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THE LOCATING-CHROMATIC NUMBER FOR CERTAIN OF TREES

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Abstract. *The locating-chromatic number is an interesting study until now, particularly in tree. In this article, we will discuss the locating-chromatic number for some certain of trees.*

1. INTRODUCTION

The locating-chromatic number of a graph was firstly studied by Chartrand *et al.* [7] in 2002. This concept is derived from graph coloring and partition dimension [9].

We use a simple graph $G = (V, E)$ and connected. Let c be a proper coloring using k colors, namely $1, 2, \ldots, k$. Let $\Pi = \{S_1, S_2, \cdots, S_k\}$ be a partition of $V(G)$, induced by c and S_i is the color classes received the color i. The color code, $c_\Pi(v) = (d(v, S_1), d(v, S_2), \ldots, d(v, S_k))$, where $d(v, S_i) = \min\{d(v, x)|x \in S_i\}$ for $i \in [1, k]$. If all vertices in $V(G)$ have different color codes, then c is called a *locating-chromatic k-coloring* of G. Minimum k such that G has a locating coloring called the locating-chromatic number, denoted by $\chi_L(G)$. A vertex $u \in S_i$, for some i, is *dominant* if $d(v, S_j) = 1$ for $j \neq i$.

Study on the locating-chromatic number is still challenging to date because there is no formula to determine the locating-chromatic number

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of tree in general. Chartrand *et al.* [7] discussed the locating-chromatic numbers paths, caterpillars, cycles, stars, double stars, and complete multipartite graphs. Next, Asmiati *et al.* [1] studied the locating-chromatic number for amalgamation of stars, non homogeneous amalgamation of stars [4], and The locating-chromatic number of firecracker graphs [2], whereas Baskoro and Purwasih [6] for corona product of graphs.

In Characterizing locating-chromatic number of a graph, Chartrand *et al.* [8] determined characterizing graph with locating-chromatic number n, $n-1$, or $n-2$. Asmiati and Baskoro [3] characterized graphs containing cycle with locating-chromatic number three and Baskoro *et al.* [5] found all trees having locating-chromatic number three.

2. THE LOCATING-CHROMATIC NUMBER OF GRAPH $nT_{k,m}$

First, we will discuss about the locating -chromatic number of graph $nT_{k,m}$, but before will be given construction of graph $nT_{k,m}$ as shown below.

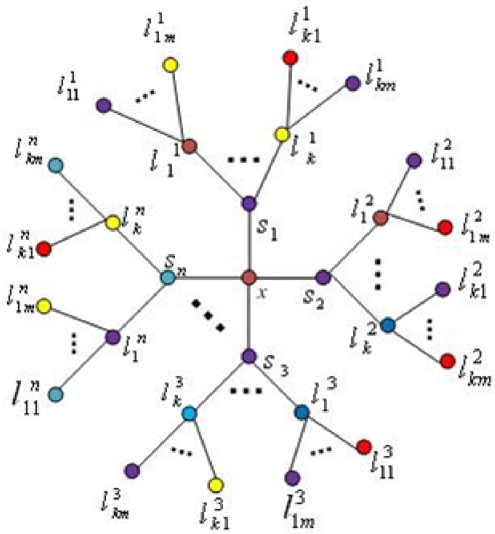

Figure 1: Construction of $nT_{k,m}$.

Theorem 2.1 *Locating-chromatic number of graph $nT_{k,m}$, for $m \geq k$ integer and $k \geq 2$ is $m+1$, where $1 \leq n \leq \lfloor \frac{m+1}{k} \rfloor$.*

Proof. We shall determine the lower bound of graph $nT_{k,m}$, for $m \geq k$ integer and $k \geq 2$. Observe that each vertex l_j^i, $j \in [1,k]$ and $i \in [1,n]$ adjacent to m leaves whose the same distance to other vertices. So, at least we need $m+1$-locating coloring of graph $nT_{k,m}$, for $m \geq k$ integer and $k \geq 2$.

