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The Relation between Almost Noetherian Module, Almost Finitely Generated Module and *T*-Noetherian Module

The Relation between Almost Noetherian Module, Almost Finitely Generated Module and T-Noetherian Module

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Abstract. In this paper we study the relation between almost Noetherian modules, almost finitely generated (a.f.g.) modules, and T-Noetherian modules. We show that if $R' = \{r \in R | rM \neq M\}$ and M is an almost Noetherian (a.f.g. *resp.*) R-module, then M is an (R')-Noetherian module. We also obtain that for any multiplicatively closed subset T of a ring R and $R' = \{r \in R | rM \neq M\}$, if M is an almost Noetherian (a.f.g. *resp.*) R-module and $T \cap R' \neq \emptyset$, then M is $(T \cap R')$-Noetherian. Moreover, we show that if M is an almost Noetherian (a.f.g. *resp.*) R-module and $T \cap R' \neq \emptyset$, then M is an T-Noetherian module for every multiplicatively closed set $T \subseteq R$. Finally, we apply the results of this paper on the structure of Generalized Power Series Module (GPSM) $M[[S]]$.

1. Introduction

Armendariz [2] introduces the concepts of an almost Noetherian module, which is a generalization of Noetherian modules. An R-module M is called almost Noetherian if every proper submodule of M is finitely generated. One of the examples of an almost Noetherian $\mathbb{Z}$-module which is not Noetherian is p-quasicyclic group $\mathbb{Z}_{p^\infty}$. In fact, based on Gilmer and O'Malley [5], any almost Noetherian $\mathbb{Z}$-module is either Noetherian or isomorphic to $\mathbb{Z}_{p^\infty}$ for a suitable prime p.

Furthermore, Weakly [10] give the definition of an a.f.g. modules, i.e., a module that is not finitely generated but every its proper submodules are finitely generated. It is clear that every a.f.g. modules are almost Noetherian modules.

In [1], Anderson and Dumitrescu introduce the definition of T-Noetherian rings and modules. For any multiplicatively closed subset T of a ring R, an R-module M is called a T-Noetherian module if for each submodule N of M, there exist an element $t \in T$ and a finitely generated submodule F of M such that $Nt \subseteq F \subseteq N$. Some properties of T-Noetherian modules studied by Baeck, Lee, and Lim [3].

In a case of Noetherian modules, Varadarajan [8] give the necessary and sufficient conditions for the module of polynomials $M[X]$, the module of Laurent polynomials $M[X, X^{-1}]$, and the module of power series $M[[X]]$ to be Noetherian modules. As a generalization of these modules, Varadarajan [9] constructed the Generalized Power Series Modules (GPSM) $M[[S]]$, i.e. a module over Generalized Power Series Rings (GPSR) $R[[S]]$ whose constructed by Ribenboim

[6]. Moreover, Varadarajan [9] determined the necessary and sufficient conditions for GPSM $M[[S]]$ to be a Noetherian module, which strengthens earlier results of Ribenboim [7].

Furthermore, Faisol, Surodjo, and Wahyuni [4] give the sufficient conditions for $R[X]$-module $M[X]$, $R[X,X^{-1}]$-module $M[X,X^{-1}]$, $R[[X]]$-module $M[[X]]$, and $R[[X,X^{-1}]]$-module $M[[X,X^{-1}]]$ to be $T[X]$-Noetherian, $T[X,X^{-1}]$-Noetherian, $T[[X]]$-Noetherian, and $T[[X,X^{-1}]]$-Noetherian, respectively, where T is a multiplicatively and also additively closed subset of ring R.

In this paper, we investigate the relation between almost Noetherian modules, a.f.g. modules, and T-Noetherian modules. As the main result of this paper, we obtain the sufficient condition for an R-module M to be T-Noetherian related to almost Noetherian R-module and a.f.g. R-module. Furthermore, we apply the main result of this paper on the structure of GPSM $M[[S]]$.

2. Main Results

In this section, we investigate the relation between almost Noetherian modules, a.f.g. modules, and T-Noetherian modules.

Now, let M be an R-module and $R' = \{r \in R | rM \neq M\}$. It is easy to show that R' is a multiplicatively closed subset of R. The following proposition shows that if M is an almost Noetherian module, then M is (R')-Noetherian.

