vertices (denoted by $l_i; i = 1, 2, ..., k$), and the j-th leaf of intermediate vertex l_i by $l_{ij}(j = 1, 2, \ldots, m-1)$. The partition dimension was firstly studied by Chartrand et al. in [1, 2], an upper bound for the partition dimension of a bipartite graph G is given in terms of the cardinalities of its partite sets, and it is shown that the bound is attained if only if G is a complete bipartite graph. Graph of order n having partition dimension $2, n$, or $n-1$ are characterized. Furthermore, graphs of order $n \geq 9$ having partition dimension $n-2$ are characterized by Tomescu [3]. Later by Chappell et al. [4] is formed a construction showing that for all integers α and β with $3 \leq \alpha \leq \beta + 1$ there exist a graph G with partition dimension α and metric dimension β, answering a question of Chartrand et al. [2]. The partition dimension of Cayley digraph was studied by Fehr et al. in [5]. Next, Baskoro and Darmaji [6] determined the partition dimension of corona product of two graphs. Before presenting main result, we learn the following lemma is discussed by Chartrand et al. in [1].

Lemma 1.1 *Let* $\prod$ *be a resolving partition of* $V(G)$ *and* $u, v \in V(G)$*. If* $d(u,w) = d(v,w)$ *for all* $w \in V(G) - \{u, v\}$ *then* u *and* v *belong to distinct classes of* Π.

2. MAIN RESULTS

In this section, we determine the partition dimension of amalgamation of stars.

Lemma 2.2 *Let* Π *be a partition of* $S_{k,m}, k, m \geq 2$*, with* $|\Pi| \geq m-1$*. Partition* Π *is a resolving partition of* $S_{k,m}$ *if and only if* l_i *and* $l_k, i = k$ *are in the same class of* Π *implies class combinations of* $l_{ij}|j = 1, 2, \ldots, m-1$ *and* $l_{kj}|j = 1, 2, \ldots, m-1$ *are distinct.*

Proof 2.1 *Let* Π *be a resolving partition of* $S_{k,m}, k, m \geq 2$*,* $|\Pi| \geq m-1$ *and* $l_i, l_k, i = j$ *are in the same class of* Π*. Suppose that class combinations of* $lij|j = 1, 2, \ldots, m-1$ *and* $\{l_{kj}|j = 1, 2, \ldots, m-1\}$ *are the same. Because* $d(l_i, u) = d(l_k, u)$ *for every* $u \in V\{\{l_{ij}|j = 1, 2, ..., m-1\} \cup \{l_{kj}|j = 1, 2, ..., m-1\}\}$ *then the representation of* l_i *and* l_k *are the same. So* Π *is not a resolving partition. A contradiction.*

Let Π be a partition of $S_{k,m}, k, m \geq 2$, with $|\Pi| \geq m-1$. Let A to denote class combinations of $\{l_{ij}|j = 1, 2, ..., m-1\}$ and B to denote class combinations of $\{l_{kj}|j = 1, 2, ..., m-1\}$. Consider l_i and $l_k, i = j$ are in the same class of Π. Since $A = B$, then there are S_m and S_n such that $S_m \in A, S_m \notin B$ and $S_n \in B, S_n \notin A$. We will show that representation for every $v \in V(S_{k.m})$ is unique.

- *Clearly, $r(l_i|\Pi) = r(l_k|\Pi)$ because their representation different in the mth-ordinate and nth-ordinate.*
- *If l_{ij} and l_{km} are in the same class of Π with $l_i = l_k$, we will show that $r(l_{ij}|\Pi) = r(l_{km}|\Pi)$. We divide into two cases.*
  - *Case 1, if l_i and l_k are in the same class of Π then by the premise of this theorem, $A = B$. So $r(l_{ij}|\Pi) = r(l_{km}|\Pi)$.*
  - *Case 2, let $l_i \in S_x$ and $l_k \in S_y, S_x = S_y$. Then $r(l_{ij}|\Pi)$ and $r(l_{km}|\Pi)$ are different in the xth-ordinate and yth-ordinate. So $r(l_{ij}|\Pi) = r(l_{km}|\Pi)$.*
- *If l_i and l_{kj}, $l_i = l_k$ are in the same class of Π, then the representation $r(l_i|\Pi)$ contains at least one components of value 1. Whereas $r(l_{kj}|\Pi)$ contains exactly one component of value 1. Thus $r(l_i|\Pi) = r(l_{kj}|\Pi)$.*
- *If x and l_{ij} are in the same class of Π, then the representation $r(x|\Pi)$ contains at least one component of value 1. Whereas $r(l_{ij}|\Pi)$ contains exactly one component of value 1. Thus $r(x|\Pi) = r(l_i|\Pi)$.*
- *If x and l_i are in the same class of Π. We divide into two cases.*
  - *Case $3 \leq k < m-1$. Representation $r(l_i|\Pi)$ contains $(m-2)$ components of value 1. Whereas $r(x|\Pi)$ contains less than $(m-2)$ components of value 1. So $r(x|\Pi) = r(li|\Pi)$.*
  - *Case $k \geq m+1$. Because $d(x, m) = d(l_i, m)$ then $r(x|\Pi) = r(l_i|\Pi)$.*