Next, We determine the upper bound of graph $nT_{k,m}$, for $m \geq k$ integer and $k \geq 2$. Consider the $(m+1)$-coloring c on $nT_{k,m}$. Without loss of generality, we assign $c(x) = 1$ and s_i, for $i \in [1,n]$ is colored by $2, 3, \ldots, \lfloor \frac{m+1}{k} \rfloor$. The vertices l_j^i, $c(l_j^i) = \{1, 2, \ldots, m+1\} \backslash \{c(s_i)\}$, for $j \in [1,k]$ and $i \in [1,n]$. To make sure that the leaves will have distinct color code, we assign combination of $c(l_{jt}^i) = \{1, 2, \ldots, m+1\} \backslash \{c(l_j^i)\}$, for $j \in [1,k]$, $i \in [1,n]$, and $t \in [1,m]$. We show that the color codes for all vertices in $nT_{k,m}$, for $m \geq k$ integer and $k \geq 2$, are different.

- If $c(x) = c(l_j^i)$, then $c_\Pi(x)$ contains $\lfloor \frac{m+1}{k} \rfloor$ components have value 1, whereas $c_\Pi(l_j^i)$ contains m components have value 1. We know that $\lfloor \frac{m+1}{k} \rfloor < m$. So, $c_\Pi(x) \neq c_\Pi(l_{jt}^i)$.
- Consider $c(s_i) = m$. If $c(x) = c(l_{jt}^i)$, then m^{th}-component in $c_\Pi(x)$ has value 1, whereas in $c_\Pi(l_{jt}^i)$ has value 2. As a result, $c_\Pi(x) \neq c_\Pi(l_{jt}^i)$.
- If $c(s_r) = c(l_j^i)$, where $r \neq i$, then l_j^i must be a dominant vertex, but s_r is not. So, $c_\Pi(s_r) \neq c_\Pi(l_j^i)$.
- If $c(s_r) = c(l_{jt}^i)$, then $c_\Pi(s_r)$ contains at least two components have value 1, whereas $c_\Pi(l_{jt}^i)$ contains exactly one component has value 1. So, $c_\Pi(k_r) \neq c_\Pi(l_{jt}^i)$.
- If $c(l_{jt}^i) = c(l_{rp}^q)$, then $c_\Pi(l_{jt}^i) \neq c_\Pi(l_{rp}^q)$, since $c(l_r^p) \neq c(l_j^i)$.
- If $c(l_n^i) = c(l_{jt}^m)$, then $c_\Pi(l_n^i)$ contains exactly m components have value 1, whereas $c_\Pi(l_{jt}^m)$ contains exactly one component has value 1. So, $c_\Pi(l_n^i) \neq c_\Pi(l_{jt}^m)$.

From all above cases, We see that the color code for each vertex in $nT_{k,m}$ is unique, then c is a locating-coloring. Therefore, $\chi_L(nT_{k,m}) \leq m+1$, for $m \geq k$ integer and $k \geq 2$. □

3. THE LOCATING-CHROMATIC NUMBER OF $F^*_{n,k}$

A firecracker graph $F_{n,k}$, namely the graph obtained by the concatenation of n stars S_k by linking one leaf from each star [2]. Let $V(F_{n,k}) = \{x_i, m_i, l_{ij} | i \in [1,n]; j \in [1,k-2]\}$, and $E(F_{n,k}) = \{x_i x_{i+1} | i \in [1,n-1]\} \cup \{x_i m_i, m_i l_{ij} | i \in [1,n]; j \in [1,k-2]\}$. If we give subdivision one vertex y_i in edge $x_i m_i$, we denote $F^*_{n,k}$ with n,k natural numbers.

