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Far East Journal of Mathematical Sciences (FJMS) © 2017 Pushpa Publishing House, Allahabad, India http://www.pphmj.com http://dx.doi.org/10.17654/MS102030645 Volume 102, Number 3, 2017, Pages 645-654 ISSN: 0972-0871 Received: March 24, 2017; Accepted: June 5, 2017 2010 Mathematics Subject Classification: 62J12. Keywords and phrases: longitudinal data, outlier, regression model.

A ROBUST PROCEDURE FOR GEE MODEL Netti Herawati and Khoirin Nisa Department of Mathematics University of Lampung Indonesia Abstract In longitudinal studies, multiple measurements are taken on the same subject at different points in time. Thus, observations for the same subject are correlated. This pape r proposes a robust procedure for estimating parameters of regression model when generalized estimating equation (GEE) applied to longitudinal data that contains outliers.

The procedure is a combination of the iteratively reweighted least square (IRLS) and least tri mmed square (LTS) methods and is called iteratively reweighted least trimmed square (IRLTS). We conducted a simulation study for gamma model and Poisson model using the proposed method, the result shows that our approach can provide a better result than the classical GEE. 1.

Introduction In statistics, generalized estimati ng equation (GEE) [5] is used to estimate the parameters of a generalized linear model (GLM) [6] with a possible unknown correlation between outcomes. It is a general statistical approach to fit a marginal model for longitudinal data analysis, and it has been popularly applied into clinical trials and biomedical studies.

GEEs belong to a class of regression techniques that are referred to as Netti Herawati and Khoirin Nisa 646 semiparametric because they rely on specification of only the first two moments. Under correct model specification and mild regularity conditions, parameter estimates from GEEs are cons istent. The generalized estimating equation approach requires correct specification of the first two moments of a model.

However, these moment assumptions can be distorted by contaminated or irregular measurements namely outliers. As a result, the generalized estimating equation method fa ils to give consistent estimators, and more seriously this will lead to incorrect conclusions [1, 8]. In this situation, we need a robust method that can minimize the effect of outliers.

In recent years a few studies have considered robust methods for longitudinal data analysis, see e.g. [1, 2, 4, 8, 11]. In this paper, we combine the IRLS and LTS for obtaining a robust estimation of GEE when data contain outliers. We have shown the effectiveness of this procedure for normal model [7]. In this paper we apply the proposed procedure to gamma and Poisson models. 2.

Generalized Estimating Equation Let the vector of measurements on the ith subject be [] T niiii YY ...,, 1 =Y with corresponding vector of means [] T niiii μμ= ...,, 1 μ and = i X [] T niii XX ...,, 1 be the pn i × matrix of covariates. In general, the components of i Y are correlated but i Y and k Y are independent for any .ki ? To model the relation between th e response and covariates, we can use a regression model similar to the generalized linear models: ( ) ,ßX?μ iii g = = where () ,E iii XYμ |= g is a specified link function, and [] T p ßß= ...,, 1 ß is a vector of unknown regression coefficients to be estimated.

The GEE for estimating the 1×p vector of regression parameter ß is given by: () () [] ? = - =-? ?= K i iii T i S 1 1 ,0ßμYVß μß (1) A Robust Procedure for GEE Model 647 where i V be the covariance matrix of i Y modeled as () , 2121 iii AaRAV ?= i A is a diagonal matrix of variance functions ( ) , ij v μ and ( ) a R is the working correlation matrix of i Y indexed by a vector of parameters .a Solutions to equation (1) are obtained by alternating between estimation of ? and .a

There are several specific choices of the form of working correlation matrix () aR i commonly used to model the correlation matrix of , i Y among them are exchangeable and autoregressive correlation matrices. Solving for ß is done with iteratively rewe ighted least squares (IRLS). The following is an algorithm for fitting the specified model using GEEs as described in [3] and [8]: 1.

Compute an initial estimate of ,ˆ GEE ß for example with an ordinary generalized linear model assuming independence. 2. A current estimate GEE ßˆ is updated by regressing the working response vector () µyß µßXZ ˆˆ -? ?+= * on X. A new estimate new ßˆ is obtained by: () ,ˆ 1 ** - * = ZWXXWXß TTnew where * W is a block diagonal weight matrix whose ith block is the ii nn × matrix () .ˆ

1111 ----* ? ? ?? ? ? ? ?? ? ?? ? ? ? ?= ß µAaRAß µW iiiiii 3. Use new ßˆ to update ,ˆˆ * == HZßX? new where () . 1 * - * = WXXWXXH TT 4. Iterate until convergence. 3. Iterated Reweighted Least Trimmed Square Let us briefly recall that the robust estimation of regression parameters Netti Herawati and Khoirin Nisa 648 using LTS [9] method is given by: ? = = h i iLTS e 1 2,minargˆß where 222 2 2 1 nh eeee ===== "" are the ordered squared residuals, from smallest to largest.

LTS is calculated by minimizing the h ordered squares residuals, where h can be chosen between the range ,4 1 4 312 ++==+ nnhn with n being sample size and number of parameters, respectively. One can refer to e.g. [9, 10] for some details on LTS method. The IRLTS procedure is stated in the following short algorithm. To motivate this method, it is convenient to write the algorithm with involving the residuals. 1.

Compute an initial estimate of GEE ßˆ using IRLS, use the estimate to calculate fitted value: () .ˆˆ 1 ßXµ ii g - = 2. Calculate residuals: .ˆ ijijij Ye µ - = Sort ij e for ti ...,,2,1 = nj ...,,2,1= in ascending order: . 1211 m eee = = = " 3. Choose h observations which have the lowest h-residuals, we denote as subset H. 4. Improve estimates of ß new ßˆ based on subset H using IRLS. 5. Iterate until convergence.

4. Simulation Study We compare the performances of IRLTS and IRLS through simulation study. Two types of outcomes are considered, continuous and count responses. The models for data generation are as follows: A Robust Procedure for GEE Model 649 ,1 22110 ijijij xx ß + ß + ß = µ () , 22110 ijijij xxLog ß + ß + ß = µ where k ß 's for 2,1,0=k are randomly generated, 200...,,2,1 = i and .5...,,2,1=j The covariates ij x1 are i.i.d.

from a uniform distribution () ,5,1Unif and 2 x is the measurement time variable, i.e., .5,4,3,2,1 2 = i x For each scenario, we generate th e data based on the underlying true correlation structures as exchangeable and autoregressive with .5.0 = a = e Table 1. The expected values, standard errors and MSEs of i ßˆ for gamma distributed model with exchangeable correlation matrix Method Classical GEE IRLTS 0 ˆß e () 0 ˆ ß E () 0 ˆ ß SE MSE () 0 ˆ ß E () 0 ˆ ß SE MSE 5% 1.833421 0.417356 0.419098 1.292539 0.280084 0.080563 10% 1.963008 0.551804 0.694453 1.343505 0.345998 0.119739 20% 2.105067 0.522999 0.861099 1.302799 0.245013 0.061308 30% 2.158723 0.712847 1.180859 1.432421 0.581205 0.346613 Netti Herawati and Khoirin Nisa 650 1 ˆ ß e () 1 ˆ ßE () 1 ˆ ßSE MSE () 1 ˆ ßE () 1 ˆ ßSE MSE 5% 0.664892 0.297197 0.205406 0.957103 0.034595

0.003693 10% 0.587415 0.348410 0.297492 0.968876 0.075239 0.007119 20% 0.314225 0.347369 0.600688 0.964959 0.039372 0.003323 30% 0.297561 0.338291 0.617832 0.937923 0.088894 0.012682 2 ˆ ß e () 2 ˆ ßE () 2 ˆ ßSE MSE () 2 ˆ ßE () 2 ˆ ßSE MSE 5% 0.743453 0.289166 0.174850 0.974127 0.039936 0.006689 10% 0.549449 0.423348 0.425292 0.958451 0.018566 0.007922 20% 0.355655 0.349199 0.597828 0.975630 0.032734 0.005954 30% 0.332191 0.336312 0.621917 0.969043 0.071512 0.010960 Table 2.