Proposition 2.1 *Let R be a ring, M an R-module and $R' = \{r \in R | rM \neq M\}$. If M is almost Noetherian, then M is (R')-Noetherian.*

Proof. For any $r \in R'$ and every submodule N of M,

$$rN \subseteq rM \subset M.$$

Hence, rN is a proper submodule. Since M is almost Noetherian, rN is a finitely generated submodule of M. Therefore, for any submodule N of M, there exist an element $r \in R'$ and a finitely generated submodule $F = rN$ of M such that

$$rN \subseteq F \subseteq N.$$

So, M is (R')-Noetherian. ■

The consequence of Proposition 2.1 is given by the following corollary.

Corollary 2.2 *Let R be a ring, M an R-module and $R' = \{r \in R | rM \neq M\}$. If M is an a.f.g. module, then M is (R')-Noetherian.*

Next, for any multiplicative subset T of R, clearly $T \cap R'$ is also a multiplicative subset of R. The following proposition shows that if M is almost Noetherian as an R-module and $(T \cap R') \neq 0$, then M is $(T \cap R')$-Noetherian.

Proposition 2.3 *Let R be a ring, T a multiplicative subset of R, M an R-module and $R' = \{r \in R | rM \neq M\}$. If M is an almost Noetherian module and $(T \cap R') \neq \emptyset$, then M is $(T \cap R')$-Noetherian.*

Proof. For any $a \in (T \cap R')$ and every submodule N of M,

$$aN \subseteq aM \subset M.$$

Hence, aN is a proper submodule. Since M is a.f.g., aN is a finitely generated submodule of M. Therefore, for any submodule N of M, there exist an element $a \in (T \cap R')$ and a finitely generated submodule $F = aN$ of M such that

$$aN \subseteq F \subseteq N.$$

So, M is $(T \cap R')$-Noetherian. ■

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The consequence of Proposition 2.3 is given by the following corollary.

Corollary 2.4 *Let R be a ring, T a multiplicative subset of R, M an R-module and $R' = \{r \in R | rM \neq M\}$. If M is an a.f.g. module and $(T \cap R') \neq \emptyset$, then M is $(T \cap R')$-Noetherian.*

Next, we recall the properties of a Noetherian R-module related to two multiplicative subsets of ring R.

Lemma 2.5 *[3] Let T_1, T_2 are multiplicative subsets of a ring R. If $T_1 \subseteq T_2$, then any T_1-Noetherian R-module is T_2-Noetherian.*

By using Lemma 2.5, we obtain the sufficient conditions for an R-module M to be T-Noetherian related to almost Noetherian R-module and a.f.g. R-module as follows.

Theorem 2.6 *Let R be a ring, T a multiplicative subset of R, M an R-module and $R' = \{r \in R | rM \neq M\}$. If M is an almost Noetherian R-module and $(T \cap R') \neq \emptyset$, then M is T-Noetherian.*

Proof. Since M is an almost Noetherian R-module, based on Proposition 2.3, M is $(T \cap R')$-Noetherian. Furthermore, since $(T \cap R') \subseteq T$, based on Lemma 2.5, M is a T-Noetherian R-module. ■

Theorem 2.7 *Let R be a ring, T a multiplicative subset of R, M an R-module and $R' = \{r \in R | rM \neq M\}$. If M is an a.f.g. module and $(T \cap R') \neq \emptyset$, then M is T-Noetherian.*

Proof. Since M is an a.f.g. R-module, based on Corollary 2.4, M is $(T \cap R')$-Noetherian. Furthermore, since $(T \cap R') \subseteq T$, based on Lemma 2.5, M is a T-Noetherian R-module. ■

3. Application on Generalized Power Series Modules

In this section, we apply the results of the previous section on the structure of Generalized Power Series Modules (GPSM) $M[[S]]$.

Regarding ordered sets, strictly ordered monoids, Artinian and narrow sets, we will be following the terminology in [6] and [7].

An ordered set $(S, \leq)$ is Artinian if every strictly decreasing element of S is finite, and $(S, \leq)$ is Noetherian if every strictly increasing element of S is finite.