Theorem 3.2 *Let $F^*_{n,k}$ be a subdivision firecracker graphs. Then,*

i. $\chi_L(F^*_{n,4}) = 4$, *for* $n \geq 2$.

ii. For $k \geq 5$,

$$\chi_L(F^*_{n,k}) = \begin{cases} k-1, & \text{if } 1 \leq n \leq k-1, \\ k, & \text{otherwise.} \end{cases}$$

Proof

First, we determine the lower bound of $F^*_{n,4}$, for $n \geq 2$. It is clearly that $\chi_L(F^*_{n,4}) \geq 3$. For a contradiction assume that there exists a 3-locating coloring c on $F^*_{n,4}$, $n \geq 1$. If the colors are $1, 2,$ and 3, then $\{c(m_1), c(l_{11}), c(l_{12})\} = \{c(m_2), c(l_{21}), c(l_{22})\} = \{1,2,3\}$ but $c(m_1) \neq c(m_2)$. Now consider $c(y_i)$ for $i = 1, 2$. Since we have only 3 colors, then $c(y_i) = c(l_{ij})$ for some $j = \{1,2\}$. Therefore $c_\Pi(y_i) = c_\Pi(l_{ij})$ for some $i, j = \{1,2\}$, a contradiction. So, $\chi_L(F^*_{n,4}) \geq 4$ for $n \geq 2$.
To show that $\chi_L(F^*_{n,4}) \leq 4$ for $n \geq 2$, consider the 4-coloring c on $F^*_{n,4}$ as follows:

- $c(x_i) = 3$ for odd i, 2 for even i;
- $c(y_i) = 2$ for odd i, 1 for even i;
- $c(m_i) = 3$ for odd i and 2 for even i;
- for all vertices l_{ij}, define

$$c(l_{ij}) = \begin{cases} 4 & \text{if } i = 1, j = 1, \\ 1 & \text{if } i \geq 2, j = 1, \\ 2 & \text{if } i \textit{ is odd}, j = 2, \\ 3 & \text{if } i \textit{ is even}, j = 2. \end{cases}$$

The coloring c will create a partition Π on $V(F^*_{n,4})$. We shall show that all vertices in $F^*_{n,4}$ have different color codes. We have $c_\Pi(x_i) = (2, 1, 0, i+2)$ for odd i and $c_\Pi(x_i) = (1, 0, 1, i+2)$ for even i. For y_1, $c_\Pi(y_1) = (3, 0, 1, 2)$ and for $i \geq 2$ $c_\Pi(y_i) = (3, 0, 1, i+3)$ for odd i, $c_\Pi(y_i) = (0, 1, 2, i+3)$ for even i. Next, we have $c_\Pi(m_i) = (1, 1, 0, i+4)$ for odd i and $c_\Pi(m_i) = (1, 0, 1, i+4)$ for even i. For vertices $l_{i,j}$, $c_\Pi(l_{11}) = (5, 2, 1, 0)$ and $c_\Pi(l_{12}) = (5, 0, 1, 2)$. For $i \geq 2$, $c_\Pi(l_{i1}) = (0, 1, 2, i+5)$, $c_\Pi(l_{ij}) = (0, 2, 1, i+5)$ for odd i and $c_\Pi(l_{ij}) = (2, 1, 0, i+5)$ for even i. Since all vertices in $F^*_{n,4}$ have different color codes, thus c is a locating-chromatic coloring. So $\chi_L(F^*_{n,4}) \leq 4$.
Next, we show that for $k \geq 5$, $\chi_L(F^*_{n,k}) = k$ if $n \geq k$, and $\chi_L(F^*_{n,k}) = k-1$ if $1 \leq n \leq k-1$. To show this, we divide two cases.

Case 1. For $k \geq 5$ and $1 \leq n \leq k-1$.

Since each vertex l_i is adjacent to $(k-2)$ leaves, clearly that $\chi_L(F^*_{n,k}) \geq k-1$ for $k \geq 5$ and $1 \leq n \leq k-1$.
Next, Define a $(k-1)$-coloring c of $F^*_{n,k}$, as follows. Assign $c(m_i) = i$, for $i \in [1, n]$. Leaves, $\{l_{ij} | j \in [1, k-2]$ by $\{1, 2, \ldots, k-1\} \backslash \{i\}$, for any i. Next, $c(y_i) = 2$, for odd i and 1 for even i. $c(x_i) \neq c(m_i)$ for $i \in [1, k-1]$. As a result, coloring c will create a partition $\Pi = \{U_1, U_2, \cdots, U_{k-1}\}$ on $V(F^*_{n,k})$, where U_i is the set of the vertices by color i.
We show that all vertices in $F^*_{n,k}$ for $k \geq 5$, $n \leq k-1$ have different color codes. Let $u, v \in V(F_{n,k})$ and $c(u) = c(v)$. Then, for some i, j, h, l and $i \neq j$ consider the following cases.