The expected values, standard errors and MSEs of i ßˆ for gamma distributed model with autoregressive correlation matrix Method Classical GEE IRLTS 0 ˆ ß e () 0 ˆ ßE () 0 ˆ ßSE MSE () 0 ˆ ßE () 0 ˆ ßSE MSE 5% 1.713422 0.174403 0.127509 1.169724 0.219998 0.102270 10% 2.051958 0.292341 0.476604 1.172668 0.169948 0.081395 20% 2.096749 0.279746 0.561178 1.182726 0.281815 0.127424 30% 2.208355 0.570003 0.975395 1.094088 0.105856 0.105908 1 ˆ ß e () 1 ˆ ßE () 1 ˆ ßSE MSE () 1 ˆ ßE () 1 ˆ ßSE MSE 5% 0.746269 0.169773 0.087549 0.977105 0.032371 0.001180 10% 0.493787 0.257591 0.311197 0.974237 0.026179 0.000892 20% 0.290721 0.257720 0.553459 0.982542 0.035247 0.001279 30% 0.228004 0.231963 0.595925 0.981851 0.023087 0.000579 2 ˆ ß e () 2 ˆ ßE () 2 ˆ ßSE MSE () 2 ˆ ßE () 2 ˆ ßSE MSE 5% 0.739142 0.224797 0.132593 0.981539 0.021532 0.002405 10% 0.511909 0.302973 0.355672 0.981291 0.028264 0.002762 20% 0.281145 0.270400 0.627330 0.983924 0.028529 0.002551 30% 0.225163 0.377600 0.646722 1.003760 0.035796 0.001758 A Robust Procedure for GEE Model 651 As shown in Table 1 and Table 2, our approach (IRLTS) performs better than the classical GEE.

The MSEs of IRLTS are smaller than the MSEs of classical GEE, the outliers influence the estimation of 10 ˆ,ˆ ßß and .ˆ 2 ß The parameter estimates of classical GEE are much more influenced than the parameter estimates of IRLTS. The more outliers contained in the data the larger the deviation of classical GEE estimates from the parameter value.

In Table 3 and Table 4, the behavior of MSEs of both methods is the same as the first case, here we can see that IRLTS performs better than the classical GEE because the MSEs of IRLTS are sm aller than the MSEs of classical GEE. Table 3. The expected values, standard errors and MSEs of i ßˆ for Poisson distributed model with exchangeable correlation matrix Method Classical GEE IRLTS 0 ˆ ß e () 0 ˆ ßE () 0 ˆ ßSE MSE () 0 ˆ ßE () 0 ˆ ßSE MSE 5% 7.851928 0.578389 49.012922 2.242959 0.411831 1.041084 10% 9.299508 0.776268 71.575965 2.084935 0.454712 1.670854 20% 10.710302 0.425014 96.915019 2.351178 0.546663 1.478125 30% 11.413164 0.621123 111.439999 1.988002 0.486656 1.475743 1 ˆ ß e () 1 ˆ ßE () 1 ˆ ßSE MSE () 1 ˆ ßE () 1 ˆ ßSE MSE 5% 0.309635 0.108437 0.418891 1.052717 0.059291 0.014543 10% 0.194983 0.044670 0.568586 1.011790 0.074132 0.009602 20% 0.097441 0.077894 0.729016 0.918343 0.038557 0.002349 30%

0.092040 0.031461 0.733153 0.809874 0.101063 0.029211 2 ˆ ß e () 2 ˆ ßE () 2 ˆ ßSE MSE () 2 ˆ ßE () 2 ˆ ßSE MSE 5% 0.281547 0.064601 0.478304 0.947537 0.021808 0.000986 10% 0.159566 0.060507 0.660655 0.967633 0.065401 0.004284 20% 0.087528 0.031481 0.779957 0.900333 0.028748 0.005696 30% 0.067109 0.035320 0.816673 0.825752 0.057643 0.024164 Netti Herawati and Khoirin Nisa 652 The result for Poisson model shows similar behavior to the result for gamma model.

For the result of Poisson model in Table 3 and Table 4, IRLTS also performs better than the classical GEE. The MSEs of IRLTS are smaller than the MSEs of classical GEE, the outliers influence the estimation of ,ˆ 0 ß 1 ˆß and .ˆ 2 ß The parameter estimates of classical GEE are much more influenced than the parameter estimates of IRLTS.

The more outliers contained in the data the larger the deviation of classical GEE estimates from the parameter value. In Table 3 and Ta ble 4, the behavior of MSEs of both methods is the same as the first case, here we can see that IRLTS performs better than the classical GEE. The estimation of IRLTS yields better results than classical GEE for both cases we considered here.

The MSEs of IRLTS is smaller than classical GEE, this means that IRLTS can reduce the influence of the high leverage points better than the classical GEE. Table 4. The expected values, standard errors and MSEs of i ßˆ for Poisson distributed model with autoregressive correlation matrix Method Classical GEE IRLTS 0 ˆ ß e () 0 ˆ ßE () 0 ˆ ßSE MSE () 0 ˆ ßE () 0 ˆ ßSE MSE 5% 8.493574 0.689463 40.055071 1.614313 0.331758 0.455831 10% 9.788645 0.962589 58.478701 1.765065 0.204619 0.523307 20% 10.525858 0.877176 70.050502 2.356166 0.470975 0.724548 30% 10.635832 0.570803 71.449706 3.071576 0.974599 1.054251 1 ˆ ß e () 1 ˆ ßE () 1 ˆ ßSE MSE () 1 ˆ ßE () 1 ˆ ßSE MSE 5% 0.309635 0.108437 0.418891 1.052717 0.059291 0.014543 10% 0.194983 0.044670 0.568586 1.011790 0.074132 0.009602 20% 0.097441 0.077894 0.729016 0.918343 0.038557 0.002349 30% 0.092040 0.031461 0.733153 0.809874 0.101063 0.029211 A Robust Procedure for GEE Model 653 2 ˆ ß e () 2 ˆ ßE () 2 ˆ ßSE MSE () 2 ˆ ßE () 2 ˆ ßSE MSE 5% 0.281547 0.064601 0.478304 0.947537 0.021808 0.000986 10% 0.159566 0.060507 0.660655 0.967633 0.065401 0.004284 20% 0.087528 0.031481 0.779957 0.900333 0.028748 0.005696 30% 0.067109 0.035320 0.816673 0.825752 0.057643 0.024164 5.

Concluding Remark In this paper, we have shown that our proposed procedure can minimize the effect of outliers on parameter estimation; IRLTS can produce a relatively efficient and consistent estim ator compared to the classical GEE (IRLS). Based on the MSE, IRLTS performs much better than the classical GEE for gamma and Poisson models.

Acknowledgement The authors would like to thank the anonymous referees for their valuable suggestions and comments th at led to a considerably improved manuscript.
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Applied Mathematical Sciences, Vol. 10, 2016, no. 63, 3107 - 3118 HIKARI Ltd, www.m-hikari.com https://doi.org/10.12988/ams.2016.69238 Empirical Comparison of ML and UMVU Estimators of the Generalized Variance for some Normal Stable Tweedie Models: a Simulation Study Khoirin Nisa1 Department of Mathematics, Lampung University, Indonesia C´elestin C.

Kokonendji Laboratoire de Math´ematiques de Besan¸con Universit´e Bourgogne Franche-Comt´e, France Asep Saefuddin, Aji Hamim Wigena Department of Statistics, Bogor Agricultural University, Indonesia I Wayan Mangku Department of Mathematics, Bogor Agricultural University, Indonesia Copyright c 2016 Khoirin Nisa, C´elestin C. Kokonendji, Asep Saefuddin, Aji Hamim Wigena and I Wayan Mangku.

This article is distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited. Abstract This paper discuss a comparison of the maximum likelihood (ML) estimator and the uniformly minimum variance unbiased (UMVU) es- timator of generalized variance for some normal stable Tweedie models through simulation study. We describe the estimation of some particular cases of multivariate NST models, i.e.

normal gamma, normal Poisson 1Also a?liated to Bogor Agricultural University, Indonesia and Universit´e Bourgogne Franche-Comt´e, france 3108 Khoirin Nisa et al. and normal invers-Gaussian. The result shows that UMVU method pro- duces better estimations than ML method on small samples and they both produce similar estimations on large samples.

Mathematics Subject Classi?cation: 62H12 Keywords: Multivariate natural exponential family, variance function, maximum likelihood, uniformly minimum variance unbiased 1 Introduction Normal stable Tweedie (NST) models were introduced by Boubacar Ma¨inassara and Kokonendji [3] as the extension of normal gamma [5] and normal inverse Gaussian [4] models.