Lemma 3.1 *[6] Let $(S, \leq)$ be any ordered set.*

(1) If S is Artinian (Noetherian) and $X \subseteq S$, then X is Artinian (Noetherian).

(2) If $X_1, X_2, ..., X_n$ are Artinian (Noetherian) subsets of S, then $\bigcup_{i=1}^{n} X_i$ is Artinian (Noetherian).

An ordered set $(S, \leq)$ is said to be *narrow* if S does not contain an infinite subset consisting of pairwise incomparable elements.

Lemma 3.2 *[6] Let $(S, \leq)$ be any order set.*

(1) If S narrow and $X \subseteq S$, then X is narrow.

(2) If $X_1, X_2, ..., X_n$ are narrow subsets of S, then $\bigcup_{i=1}^{n} X_i$ is narrow.

Lemma 3.3 *[7] If X, Y are Artinian and narrow subsets of $(S, \leq)$, then $X + Y = \{s + t | s \in X, t \in Y\}$ is also Artinian and narrow.*

Now, we recall the construction of GPSR and GPSM as follows from Ribenboim [6] and Varadarajan [9].

Let $(S, \leq)$ be a strictly ordered monoid, R a commutative ring with an identity element and M an R-module. Let $R^S = \{f|f : S \to R\}$ and

$$R[[S]] = \{f \in R^S|supp(f) \text{ is Artinian and narrow }\},$$

where $supp(f) = \{s \in S|f(s) \neq 0\}$.

For any $f, g \in R[[S]]$, $supp(f+g) \subseteq supp(f) \cup supp(g)$, $supp(-f) = supp(f)$, and $supp(fg) \subseteq supp(f) + supp(g)$. Therefore, under pointwise addition and convolution multiplication defined by

$$(fg)(s) = \sum_{(x,y)\in\chi_s(f,g)} f(x)g(y), \tag{1}$$

for all $f, g \in R[[S]]$ where

$$\chi_s(f, g) = \{(x, y) \in supp(f) \times supp(g)|xy = s\}$$

is finite, $R[[S]]$ becomes a ring which is known as Generalized Power Series Ring (GPSR).

Next, let $M^S = \{\alpha|\alpha : S \to M\}$ and

$$M[[S]] = \{\alpha \in M^S|supp(\alpha) \text{ is Artinian and narrow }\},$$

where $supp(\alpha) = \{s \in S|\alpha(s) \neq 0\}$.

For any $\alpha, \beta \in M[[S]]$, $supp(\alpha + \beta) \subseteq supp(\alpha) \cup supp(\beta)$, $supp(-\alpha) = supp(\alpha)$, and $supp(\alpha\beta) \subseteq supp(\alpha) + supp(\beta)$. Therefore, under pointwise addition and scalar multiplication defined by

$$(f\alpha)(s) = \sum_{(x,y)\in\chi_s(f,\alpha)} f(x)\alpha(y), \tag{2}$$

for all $f \in R[[S]]$ and $\alpha \in M[[S]]$ where

$$\chi_s(f, \alpha) = \{(x, y) \in supp(f) \times supp(\alpha)|xy = s\}$$

is finite, $M[[S]]$ acquires the structure of an $R[[S]]$-module. Next, this module is called Generalized Power Series Module (GPSM).

For any $r \in R$ and any $s \in S$, we associated the maps $c_r, e_s \in R[[S]]$, defined by

$$c_r(t) = \begin{cases} r & \text{if } t = 1 \\ 0 & \text{if } t \neq 1, \end{cases}$$

and

$$e_s(t) = \begin{cases} 1 & \text{if } t = s \\ 0 & \text{if } t \neq s. \end{cases}$$

For any $m \in M$ and any $s \in S$, we define $d_m^s \in M[[S]]$ by

$$d_m^s(t) = \begin{cases} m & \text{if } t = s \\ 0 & \text{if } t \neq s. \end{cases}$$

Then, it is clear that $r \mapsto c_r$ is a ring embedding of R into $R[[S]]$, and $s \mapsto e_s$ is a monoid embedding of S into the multiplicative monoid of the ring $R[[S]]$, and also $m \mapsto d_m^0$ is a module embedding of M into $M[[S]]$.