Lemma 2.3 *Let Π be a resolving $(m+a)$-partition of $S_{k,m}$, for $a \geq 1$, then $k \leq (m+a)^2 - 1$.*

Proof 2.2 *Let Π be a $(m+a)$-resolving partition of $S_{k,m}$, for $a \geq 1$. For fixed i, let l_i, then the number partition combinations can be used by $\{l_{ij}|j = 1, 2, ..., m-1\}$ is $(m+a)$. Because, there are $(m+a)$ partitions for l_i, for every $i = 1, 2, ..., (m+a)$, by Lemma 2.2, we obtain number of k is*

$(m+a)^2$. However, if x, l_b are in the same class partition and $N(x) = N(l_b)$, then we must cut l_b, to make sure that the all vertices will have distinct representation. So, the maximum number of k is $(m+a)^2 - 1$.

Theorem 2.1

$$pd(S_{k,m}) = \begin{cases} m-1, & \text{for } 2 \leq k \leq m-2, m \geq 4; \\ m, & \text{for } m-1 \leq k \leq m^2-1; \\ m+a, & \text{for } (m+a-1)^2 - 1 < k \leq (m+a)^2 - 1, a \geq 1. \end{cases} \quad (1)$$

We will determine lower bound trivial. By Lemma 1.1, we have that for every $i, i = 1, 2, ..., k$ two vertices l_{ij} and $l_{ik}, k \neq j$ must lie in different classes. Thus $pd(S_{k,m}) \geq m - 1$.

We determine the upper bound for $2 \leq k < m - 1, m \geq 4$. Let $\Pi_1 = S_1, S_2, ..., S_k, ..., S_{m-1}$ be the partition of $V(S_{k,m})$. Without loss of generality for center x can be in S_1 and $\{l_{ij} | j = 1, 2, ..., m - 1\}$, for each $i = 1, 2, ..., k$ must be in $S_1, S_2, ..., S_{m-1}$, respectively. To make sure that the leaves will have distinct representation, we must put l_i, for each $i = 1, 2, ..., k$ in different classes. So Π_1 is a resolving k-partition of $V(S_{k,m})$. Hence $pd(S_{k,m}) \geq m - 1$. So $pd(S_{k,m}) = m - 1, m \geq 4$.

We improve the lower bound for $m - 1 \leq k \leq m^2 - 1$. Consider $\Pi' = C_1, C_2, ..., C_{m-1}$ a partition of $V(S_{k,m})$. If there are l_{io} and l_{jo} that belong to the same class, say C_{ko}. As a result $r(l_{io}|\Pi')$ and $r(ljo|\Pi')$ both have 0 in the koth-ordinate, and 1 elsewhere. Thus Π' is not resolving. So $pd(S_{k,m}) \geq m$.

To determine the upper bound for $m - 1 \leq k \leq m^2 - 1$ we recall, by lemma 1.1, that the leaves incident to the same vertex must be lie in different classes, then there are m combinations for $m - 1$ different classes, namely $K_t = \{1, 2, ..., m-1\}/t, t = 1, 2, ..., m$. For fixed i, we assign $\{l_{ij} | j = 1, 2, ..., m - 1\}$ to a K_t. Observe that if $\{l_{ij} | j = 1, 2, ..., m - 1\}$ and $\{l_{kj} | j = 1, 2, ..., m - 1\}, i = k$, are assigned to the same K_t. Then to make sure that there leaves will have distinct representation, we must put l_i and l_k in different classes. As assign, for $t = 2, 3, ..., m$, if K_t is assigned to $\{l_{ij} | j = 1, 2, ..., m - 1\}$ then l_i can be in any $S_j, j = 1, 2, ..., m$ and without loss of generalization for center x can be in S_1. However if K_1 is used in l_i can be in any S_j, except S_1. This is to avoid that l_i and x have the same representation. So number of l_i is $m^2 - 1$. Thus $pd(S_{k,m}) = m$ for $m - 1 \leq k \leq m^2 - 1$.