NST models are composed by a ?xed univariate stable Tweedie variable having a positive value domain, and the remaining random variables given the ?xed one are real independent Gaussian variables with the same variance equal to the ?xed component. For a k-dimensional ( k = 2) NST random vector X = ( X1 ,ÿ.ÿ.ÿ.ÿ,ÿXk)>, the generating s-?nite positive mea- sure ?a,t is given by ?a,t( dx) = ?a,t( dx1) k Y j=2 ?2,x 1( dxj) , (1) where ?a,t is the well-known probability measure of univariate positive s-stable distribution generating L´evy process ( Xa t )t>0 which was introduced by Feller [7] as follows ?a,t( dx) = 1 pÿx 8 X r=0 trG(1 + aÿr) sin(- rÿpÿa) r! ar( a- 1)-r [(1- a) x]ar 1x>0 dx = ?a,t( x) dx. (2) Here a? (0 , 1) is the index parameter, G( .)

is the classical gamma function, and IA denotes the indicator function of any given event A that takes the value 1 if the event accurs and 0 otherwise. Paremeter a can be extended into a? (-8 , 2] [10]. For a = 2 in (2), we obtain the normal distribution with density ?2,t( dx) = 1 v2 pÿt exp _- x2 2 t _ dx. In multivariate analysis, including NST models, generalized variance has important roles in descriptive analysis and inferences.

In this paper we dis- cuss the ML and UMVU generalized variance estimators of the following NST models: Comparison of ML and UMVU e stimators ... 3109 1. Normal gamma (NG). For a = 0 in (1) one has the generating measure of normal gamma as follows: ?0,t( dx) = xt-1 1 (2 pÿx1)(k-1)/2 ?( t) exp - x1- 1 2 x1 k X j=2 x2 j ! Ix 1>0 dx1 dx2··· xk .

(3) It is a member of simple quadratic natural exponential families (NEFs) [6] and was called as "gamma-Gaussian" which was characterized by Kokonendji and Masmoedi [8]. 2. Normal invers Gaussian (NIG). For a = 1 /2 in (1) we can write the normal inverse Gaussian generating measure as follows ?1/2,t( dx) = tx-(k+2)/2 1 (2 p)k/2 exp " -1 2 x1 k X j=2 x2 j t2 + k X j=2 x2 j !# Ix 1>0 dx1 dx2··· xk .

(4) It was introduced as a variance-mean mixture of a univariate inverse Gaussian with multivariate Gaussian distribution [4] and has been used in ?nance (see e.g. [1, 2]). 3. Normal Poisson (NP). For the limit case a =-8 in (1) we have the so-called normal Poisson generating measure ?-8,t( dx) = tx 1( x1!)-1 (2 pÿx1)(k-1)/2 exp - t- 1 2 x1 k X j=2 x2 j ! Ix 1? N * dx 1( dx1) dx2··· xk .

(5) Since it is also possible to have x1 = 0 in the Poisson part, the correspond- ing normal Poisson distribution is degenerated as d0. This model is recently characterized by Nisa et al. [9] 2 Generalized Variance of NST Models The cumulant function K?a,t(?) = log R Rk exp ?Tx _ ?a,t( dx) of NST models is given by K?a,t(?) = K?a,t ?1 + 1 2 k X j=2 ?2 j ! (6) where K?a,t = log R Rk exp( ?ÿx) ?a,t( dx) is the cumulant function of the associ- ated univariate stable Tweedie distribution ?a,t.

Then for each distribution we 3110 Khoirin Nisa et al. discuss here the corresponding cumulant function is given by K?a,t(?) = ? ? ? ? ? ? ? ? ? ? ? ? t exp _ ?i + 1 2 Pk j=2 ?2 j _ , for NG - t _1 2 _ - ?i- 1 2 Pk j=2 ?2 j _ _ , for NIG - t log _ - ?i- 1 2 Pk j=2 ?2 j _ , for NP (7) (see [3, Section 2]).

The cumulant function is ?nite for? in canonical domain T( ?a,t) ={?? Rk; ?1 + 1 2 k P j=2 ?2 j? T( ?a,1)} with T( ?a,1) = ? ? ? (-8 , 0) for NG (-8 , 0] for NIG R for NP. Let G( ?a,t) = _ P (?; aÿ,ÿt);? = ( ?1 ,ÿ.ÿ.ÿ.ÿ,ÿ?k)>? T( ?a,t) be the set of prob- ability distributions P (?; aÿ,ÿt)( dx) = exp _?>x- K?a,t(?) _ ?a,t( dx) .

The vari- ance function which is the variance-covariance matrix in term of mean param- eterization; P (µ; Ga,t) := P [?(µ); ?a,t]; is obtained through the second deriva- tive of the cumulant function, i.e. V Ga,t(µ) = K00 ?a,t[?(µ)] whereµ = K0 ?a,t(?). Then calculating the determinant of the variance function will give the gen- eralized variance.

We summarize the variance function and the generalized variance of NG, NIG and NP models in Table 1. Table 1: Variance Function and Generalized Variance Model V Ga,t(µ) ? = detV Ga,t(µ) NG (1 /t)µµ> + diagk(0 ,ÿµ1 ,ÿ.ÿ.ÿ.ÿ,ÿµ1) (1 /t) µk+1 1 NIG ( µ1 /t2)µµ> + diagk(0 ,ÿµ1 ,ÿ.ÿ.ÿ.ÿ,ÿµ1) (1 /t2) µk+2 1 NP (1 /µ1)µµ> + diagk(0 ,ÿµ1 ,ÿ.ÿ.ÿ.ÿ,ÿµ1) µk 1 The ML and UMVU estimators of the generalized variance in Table 1 are stated in the following proposition. Proposition 1 Let X1 ,ÿ.ÿ.ÿ.ÿ,ÿXn be random vectors with distribution P(?; aÿ,ÿt)? G( ?p,t) in a given NST family. Denoting X = ( X1+ .ÿ.ÿ.+ Xn) /n = ( X1 ,ÿ.ÿ.ÿ.ÿ, Xk)T the sample mean with positive ?rst component X1, the ML estimator of the generalized variance of NG, NP and NIG models is given by: Tn;k;t = det V Gp,t( X) = ? ? ? ? ? (1 /t) X k+1 1 , for NG X k 1 , for NP (1 /t2) X k+2 1 , for NIG Comparison of ML and UMVU e stimators ...

3111 and the UMVU estimator is given by Un;k,t = ? ? ? ? ? ? ? ? ? ? ? ? ? ? tkG( nt)[G( nt + k + 1)]-1Sn i=1 xk+1 (1i) , for NG n-k[Sn i=1 x(1i)][Sn i=1 x(1i)- 1]··· [Sn i=1 x(i1)- k + 1] ,ÿn= k for NP tk2-1-k/2[G(1 + kÿ/2)]-1Sn i=1 x3/2 (1i) exp _( nt)2 /[2Sn i=1 x(1i)] × R Sn i =1x (1i ) 0 yk/2 1 [Sn i=1 x(1i)- y1]-3/2× exp _- y1- [( nt)2 /2[Sn i=1 x(1i)- y1]] dy1 , for NIG (see Boubacar Ma¨inassara and Kokonendji , [3]) 3 Simulation Study In order to examine the behavior of ML and UMVU estimators empirically we carried out a simulation study. We run Monte-Carlo simulations using R software.

We set several sample sizes ( n) varied from 3 to 1000 and we generated 1000 samples for each n. We consider k = 2 , 4 , 6 to see the e?ects of k on generalized variance estimations. For simplicity we set μ1 = 1. Moreover, to see the e?ect of zero values proportion within X1 in the case of normal Poisson, we also consider small mean values on the Poisson component i.e.

μ1 = 0 .5 because P( X1 = 0) = exp(- μ1). We report the numerical results of the generalized variance estimations for each model, i.e. the empirical expected value of the estimators with its standard errors (Se) and the empirical mean square error (MSE). We calculated the mean square error (MSE) of each method over 1000 data sets using the following formula: MÿSÿE( b ?) = 1 1000 1000 X i=1 n b ?i- det V Ga,t(μ) o2 (8) where b ? is the estimate of det V Ga,t(μ) using ML and UMVU estimators. 3.1

Normal gamma We generated normal gamma distribution samples using the generating s-?nite positive measure ?a,t of normal gamma in (1). Table 2 show the expected values of generalized variance estimates with their standard errors (in parentheses) and the means square error values of both ML and UMVU methods in case of normal gamma.