- If $u = l_{ih}, v = l_{jl}$, then $c_\Pi(u) \neq c_\Pi(v)$ since $d(u, U_i) \neq d(v, U_i)$.
- If $u = l_{ih}, v = y_j$, then $c_\Pi(u)$ contains exactly one component have value 1, whereas $c_\Pi(v)$ contains at least two components have value 1. So, $c_\Pi(u) \neq c_\Pi(v)$.
- If $u = l_{ih}, v = x_j$, then $c_\Pi(u)$ contains exactly one component have value 1, whereas $c_\Pi(v)$ contains at least two components have value 1. So, $c_\Pi(u) \neq c_\Pi(v)$.
- If $u = m_i, v = x_j$, then u must be a dominant vertex but v is not. Thus, $c_\Pi(u) \neq c_\Pi(v)$.
- If $u = x_i$ and $v = x_j$, then $c_\Pi(u) \neq c_\Pi(v)$ since $c(m_i) \neq c(m_j)$.
- If $v = y_i, w = x_j$, then $c_\Pi(v) \neq c_\Pi(w)$ since $d(v, U_i) \neq d(w, U_i)$.

From all above cases, we see that all vertices in $F^*_{n,k}$ for $k \geq 5$, $n \leq k-1$ have different color codes, thus $\chi_L(F^*_{n,k}) \leq k-1$.
For an illustration, we give the locating-chromatic coloring of $F^*_{4,5}$ in Figure 2.

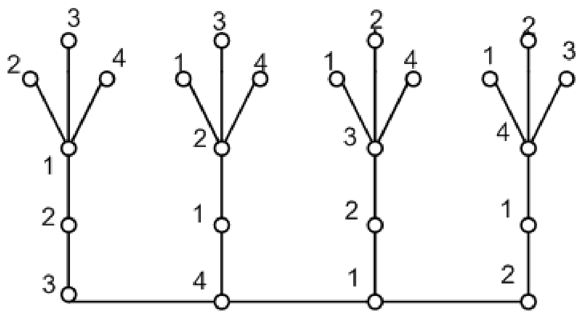

Figure 2: A locating-chromatic coloring of $F^*_{4,5}$.

Case 2. For $k \geq 5$ and $n \geq k$.

We determine the lower bound for $k \geq 5$ and $n \geq k$. It is clearly that $\chi_L(F^*_{n,k}) \geq k-1$. For a contradiction assume we have a $(k-1)$-locating coloring c on $F^*_{n,k}$ for $k \geq 5$ and $n \geq k$. Since $n \geq k$, then there are two i, j, $i \neq j$, such that $\{c(l_{it})|t = 1, 2, \cdots, k-2\} = \{c(l_{jl})|l = 1, 2, \cdots, k-2\}$. Therefore the color codes of m_i and m_j are the same, a contradiction. So, we have $\chi_L(F^*_{n,k}) \geq k-1$, for $n \geq k$.

Next, we determine the upper bound of $F^*_{n,k}$ for $k \geq 5$, $n \geq k$. To show that $F^*_{n,k} \leq k$ for $k \geq 5$ and $n \geq k$, consider the locating-coloring c on $F^*_{n,k}$ as follows:

- $c(x_i) = 1$ if i is odd and $c(x_i) = 3$ if i is even;
- $c(m_i) = 2$ for every i;
- $c(y_i) = 2$ for every i;
- If $B = \{1, 2, \ldots, k\}$, define:

$$\{c(l_{ij})|j = 1, 2, \ldots, k-2)\} = \begin{cases} B \backslash \{1, k-1\} & \text{if } i = 1, \\ B \backslash \{1, k\} & \text{otherwise.} \end{cases}$$

It is easy to verify that all vertices have different color codes. Therefore, c is a locating-chromatic coloring on $F^*_{n,k}$, and so $\chi_L(F^*_{n,k}) \leq k$, for $n \geq k$. This completes the proof. □

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