We improve the lower bound for $(m+a-1)^2 - 1 < k \leq (m+a)^2 - 1, a \geq 1$. Since $k > (m+a-1)^2 - 1$ then by Lemma 2.3, $pd(S_{k,m}) > m+a-1$. On

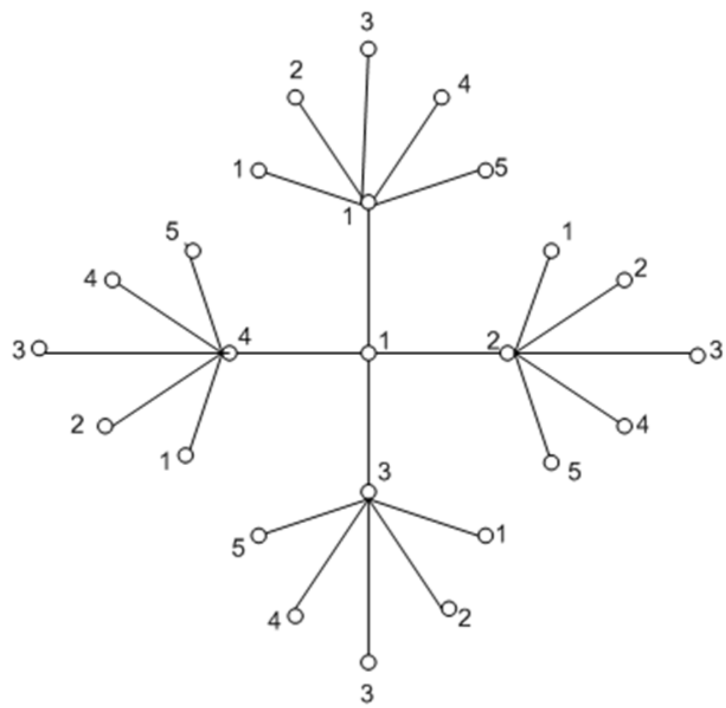

Figure 1: Minimum resolving partition of $Pd(S_{4,6})$

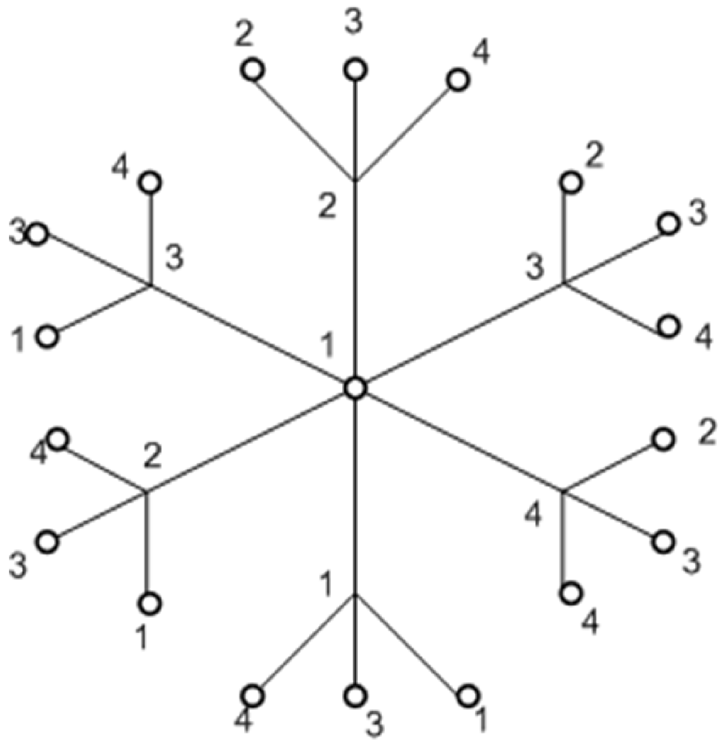

Figure 2: Minimum resolving partition of $Pd(S_{6,4})$

REFERENCES

1. Chartrand, G., Salehi, E. and Zang, P., 1998, On the partition dimension of a graph, *Congr. Numer.* 130, 157-168.
2. Chartrand, G., Salehi, E. and Zang, P., 2000, The partition dimension of a graph, *Aequationes Math.*, 59, 45-54.
3. Tomescu, I., 2008, Discrepancies between metric dimension and partition dimension of a connected graph, *Discrete Math.* 308, 5026-5031.
4. Chappell, G., Gimbel, J., and Hartman, C.,2008, Bounds on the metric and partition dimensions of a graph, *Ars Combin.* 88, 349-366.
5. Fehr, M., Gosselin, S., and Oellermann, R., 2006, The partition dimension of cayley digraphs, *Aequationes Math.* 71, 1-18.
6. Baskoro, E. and Darmaji. 2012. The partition dimension of corona product of two graphs, *Far East J. Math. Sci.* 66(12), 181-196.
7. Winston, W. L. 2004. *Operations Research: Applications and Algorithms 4th Ed.* Thomson Learning Inc., California.

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