From the result in Table 2 we can observe di?erent performances of ML estimator ( Tn;k,t) and UMVU estimator ( Un;k,p,t) of the generalized variance. The expected values of Tn;k,t converge while the values of Un;k,t do not, but Un;k,t is always closer to the parameter than Tn;k,t for small sample sizes, i.e. 3112 Khoirin Nisa et al. for n= 30, this shows that UMVU is an unbiased estimator while ML is an asymtotically unbiased estimator.

For the two methods, the standar error of the estimates decreases when the sample size increase. Table 2: The expected values (with empirical standard errors) and MSE of Tn;k,t and Un;k,t for normal-gamma with 1000 replications for given target value μk+1 1 = 1 with k?{2 , 4 , 6} . Expected values and Standard errors MSE k n T n ; k ,t U n ; k ,t T n ; k ,t U n ; k ,t 2 3 1.9805 (3.7192) 0.8912 (1.6736) 14.7935 2.8128 10 1.2878 (1.2875) 0.9756 (0.9754) 1.7405 0.9520 20 1.1648 (0.8236) 1.0085 (0.7131) 0.7054 0.5085 30 1.0998 (0.6031) 0.9978 (0.5471) 0.3736 0.2994 60 1.0380 (0.4115) 0.9881 (0.3917) 0.1708 0.1536 100 1.0231 (0.3152) 0.9931 (0.3060) 0.0999 0.0937 300 1.0036 (0.1774) 0.9936 (0.1757) 0.0315 0.0309 500 1.0076 (0.1365) 1.0016 (0.1357) 0.0187 0.0184 1000 1.0110 (0.0953) 1.0080 (0.0950) 0.0092 0.0091 4 5 4.2191 (13.3899) 0.8720 (2.7674) 189.6509 7.6750 10 2.3799 (5.0869) 0.9906 (2.1174) 27.7810 4.4837 20 1.6461 (2.0572) 1.0328 (1.2906) 4.6494 1.6668 30 1.3831 (1.3505) 1.0066 (0.9828) 1.9707 0.9660 60 1.1904 (0.8014)

1.0117 (0.6811) 0.6784 0.4640 100 1.0869 (0.5706) 0.9849 (0.5171) 0.3332 0.2676 300 1.0293 (0.2938) 0.9957 (0.2842) 0.0872 0.0808 500 1.0286 (0.2296) 1.0083 (0.2251) 0.0535 0.0507 1000 1.0137 (0.1610) 1.0036 (0.1594) 0.0261 0.0254 6 7 13.7175 (103.5833) 1.3062 (9.8634) 10891.2275 97.3811 10 6.6118 (36.8236) 1.1467 (6.3866) 1387.4736 40.8103 20 2.2455 (4.3052) 0.8670 (1.6622) 20.0860 2.7806 30 1.9055 (3.4774) 0.9905 (1.8076) 12.9123 3.2676 60 1.4151 (1.5070) 1.0092 (1.0748) 2.4434 1.1553 100 1.2248 (0.8843) 0.9972 (0.7199) 0.8325 0.5183 300 1.0606 (0.4416) 0.9894 (0.4119) 0.1986 0.1698 500 1.0182 (0.3160) 0.9765 (0.3030) 0.1002 0.0924 1000 1.0228 (0.2311) 1.0016 (0.2263) 0.0539 0.0512 To examine the consistency of the estimators we have to look at their MSE.

The result shows that when n increases the MSE of the two methods become more similar and they both produced almost the same result for n = 1000. The MSE values for n= 10 in the table are presented graphically in Figure 1. In the ?gure we can easily see that all estimators become more similar when the sample size increase.

For small sample sizes, UMVU always has smaller MSE, in this situation UMVU is preferable than ML. The ?gure also shows that the di?erence between ML and UMVU for small sample sizes increases when the dimension increases. Comparison of ML and UMVU e stimators ...

3113 (a) k=2 (b) k=4 (c) k=6 Figure 1: Bargraphs of the mean square errors of Tn;k,t and Un;k,t for normal- gamma with n?{10 , 20 , 30 , 60 , 100 , 300 , 500 , 1000} and k?{2 , 4 , 6}. 3.2 Normal inverse-Gaussian The result for normal inverse-Gaussian is presented in Table 3. Similar with normal gamma, the result for normal inverse-Gaussian shows that UMVU method produced better estimates than ML method for small sample sizes.

From the result we can conclude that the two estimators are consistent. The bargraph of MSE values for n= 10 in Table 3 is presented in Figure 2. Notice that the result for this case is similar to the normal gamma case, i.e. for small sample sizes the di?erence between the MSEs of ML and UMVU estimators for normal inverse-Gaussian also increases when k increases.

(a) k=2 (b) k=4 (c) k=6 Figure 2: Bargraphs of the mean square errors of Tn;k,t and Un;k,t for normal inverse Gaussian with n?{10 , 20 , 30 , 60 , 100 , 300 , 500 , 1000} and k?{2 , 4 , 6}. 3.3 Normal Poisson The simulation results for normal Poisson are presented in Table 4 and Table 5 for µ1 = 1 and µ1 = 0 .5 respectively. In this simulation, the proportion of zero values in the samples increases when the mean of the Poisson component becomes smaller.

For normal-Poisson distribution with μj = 0 .5, we have many zero values in the samples. However, this situation does not a?ect the 3114 Khoirin Nisa et al. Table 3: The expected values (with standar errors) and MSE of Tn;k,t and Un;k,t for normal inverse-Gaussian with 1000 replications for given target value μk+2 1 = 1 and k?{2 , 4 , 6} . Expected values and Standard errors MSE k n T n ; k ,t U n ; k ,t T n ; k ,t U n ; k ,t 2 3 2.0068 (4.9227) 0.9135 (0.8235) 25.2469 0.6856 10 1.4249 (2.8513) 1.0316 (0.4388) 8.3103 0.1935 20 1.5936 (1.8951) 1.1340 (0.3718) 3.9439 0.1562 30 1.3677 (1.0155) 1.1641 (0.2668) 1.1664 0.0981 60 1.0846 (0.5341) 1.1104 (0.1856) 0.2924 0.0466 100 1.0819 (0.5166) 1.1102 (0.1675) 0.2735 0.0402 300 1.0006 (0.2570) 1.0843 (0.0919) 0.0660 0.0156 500 1.0356 (0.1890) 1.1374 (0.0727) 0.0370 0.0242 1000 1.0156 (0.1219) 1.0116 (0.0670) 0.0151 0.0115 4 5 9.3836 (30.0947) 1.3196 (1.1323) 975.9726 1.3843 10 4.6547 (13.8643) 1.2837 (0.8153) 205.5754 0.7452 20 2.7487 (5.1845) 1.2963 (0.6189) 29.9373 0.4709 30 1.4822 (2.1166) 1.1854 (0.4572) 4.7125 0.2434 60 1.3095 (1.1051) 1.2560 (0.3054) 1.3170 0.1588 100 1.1673 (0.8467) 1.2264 (0.2671) 0.7449 0.1226 300 1.0849 (0.4296) 1.2542 (0.1520) 0.1918 0.0877 500 1.0350 (0.2839) 1.0762 (0.0914) 0.0818 0.0416 1000 1.0107 (0.2080) 1.0102 (0.1137) 0.0434 0.0337 6 7 20.4865 (113.4633) 0.9423 (0.9984) 12056.9414 1.0001 10 12.1032 (55.7841) 1.0596 (0.8610) 2329.5787 0.7449 20 3.4498 (10.3056) 1.0054 (0.5933) 112.2060 0.3520 30 2.1422 (3.2262) 1.0246 (0.4970) 11.7130 0.2476 60 1.8236 (2.6064) 1.0587 (0.3744) 7.4717 0.1436 100 1.2468 (1.1599) 1.0129 (0.2643) 1.4062 0.1170 300 1.0781 (0.4953) 1.0568 (0.1596) 0.2514 0.0929 500 1.0815 (0.4065) 1.0230 (0.1110) 0.1719 0.0922 1000 1.0207 (0.2816) 1.0204 (0.0775) 0.0798 0.0760 generalized variance estimation as we can see that Tn;k,t and Un;k,t have the same behavior for both values of μ1.