For any subset N of an R-module M, we define

$$N[[S]] = \{\gamma \in M[[S]] | \gamma(s) \in N; \forall s \in S\}.$$

The sufficient conditions of $N[[]S]]$ to be an $R[[S]]$-submodule of $M[[S]]$ are given by the following lemma.

Lemma 3.4 *Let R be a ring, M an R-module, and $(S, \leq)$ a strictly ordered monoid. If N is an R-submodule of M, then $N[[S]]$ is an $R[[S]]$-submodule of $M[[S]]$.*

Proof. First, it is clear that $N[[S]] \subseteq M[[S]]$. Next, we will show that for any $f, g \in R[[S]]$ and any $\alpha, \beta \in N[[S]]$, $f\alpha + g\beta \in N[[S]]$.

It is clear that $supp(f)$ and $supp(\alpha)$ are Artinian and narrow. Since $supp(f\alpha) \subseteq supp(f) + supp(\alpha)$ and $supp(g\beta) \subseteq supp(g) + supp(\beta)$, based on Lemma 3.3, Lemma 3.1, and Lemma 3.2 $supp(f\alpha)$ and $supp(g\beta)$ are Artinian and narrow. Furthermore, since $supp(f\alpha + g\beta) \subseteq supp(f\alpha) \cup supp(g\beta)$, based on Lemma 3.1 and 3.2 $supp(f\alpha + g\beta)$ is Artinian and narrow. In other words, $f\alpha + g\beta \in M[[S]]$.

Now, for any $s \in S$, we will show that $(f\alpha + g\beta)(s) \in N$. For any $f, g \in R[[S]]$ and any $\alpha, \beta \in M[[S]]$,

$$(f\alpha)(s) = \sum_{xy=s} f(x)\alpha(y)$$

and

$$(g\beta)(s) = \sum_{xy=s} g(x)\beta(y).$$

Since N is an R-submodule of M, we have $(f\alpha)(s) \in N$ and $(g\beta)(s) \in N$. Hence, we obtain $(f\alpha + g\beta)(s) = (f\alpha)(s) + (g\beta)(s) \in N$. In other words, it is prove that $N[[S]]$ is an $R[[S]]$-submodule of $M[[S]]$. ∎

It is clear that, if N is an R-submodule of M, then $rN \subseteq rM$ for any $r \in R$. Therefore, according to Lemma 3.4, $fN[[S]] \subseteq fM[[S]]$ for any $f \in R[[S]]$.

Next, for any subset T of a ring R, we define the set

$$T[[S]] = \{f \in R[[S]] | f(s) \in T; \forall s \in S\}.$$

It is clear that $T[[S]] \subseteq R[[S]]$. The sufficient conditions of $T[[S]]$ to be a multiplicatively closed subset of GPSR $R[[S]]$ are given by the following lemma.

Lemma 3.5 *Let R be a ring, $T \subseteq R$, $(S, \leq)$ a strictly ordered monoid, and $R[[S]]$ a GPSR. If $T \subseteq R$ is closed under the operations of R, then $T[[S]]$ is a multiplicatively closed subset of $R[[S]]$.*

Proof. For any $f, g \in T[[S]]$, we will show that $fg \in T[[S]]$. Based on the convolution multiplication in equation (1), for any $s \in S$ we obtain

$$\begin{aligned}(fg)(s) &= \sum_{(x,y)\in\chi_s(f,g)} f(x)g(y) \\ &= \sum_{xy=s} f(x)g(y)\end{aligned}$$

Since $T \subseteq R$ is closed under the operations of ring R, we have $\sum_{xy=s} f(x)g(y) \in T$. In other words, $fg \in T[[S]]$. Thus, $T[[S]]$ is a multiplicatively closed subset of $R[[S]]$. ■

As a direct result of Lemma 3.5 above, if we choose $S = \mathbb{N} \cup \{0\}$ with a trivial order $\leq$, we obtain [4, Lemma 6]. If we choose $S = \mathbb{Z}$ with a trivial order $\leq$, we obtain [4, Lemma 7(1)]. Next, if we choose $S = \mathbb{N} \cup \{0\}$ with a usual order $\leq$, we obtain [4, Lemma 7(2)]. Furthermore, if we choose $S = \mathbb{Z}$ with a usual order $\leq$, we obtain [4, Lemma 7(3)].

Now, let M be an R-module and $R' = \{r \in R | rM \neq M\}$, we defined the set

$$R'[[S]] = \{g \in R[[S]] | g(s) \in R'; \forall s \in S\}.$$

It is clear that $R'[[S]] \subseteq R[[S]]$, and it is easy to show that $R'[[S]]$ is a multiplicatively closed subset of $R[[S]]$.

The following lemma shows that if $M[[S]]$ is an $R[[S]]$-module, then $gM[[S]] \neq M[[S]]$ for any $g \in R'[[S]]$.

Lemma 3.6 *Let R be a ring, M an R-module, $R' = \{r \in R | rM \neq M\}$, $(S, \leq)$ a strictly ordered monoid. Then for any $g \in R'[[S]]$, $gM[[S]] \neq M[[S]]$.*

Proof. Suppose that for any $g \in R'[[S]]$ and for an $R[[S]]$-module $M[[S]]$, $gM[[S]] = M[[S]]$. In other words, $gM[[S]] \subset M[[S]]$ and $M[[S]] \subset gM[[S]]$. Therefore, for any $\alpha \in M[[S]]$, then $\alpha \in gM[[S]]$. Hence, for any $s \in S$ and $\beta \in M[[S]]$, we obtain

$$\alpha(s) = (g\beta)(s) = \sum_{xy=s} g(x)\beta(y) \in M. \tag{3}$$

In the other side, since $g(s) \in R'$ for all $s \in S$, we have $g(s)M \neq M$. Therefore, $\sum_{xy=s} g(x)\beta(y)$ in equation (3) not necessary in M. Thus, a contradiction. ■

Next, we apply Proposition 2.1 to the structure of GPSM $M[[S]]$ and we obtain the following proposition.

Proposition 3.7 *Let R be a ring, M an R-module, N a proper R-submodule of M, $R' = \{r \in R | rM \neq M\}$, and $(S, \leq)$ a strictly ordered monoid. If $M[[S]]$ is an almost Noetherian $R[[S]]$-module, then $M[[S]]$ is $R'[[S]]$-Noetherian.*

Proof. Based on Lemma 3.4, it is clear that for any R-submodule N of M, $N[[S]]$ is an $R[[S]]$-submodule of $M[[S]]$. Therefore, based on Lemma 3.6,

$$gN[[S]] \subseteq gM[[S]] \subset M[[S]]$$

for any $g \in R'[[S]]$. Then, $gN[[S]]$ is a proper $R[[S]]$-submodule of $M[[S]]$.

Since $M[[S]]$ is almost Noetherian, we have $gN[[S]]$ is a finitely generated $R[[S]]$-submodule of $M[[S]]$. Therefore, for any $R[[S]]$-submodule $N[[S]]$ of $M[[S]]$, there exist an element $g \in R'[[S]]$ and a finitely generated submodule $K = gN[[S]]$ of $M[[S]]$ such that

$$gN[[S]] \subseteq K \subseteq N[[S]].$$

So, $M[[S]]$ is an $R'[[S]]$-Noetherian $R[[S]]$-module. ■

Now, we apply Proposition 2.3 on the structure of GPSM $M[[S]]$, and we obtain the following proposition.

Proposition 3.8 *Let R be a ring, T an additively and a multiplicatively closed subset of ring R, M an R-module, N a proper R-submodule of M, $R' = \{r \in R | rM \neq M\}$, and $(S, \leq)$ a strictly ordered monoid. If $M[[S]]$ is an almost Noetherian $R[[S]]$-module and $T[[S]] \cap R'[[S]] \neq \emptyset$, then $M[[S]]$ is $(T[[S]] \cap R'[[S]])$-Noetherian.*

Proof. Based on Lemma 3.4, it is clear that for any R-submodule N of M, $N[[S]]$ is an $R[[S]]$-submodule of $M[[S]]$. Based on Lemma 3.5, $T[[S]]$ is a multiplicative subset of $R[[S]]$. Therefore, based on Lemma 3.6,

$$gN[[S]] \subseteq gM[[S]] \subset M[[S]]$$

for any $g \in (T[[S]] \cap R'[[S]])$. Then, $gN[[S]]$ is a proper $R[[S]]$-submodule of $M[[S]]$.