The MSE in Table 4 and 5 for n= 10 are displayed as bargraphs presented in Figure 3 and Figure 4. From those ?gures we see that UMVU is preferable than ML because it always has smaller MSE values when sample sizes are small ( n 6 30). 4 Conclusion In this paper we have discussed the generalized variance estimator of normal gamma, normal inverse-Gaussian and normal Poisson models using ML and UMVU methods. The simulation studies of the generalized variance estimators Comparison of ML and UMVU e stimators ...

3115 Table 4: The expected values (with standar errors) and MSE of Tn;k,t and Un;k,t for normal Poisson with 1000 replications for given target value μk 1 = 1 and k?{2 , 4 , 6} . Expected values and Standard errors MSE k n T n ; k ,t U n ; k ,t T n ; k ,t U n ; k ,t 2 3 1.3711 (1.4982) 1.0349 (1.3130) 2.3824 1.7252 10 1.0810 (0.6589) 0.9817 (0.6286) 0.4407 0.3955 20 1.0424 (0.4471) 0.9925 (0.4363) 0.2017 0.1904 30 1.0329 (0.3817) 0.9996 (0.3756) 0.1468 0.1411 60 1.0184 (0.2661) 1.0017 (0.2639) 0.0711 0.0697 100 1.0066 (0.2016) 0.9966 (0.2006) 0.0407 0.0403 300 1.0112 (0.1153) 1.0079 (0.1151) 0.0134 0.0133 500 0.9986 (0.0942) 0.9966 (0.0941) 0.0089 0.0089 1000 0.9998 (0.0641) 0.9988

(0.0641) 0.0041 0.0041 4 5 2.6283 (5.0058) 1.0721 (2.7753) 27.7093 7.7075 10 1.7362 (2.2949) 1.0422 (1.6267) 5.8085 2.6480 20 1.3276 (1.1713) 1.0073 (0.9588) 1.4793 0.9193 30 1.2274 (0.8892) 1.0167 (0.7750) 0.8424 0.6008 60 1.1111 (0.5643) 1.0085 (0.5250) 0.3308 0.2757 100 1.0647 (0.4448) 1.0038 (0.4260) 0.2021 0.1815 300 1.0245 (0.2389) 1.0043 (0.2354) 0.0577 0.0554 500 1.0092 (0.1889) 0.9972 (0.1872) 0.0358 0.0351 1000 1.0013 (0.1272) 0.9953 (0.1267) 0.0162 0.0161 6 7 4.5153 (12.8404) 0.9378 (4.0255) 177.2319 16.2084 10 3.6865 (8.1473) 1.1642 (3.3992) 73.5952 11.5816 20 1.9674 (2.9034) 1.0227 (1.7467) 9.3656 3.0514 30 1.5605 (1.8825) 0.9901 (1.3133) 3.8580 1.7250 60 1.2954 (1.0360) 1.0220 (0.8541) 1.1606 0.7300 100 1.2084 (0.7824) 1.0462 (0.6957) 0.6556 0.4861 300 1.0621 (0.3793) 1.0109 (0.3641) 0.1477 0.1327 500 1.0294 (0.2778) 0.9992 (0.2710) 0.0780 0.0734 1000 1.0185 (0.1939) 1.0034 (0.1915) 0.0379 0.0367 (a) k=2 (b) k=4 (c) k=6 Figure 3: Bargraphs of the mean square errors of Tn;k,t and Un;k,t for normal Poisson with μ1 = 1, n?{10 , 20 , 30 , 60 , 100 , 300 , 500 , 1000} and k?{2 , 4 , 6}. 3116 Khoirin Nisa et al.

Table 5: The expected values (with standard errors) and MSE of Tn;k,t and Un;k,t for normal Poisson with 1000 replications for given target value μk 1 = 0 .5k and k?{2 , 4 , 6} . Expected values and Standard errors MSE k n T n ; k ,t U n ; k ,t T n ; k ,t U n ; k ,t 2 3 0.3930 (0.5426) 0.2320 (0.4223) 0.3148 0.1787 10 0.2868 (0.2421) 0.2378 (0.2212) 0.0600 0.0491 20 0.2652 (0.1660) 0.2407 (0.1583) 0.0278 0.0251 30 0.2642 (0.1374) 0.2476 (0.1332) 0.0191 0.0177 60 0.2598 (0.0903) 0.2514 (0.0888) 0.0083 0.0079 100 0.2534 (0.0712) 0.2484 (0.0705) 0.0051 0.0050 300 0.2495 (0.0418) 0.2478 (0.0417) 0.0017 0.0017 500 0.2491 (0.0313) 0.2482 (0.0313) 0.0010 0.0010 1000 0.2495 (0.0221) 0.2490 (0.0221) 0.0005 0.0005 4 5 0.2999 (0.8462) 0.0685 (0.3474) 0.7724 0.1207 10 0.1696 (0.3115) 0.0689 (0.1750) 0.1085 0.0306 20 0.1089 (0.1541) 0.0658 (0.1097) 0.0259 0.0120 30 0.0886 (0.0894) 0.0617 (0.0689) 0.0087 0.0048 60 0.0774 (0.0559) 0.0642 (0.0487) 0.0033 0.0024 100 0.0704 (0.0403) 0.0627 (0.0370) 0.0017 0.0014 300 0.0643 (0.0207) 0.0618 (0.0201) 0.0004 0.0004 500 0.0635 (0.0158) 0.0620 (0.0156) 0.0003 0.0002 1000 0.0631 (0.0115) 0.0624 (0.0114) 0.0001 0.0001 6 7 0.2792 (1.2521) 0.0268 (0.2274) 1.6371 0.0519 10 0.1212 (0.3918) 0.0165 (0.0858) 0.1646 0.0074 20 0.0427 (0.0883) 0.0124 (0.0345) 0.0085 0.0012 30 0.0356 (0.0539) 0.0151 (0.0271) 0.0033 0.0007 60 0.0236 (0.0281) 0.0149 (0.0196) 0.0009 0.0004 100 0.0211 (0.0183) 0.0159 (0.0145) 0.0004 0.0002 300 0.0173 (0.0089) 0.0157 (0.0082) 0.0001 0.0001 500 0.0166 (0.0068) 0.0157 (0.0064) 0.0000 0.0000 1000 0.0164 (0.0044) 0.0159 (0.0043) 0.0000 0.0000 for the three models show that UMVU produces better estimation than ML for small sample sizes.

However, the two methods are consistent and they become more similar when the sample size increases. References
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3117 (a) k=2 (b) k=4 (c) k=6 Figure 4: Bargraphs of the mean square errors of Tn;k,t and Un;k,t for normal Poisson with μ1 = 0 .5, n?{10 , 20 , 30 , 60 , 100 , 300 , 500 , 1000} and k?{2 , 4 , 6}. [2] J. Andersson, On the normal inverse Gaussian stochastic volatility model, Journal of Business & Economic Statistics, 19 (2001), no. 1, 44-54. https://doi.org/10.1198/07350010152472607 [3] Y. Boubacar Ma¨inassara and C. C.

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Received: September 16, 2016; Published: November 14, 2016

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JIRSS (2015) Vol. 14, No. 2, pp 37-52 DOI:10.7508 / jirss.2015.02.003 Characterizations of Multivariate Normal-Poisson Model Khoirin Nisa 1 , 2 , 3 , Célestin C. Kokonendji 1 , Asep Saefuddin 2 1 University of Franche-Comté, Besançon, France. 2 Bogor Agricultural University, Bogor, Indonesia. 3 Lampung University, Bandar Lampung, Indonesia. Abstract.

Multivariate normal-Poisson model has been recently introduced as a special case of normal stable Tweedie models. The model is composed of a univariate Poisson variable, and the remaining variables given the Poisson one are independent Gaussian variables with variance the value of the Poisson component.

Two characterizations of this model are shown, ?rst by variance function and then by generalized variance function which is the determinant of the variance function. The latter provides an explicit solution of a particular Monge-Ampère equation. Keywords. Generalized variance, In?nitely divisible measure, Monge-Ampère equa- tion, Multivariate exponential family, Variance function. MSC: 62H05; 60E07.

1 Introduction Motivated by normal gamma and normal inverse Gaussian models, Boubacar Maïnas- sara and Kokonendji (2014) introduced a new form of generalized variance functions which are generated by the so-called normal stable Tweedie (NST) models of k -variate Khoirin Nisa( )(khoirin.nisa@univ-fcomte.fr), Célestin C. Kokonendji (celestin.kokonendji@univ- fcomte.fr), Asep Saefuddin (asaefuddin@ipb.ac.id) 38 Nisa et al. distributions ( k > 1 ).

The generating s -?nite positive measure µ a, t on R k of NST models is composed by the well-known probability measure ? a, t of univariate positive s -stable distribution generating L process ( X a t ) t > 0 which was introduced by Feller (1971) as follows ? a, t ( dx ) = 1 p x 8 ? r = 0 t r G (1 + a r ) sin ( - r pa ) r ! a r ( a - 1) - r [(1 - a ) x ] a r 1 x > 0 dx = ? a, t ( x ) dx , where a ? (0 , 1) is the index parameter, G ( .

) is the classical gamma function, and I A denotes the indicator function of any given event A that takes the value 1 if the event accurs and 0 otherwise. Paremeter a can be extended into a ? ( -8 , 2] (see Tweedie, 1984). For a = 2 , we obtain the normal distribution with density ? 2 , t ( dx ) = 1 v 2 p t exp ( - x 2 2 t ) dx . For a k -dimensional NST random vector X = ( X 1 ,ÿ.ÿ.ÿ.ÿ, X k ) ? , the generating s -?nite positive measure µ a, t is given by µ a, t ( d x ) = ? a, t ( dx 1 ) k ? j = 2 ? 2 , x 1 ( dx j ) , (1.1) where X 1 is a univariate (non-negative) stable Tweedie variable and ( X 2 ,ÿ.ÿ.ÿ.ÿ,ÿ, X k ) ? = : X c 1 given X 1 are k - 1 real independent Gaussian variables with variance X 1 .

Normal-Poisson model is a special case of NST models; it is new and the only model which has a discrete component. Among NST models, normal-gamma is the only model which is also a member of simple quadratic natural exponential families (NEFs) of Casalis (1996); she called it “gamma-Gaussian“ and it has been characterized by variance and generalized variance functions.

See Casalis (1996) or Kotz et al (2000, Chapter 54) for characterization by variance function, and Kokonendji and Masmoudi (2013) for characterization by generalized variance function which is the determinant of covariance matrix expressed in terms of the mean vector. In contrast to normal-gamma which is the same to gamma-Gaussian; normal- Poisson and Poisson-Gaussian (Kokonendji and Masmoudi , 2006; Koudou and Pom- meret , 2002) are two completely di ? erent models. Indeed, for any value of j ?ÿ{ 1 ,ÿ.ÿ.ÿ.ÿ, k } , normal-Poisson model has only one Poisson component and k - 1 Gaussian compo- nents, while a Poisson-Gaussian j model has j Poisson components and k - j Gaus- sian components which are all independent. Poisson-Gaussian is a particular case of the simple quadratic NEFs with variance function V F ( m ) = diag k ( m 1 ,ÿ.ÿ.ÿ.ÿ, m j , 1 ,ÿ.ÿ.ÿ.ÿ, 1) Characterizations of Multivariate Normal-Poisson Model 39 where m = ( m 1 ,ÿ.ÿ.ÿ.ÿ, m k ) ? is the mean vector, and its generalized variance function is det V F ( m ) = m 1 .ÿ.ÿ. m j .

Some characterizations of Poisson-Gaussian j models have been done by several authors such as Letac (1989) for variance function, Kokonendji and Masmoudi (2006) for generalized variance function, and Koudou and Pommeret (2002) for convolution-stability. Also one can see Kokonendji and Seshadri (1996); Kokonendji and Pommeret (2007) for the generalized variance estimators of Poisson-Gaussian.

This normal-Poisson is also di ? erent from the purely discrete "Poisson-normal" model of Steyn (1976), which can be de?ned as a multiple mixture of k independent Poisson distributions with parameters m 1 , m 2 ,ÿ.ÿ.ÿ.ÿ, m k and those parameters have a multivariate normal distribution. Three generalized variance estimators of normal Poisson model have been intro- duced (Kokonendji and Nisa , 2016).

In this paper we present the characterizations of multivariate normal-Poisson model by variance function and by generalized vari- ance function which is connected to the Monge-Ampère equation (Gutiérrez , 2001). In Section 2 we present some properties of normal-Poisson model. We present the characterizations of normal-Poisson model by variance function in Section 3 and the characterization by generalized variance in Section 4.

2 Normal-Poisson model By introducing "power variance" parameter p de?ned by ( p - 1)(1 - a ) = 1 and equiv- alent to p = p ( a ) = a - 2 a - 1 or a = a ( p ) = p - 2 p - 1 (see Jorgensen , 1997, Chapter 4, for complete description of the power unit variance function of univariate stable Tweedie distributions), in the case of a ?ÿ-8 or p = p ( -8 ) = 1 , Expression (1.1) will lead to k -variate normal-Poisson model.

Replacing a ( p ) with p ( a ) the generating measure of normal-Poisson model can be express as follows µ t ( d x ) = µ 1 , t ( d x ) = ? 1 , t ( dx 1 ) k ? j = 2 ? 0 , x 1 ( dx j ) . (2.1) Then by (2.1), for a ?xed power of convolution t > 0 , denote F t = F ( µ t ) the multi- variate NEF (Kotz et al , 2000, Chapter 54) of normal-Poisson with µ t : = µ * t , the NEF of a k -dimensional normal-Poisson random vector X is generated by µ t ( d x ) = t x 1 ( x 1 !) - 1 (2 p x 1 ) ( k - 1) / 2 exp ? ? ? ? ? ? ? ? - t - 1 2 x 1 k ? j = 2 x 2 j ? ? ? ? ? ? ? ? ? 1 x 1 ? N \{ 0 } d x 1 ( dx 1 ) k ? j = 2 dx j . (2.2) 40 Nisa et al.

Since t > 0 then µ t is known to be an in?nitely divisible measure; see, e.g., Sato (1999). The cumulant function which is the logarithm of the Laplace transform of µ t , i.e. K µ t ( ? ) = log ? R k exp( ? ? x ) µ t ( d x ) , is given by K µ t ( ? ) = t exp ? ? ? ? ? ? ? ? ? 1 + 1 2 k ? j = 2 ? 2 j ? ? ? ? ? ? ? ? ? . (2.3) The function K µ t ( ? ) is ?nite for all ? in the canonical domain T ( µ t ) = ? ? ? ? ? ? ? ? ? ? ? R k ; ? ? ˜ ? c 1 : = ? 1 + 1 2 k ? j = 2 ? 2 j < 0 ? ? ? ? ? ? ? ? ? (2.4) with ? = ( ? 1 ,ÿ.ÿ.ÿ.ÿ,ÿ? k ) ? and ˜ ? c 1 : = (1 ,ÿ? 2 ,ÿ.ÿ.ÿ.ÿ,ÿ? k ) ? . (2.5) The probability distribution of normal-Poisson which is a member of NEF is given by P ( ? ; µ t )( d x ) = exp { ? ? x - K µ t ( ? ) } µ t ( d x ) . From (2.3) we can calculate the ?rst derivative of the cumulant function that pro- duces a k -vector as the mean vector of F t , and also its second derivative which is a k × k matrix that represents the covariance matrix. Using notations in (2.5) we obtain K ' µ t ( ? ) = K µ t ( ? ) × ˜ ? c 1 and K '' µ t ( ? ) = K µ t ( ? ) × [ ˜ ? c 1 ˜ ? c ? 1 + I 0 1 k ] , with I 0 1 k = diag k (0 , 1 ,ÿ.ÿ.ÿ.ÿ, 1) .

The cumulant function presented in (2.3) and its derivatives are functions of the canonical parameter ? . For practical calculation we need to use the following mean parameterization: P ( m ; F t ) : = P ( ? ( m ); μ t ) , where ? ( m ) is the solution in ? of the equation m = K ' μ t ( ? ) .

The variance function of normal-Poisson model which is the variance-covariance matrix in term of mean parameterization is obtained through the second derivative of the cumulant function, i.e. V F t ( m ) = K '' μ t ( ? ( m )) . Then we have V F t ( m ) = 1 m 1 mm ? + diag k (0 , m 1 ,ÿ.ÿ.ÿ.ÿ, m 1 ) (2.6) Characterizations of Multivariate Normal-Poisson Model 41 on its support M F t = { m ? R k ; m 1 > 0 and m j ? R for j = 2 ,ÿ.ÿ.ÿ.ÿ, k } . (2.7) Consequently, its generalized variance function is det V F t ( m ) = m k 1 with m ? M F t . (2.8) Equation (2.8) expresses that the generalized variance of normal-Poisson model de- pends mainly on the mean of the Poisson component.