Since $M[[S]]$ is almost Noetherian, we have $gN[[S]]$ is a finitely generated $R[[S]]$-submodule of $M[[S]]$. Since $T[[S]] \cap R'[[S]] \neq \emptyset$, there exist an element $g \in T[[S]] \cap R'[[S]]$ and a finitely generated submodule $K = gN[[S]]$ of $M[[S]]$ such that

$$gN[[S]] \subseteq K \subseteq N[[S]],$$

for every $R[[S]]$-submodule $N[[S]]$ of $M[[S]]$. So, $M[[S]]$ is an $(T[[S]] \cap R'[[S]])$-Noetherian $R[[S]]$-module. ■

Finally, we apply Theorem 2.6 on the structure of GPSM $M[[S]]$, and we get the following theorem.

Theorem 3.9 *Let R be a ring, T an additively and a multiplicatively closed subset of ring R, M an R-module, N a proper R-submodule of M, $R' = \{r \in R | rM \neq M\}$, and $(S, \leq)$ a strictly ordered monoid. If $M[[S]]$ is an almost Noetherian $R[[S]]$-module and $T[[S]] \cap R'[[S]] \neq \emptyset$, then $M[[S]]$ is $T[[S]]$-Noetherian.*

Proof. Since $M[[S]]$ is an almost Noetherian $R[[S]]$-module, based on Proposition 3.8 $M[[S]]$ is an $(T[[S]] \cap R'[[S]])$-Noetherian $R[[S]]$-module. Furthermore, since $(T[[S]] \cap R'[[S]]) \subseteq T[[S]]$, then based on Lemma 2.5, we obtain $M[[S]]$ is $T[[S]]$-Noetherian. ■

4. Conclusion

An R-module M is T-Noetherian if M is almost Noetherian (a.f.g. resp.) and $T \cap R' \neq \emptyset$, for any multiplicatively closed subset T of ring R and $R' = \{r \in R | rM \neq M\}$.

On the structure of GPSM, $M[[S]]$ is $T[[S]]$-Noetherian if $M[[S]]$ is almost Noetherian and $T[[S]] \cap R'[[S]] \neq \emptyset$, for an additively and a multiplicatively closed subset T of ring R and $R' = \{r \in R | rM \neq M\}$.

References

[1] Anderson, D., D., and Dumitrescu, T., S-Noetherian Rings, *Communications in Algebra*, **30** (2002), 4407-4416.
[2] Armendariz, E.P., Rings with an Almost Noetherian Ring of Fraction, *Math. Scand.*, **41** (1977), 15-18.
[3] Baeck, J., Lee, G., and Lim, J.W., S-Noetherian Rings and Their Extensions, *Taiwanese Journal of Mathematics*, **20(6)** (2016), 1231-1250.
[4] Faisol, A., Surodjo, B., and Wahyuni, S., The Sufficient Conditions for $R[X]$-module $M[X]$ to be $S[X]$-Noetherian, *European Journal of Mathematical Sciences*, **5(1)** (2019), 1-13.
[5] Gilmer, R., and O'Malley, M., Non-Noetherian Rings for which each Proper Subring is Noetherian, *Math. Scand.*, **31** (1972), 118-122.
[6] Ribenboim, P., Generalized Power Series Rings, *In Lattice, Semigroups and Universal Algebra*, Plenum Press, New York, (1990), 271-277.
[7] Ribenboim, P., Noetherian Rings of Generalized Power Series, *Journal of Pure and Applied Algebra*, **79** (1992), 293-312.
[8] Varadarajan, K., A Generalization of Hilbert's Basis Theorem, *Communications In Algebra*, **10** (1982), 2191-2204.
[9] Varadarajan, K., Generalized Power Series Modules, *Communications In Algebra*, **29(3)** (2001), pp 1281-1294.
[10] Weakley, W.D., Modules whose Proper Submodules are Finitely Generated, *J. Algebra*, **84** (1983), 189-219.