The in?nite divisibility property of normal-Poisson is very useful for its character- ization by generalized variance. Regarding to this property, Hassairi (1999) found an interesting representation as stated in the following proposition (without proof). Proposition 2.1. If μ is an in?nitely divisible measure generating an NEF F = F ( μ ) on R k , then there exists a positive measure ? ( μ ) on R k such that det K '' μ ( ? ) = exp K ? ( μ ) ( ? ) , for all ? ? T ( μ ) . The positive measure ? ( μ ) is called the modi?ed Lévy measure of μ .

For F t of normal-Poisson model, the modi?ed Lévy measure that satis?es Proposi- tion 2.1 is given by ? ( μ t ) = t k ? ? ? ? ? ? ? ? ? d e 1 k ? j = 2 N (0 , 1) ? ? ? ? ? ? ? ? * k , (2.9) where ( e 1 ) an orthonormal basis of R k and N (0 , 1) is the standard univariate normal probability measure.

It comes from the cumulant function of ? ( μ t ) which is K ? ( μ t ) ( ? ) = k t ? ? ? ? ? ? ? ? ? 1 + 1 2 k ? j = 2 ? 2 j ? ? ? ? ? ? ? ? ? t = : ( k × ? ? ˜ ? c 1 ) t . By implementing Proposition 2.1 into normal-Poisson model we obtain det K '' μ t ( ? ) = t exp ( k × ? ? ˜ ? c 1 ) . (2.10) We use (2.10) for characterizing normal-Poisson by generalized variance. The problem in this characterization is that for given information in the right-hand side of (2.10), we need to ?nd the cumulant function K in the left-hand side of (2.10) such that the determinant of its second derivative equals to the Laplace transform exp K ? ( μ t ) ( ? ) .

So, this problem becomes a particular case of the Monge-Ampère equation (see equation (4.1) in Section 4 ). 42 Nisa et al. 3 Characterization by variance function In order to characterize normal-Poisson model through its generalized variance func- tion from (2.10) back to (2.3) and then to (2.2), it is also interesting to have their characterizations by variance functions from (2.6) back to (2.3), up to some elementary operations of NEFs.

We here state the ?rst result as follows. Theorem 3.1. Let k ? { 2 , 3 ,ÿ.ÿ.ÿ. } and t > 0 . If an NEF F t satis?es (2.6), then, up to a ? nity, F t is normal-Poisson model. The proof is established by analytical calculations and using the well-known prop- erties of NEFs described in Proposition 3.1 below. Proposition 3.1.

Let μ and e μ be two s -?nite positive measures on R k such that F = F ( μ ) , e F = F ( e μ ) and m ? M F . (i) If there exists ( d , c ) ? R k × R such that e μ ( d x ) = exp {? d , x ? + c } μ ( d x ) , then F = e F : T e μ = T μ - d and K e μ ( ? ) = K μ ( ? + d ) + c ; for e m = m ? M F , V e F ( e m ) = V F ( m ) and det V e F ( e m ) = det V F ( m ) .

(ii) If e μ = f * μ with f ( x ) = Ax + b , then: T ( e μ ) = A ? T ( μ ) and K e μ ( ? ) = K μ ( A ? ? ) + b ? ? ; for e m = Am + b ? f ( M F ) , V e F ( e m ) = AV F ( f - 1 ( e m )) A ? and det V e F ( e m ) = ( det A ) 2 det V F ( m ) . (iii) If e μ = μ * t is the t -th convolution power of μ for t > 0 , then, for e m = t m ? t M F , V e F ( e m ) = t V F ( t - 1 e m ) and det V e F ( e m ) = t k det V F ( m ) . Proposition 3.1

shows that the generalized variance function of F , det V F ( m ) , is in- variant for any element of its generating measure (Part (i)) and for a ? ne transformation f ( x ) = Ax + b such that det A = ± 1 , in particular for a translation x 7? x + b (Part (ii)). Sometimes we use terminology type to call an NEF F as a particular model up to a ? nity (Part (ii)) and convolution power (Part (iii)).

Characterizations of Multivariate Normal-Poisson Model 43 Proof. Without loss of generality, ?rst we assume that t = 1 with ?ashback to the identi?ability of Poisson component. Let F = F ( μ ) be an NEF satis?es (2.6) and (2.7) for t = 1 . Using the blockwise inversion into V F ( m ) in (2.6), one has : [ V F ( m )] - 1 = ? ? ? ? ? ? ? ? ? 1 m 1 + 1 m 3 1 ? k j = 2 m 2 j - 1 m 2 1 ( m c 1 ) ? - 1 m 2 1 ( m c 1 ) 1 m 1 I k - 1 ? ? ? ? ? ? ? ? ? (3.1) with m 1 > 0 and m c 1 : = ( m 2 ,ÿ.ÿ.ÿ.ÿ, m k ) ? ? R k .

Since m = K ' μ ( ? ) and V F ( m ) = K '' μ ( ? ) , then by writing ? in terms of m one gets V F ( m ) = [ ? ' ( m )] - 1 which implies ? ( m ) = ? [ V F ( m )] - 1 d m . For ? ? T : = ? ( M F ) such that M F has the same elements as M F t in (2.7), there exists a function f : R k ? R such that ? ' ( m ) = [ ? 2 f ( m ) ? m i ? m j ] i , j = 1 , 2 ,..., k . (3.2) Using (3.2) into (3.1) for getting the ?rst information on Poisson component, we have ? 2 f ( m ) ? m 2 1 = 1 m 1 + 1 m 3 1 k ? j = 2 m 2 j and then ?f ( m ) ? m 1 = log m 1 - 1 2 m 2 1 k ? j = 2 m 2 j + f ( m 2 ,ÿ.ÿ.ÿ.ÿ, m k ) , (3.3) where f : R k - 1 ? R is an analytical function to be determined.

Note that since m 1 > 0 then log m 1 and 1 / (2 m 2 1 ) in (3.3) are well-de?ned. Derivative of (3.3) with respect to m j gives ? 2 f ? m 1 m j = - m j m 2 1 + ? f ( m 2 ,ÿ.ÿ.ÿ.ÿ, m k ) ? m j . (3.4) 44 Nisa et al. Expression (3.4) is equal to the (1, j )th element of

[ V F ( m )] - 1 in (3.1), that is - m j m 2 1 + ? f ( m 2 ,ÿ.ÿ.ÿ.ÿ, m k ) ? m j = - m j m 2 1 ; therefore, ? f ( m 2 ,ÿ.ÿ.ÿ.ÿ, m k ) /? m j = 0 for all j ?ÿ{ 2 ,ÿ.ÿ.ÿ.ÿ, k } , this implies f ( m 2 ,ÿ.ÿ.ÿ.ÿ, m k ) = c 1 (a real constant). Thus, (3.3) becomes ?f ? m 1 = log m 1 - 1 2 m 2 1 k ? j = 2 m 2 j + c 1 (3.5) and by integration with respect to m 1 , one gets f ( m ) = m 1 log m 1 - m 1 + 1 2 m 1 k ? j = 2 m 2 j + c 1 m 1 + h ( m 2 ,ÿ.ÿ.ÿ.ÿ, m k ) , (3.6) where h : R k - 1 ? R is an analytical function to be determined. From now on, complete information of the model (i.e. normal components) begin to show itself.

The ?rst and second derivatives of (3.6) with respect to m j give, respectively, ?f ( m ) ? m j = m j m 1 + ? h ( m 2 ,ÿ.ÿ.ÿ.ÿ, m k ) ? m j , ? j ?ÿ{ 2 ,ÿ.ÿ.ÿ.ÿ, k } (3.7) and ? 2 f ( m ) ? m 2 j = 1 m 1 + ? h 2 ( m 2 ,ÿ.ÿ.ÿ.ÿ, m k ) ? m 2 j , ? j ?ÿ{ 2 ,ÿ.ÿ.ÿ.ÿ, k } . (3.8) Expression (3.8) is equal to the diagonal ( j , j ) th element of [ V F ( m )] - 1 in (3.1) for all j ?ÿ{ 2 ,ÿ.ÿ.ÿ.ÿ, k } , hence we have 1 m 1 + ? 2 h ( m 2 ,ÿ.ÿ.ÿ.ÿ, m k ) ? m 2 j = 1 m 1 . Consequently, ? 2 h ( m 2 ,ÿ.ÿ.ÿ.ÿ, m k ) /? m 2 j = 0 and ? h ( m 2 ,ÿ.ÿ.ÿ.ÿ, m k ) /? m j = c j (a real constant) for all j ?ÿ{ 2 ,ÿ.ÿ.ÿ.ÿ, k } . Then, equation (3.7) becomes ?f ( m ) ? m j = m j m 1 + c j ? j ?ÿ{ 2 ,ÿ.ÿ.ÿ.ÿ, k } . (3.9) Characterizations of Multivariate Normal-Poisson Model 45 Using equation (3.5) and (3.9) one obtains ? ( m ) = ? ? ? ? ? ? ? ? log m 1 - 1 2 m 2 1 k ? j = 2 m 2 j , m 2 m 1 ,ÿ.ÿ.ÿ.ÿ, m k m 1 ? ? ? ? ? ? ? ? ? ? + ( c 1 ,ÿ.ÿ.ÿ.ÿ, c k ) ? or ? ( m ) = ? ? ? ? ? ? ? ? ? ? ? ? ? ? 1 = log m 1 - 1 2 m 2 1 ? k j = 2 m 2 j + c 1 ? j = m j m 1 + c j , j = 2 ,ÿ.ÿ.ÿ.ÿ, k . (3.10) From (3.10), each ? j belongs to R for j ?ÿ{ 1 , 2 ,ÿ.ÿ.ÿ.ÿ, k } because m 1 > 0 and m j ? R for j ?ÿ{ 2 ,ÿ.ÿ.ÿ.ÿ, k } .

Thus, one has T ( M F ) = : T ? R k and also m 1 = exp ? ? ? ? ? ? ? ? ? ( ? 1 - c 1 ) + 1 2 k ? j = 2 ( ? j - c j ) 2 ? ? ? ? ? ? ? ? ? ? , (3.11) m j = ( ? j - c j ) exp ? ? ? ? ? ? ? ? ? ( ? 1 - c 1 ) + 1 2 k ? j = 2 ( ? j - c j ) 2 ? ? ? ? ? ? ? ? ? ? . (3.12) Since m = ? K µ ( ? ) ? ? , then using (3.11) one can obtain K µ ( ? ) as follow: K µ ( ? ) = ? ? K ' µ ( ? ) ?? 1 d ? 1 = exp ? ? ? ? ? ? ? ( ? 1 - c 1 ) + 1 2 k ? l = 2 ( ? j - c j ) 2 ? ? ? ? ? ? ? ? + g ( ? 2 ,ÿ.ÿ.ÿ.ÿ,ÿ? k ) , (3.13) where g : R k - 1 ? R is an analytical function to be determined. Again, derivative of (3.13) with respect to ? j produces ? K µ ( ? ) ?? j = ( ? j - c j ) exp ? ? ? ? ? ? ? ? ? ( ? 1 - c 1 ) + 1 2 k ? j = 2 ( ? j - c j ) 2 ? ? ? ? ? ? ? ? ? ? + ? g ( ? 2 ,ÿ.ÿ.ÿ.ÿ,ÿ? k ) ?? j which is equal to (3.12); then, one gets ? g ( ? 2 ,ÿ.ÿ.ÿ.ÿ,ÿ? k ) /?? j = 0 for all j ?ÿ{ 2 ,ÿ.ÿ.ÿ.ÿ, k } implying g ( ? 2 ,ÿ.ÿ.ÿ.ÿ,ÿ? k ) = C (a real constant).

Finally, it ensues from it that we have K µ ( ? ) = exp ? ? ? ? ? ? ? ? ? ( ? 1 - c 1 ) + 1 2 k ? j = 2 ( ? j - c j ) 2 ? ? ? ? ? ? ? ? ? ? + C . 46 Nisa et al. By Proposition 3.1 one can see that, up to a ? nity, this K µ is a normal-Poisson cumulant function as given in (2.3) with t = 1 on its corresponding support (2.4). Theorem 3.1 is therefore proven by using the analytical property of K µ .

_ 4 Characterization by generalized variance function Before stating our next result, let us brie?y recall that, for an unknown smooth function K : T ? R k ? R , k > 2 , the MongeAmpère equation is de?ned by det K '' ( ? ) = g ( ? ) , (4.1) where K '' = ( D 2 i j K ) i , j = 1 ,..., k denotes the Hessian matrix of K and g is a given positive function (see e.g. Gutiérrez (2001)). The class of equation (4.1) given g has been a source of intense investigations which are related to many areas of mathematics. Note also that explicit solutions of (4.1), even if in particular situations of g , remain generally challenging problems.

We can refer to Kokonendji and Seshadri (1996); Kokonendji and Masmoudi (2006, 2013) for some details and handled particular cases. We now state the next result in the following sense. Theorem 4.1. Let F t = F ( µ t ) be an in?nitely divisible NEF on R k ( k > 1 ) such that 1. T ( µ t ) = R k , and 2. det K '' µ t ( ? ) = t exp ( k × ? ? ˜ ? c 1 ) for ? and ˜ ? c 1 given as in (2.5). Then F t is of normal-Poisson type.

To proof of this theorem is to solve the Monge-Ampère equation problem of normal-Poisson model (item 2 of the theorem). For that purpose, we need three propositions which are already used in Kokonendji and Masmoudi (2006) and Kokonendji and Masmoudi (2013) and we provide the propositions below for making the paper as selfcontained as possible. Proposition 4.1.

If µ is an in?nitely divisible measure on R k , then there exist a symmetric non-negative de?nite d × d matrix S with rank r 6 k and a positive measure ? on R k such that K '' µ ( ? ) = S + ? R k xx ? exp( ? ? x ) ? ( d x ) . (See, e.g. Gikhman and Skorokhod , 2004, page 342). Characterizations of Multivariate Normal-Poisson Model 47 The above expression of K '' µ ( ? ) is an equivalent of the Lévy-Khinchine formula (see e.g.

Sato, 1999); thus, S comes from a Brownian part and the rest L '' ? ( ? ) : = ? R k xx ? exp( ? ? x ) ? ( d x ) corresponds to jumps part of the associated Lévy process through the Lévy measure ? . Proposition 4.2. Let A and B be two k × k matrices. Then det( A + B ) = ? S ?{ 1 , 2 ,..., k } det ( A S ' ) det ( B S ) , with S ' = { 1 , 2 ,ÿ.ÿ.ÿ.ÿ, k }ÿ\ S and A S = ( a i j ) i , j ? S 2 for A = ( a i j ) i , j ?{ 1 , 2 ,..., k } 2 . (See Muir, 1960).

Proposition 4.3. Let f : R k ? R be a C 2 map. Then, f is an a ? ne polynomial if and only if ? 2 f ( ? ) /?? 2 i = 0 , for i = 1 ,ÿ.ÿ.ÿ.ÿ, k . (See Bar-Lev, et al. , 1994, Lemma 4.1). Proof. Without loss of generality, we assume t = 1 in Theorem 4.1. Letting F = F ( µ ) , we have to solve the following equation (with respect to µ t or its characteristic function): det K '' µ ( ? ) = exp ? ? ? ? ? ? ? ? ? k · ? ? ? ? ? ? ? ? ? ? 1 + 1 2 k ? j = 2 ? ? 2 j ? ? ? ? ? ? ? ? ? ? ? ? ? ? ? ? ? ? , ? ? ? R k . (4.2) From Proposition 4.1

relative to the representation of in?nitely divisible distribution, the unknown left member of Equation (4.2) can be written as det K '' μ ( ? ) = det [ S + ? R k xx ? exp( ? ? x ) ? ( d x ) ] = det [ S + L '' ? ( ? ) ] . (4.3) For S = { i 1 , i 2 ,ÿ.ÿ.ÿ.ÿ, i j } , with 1 6 i 1 < i 2 < ·ÿ·ÿ· < i j 6 k , a non-empty subset of { 1 , 2 ,ÿ.ÿ.ÿ.ÿ, k } , and t S : R k ? R j the map de?ned by t S ( x ) = ( x i 1 , x i 2 ,ÿ.ÿ.ÿ.ÿ, x i j ) ? , we de?ne ? S the image measure of H j ( d x 1 ,ÿ.ÿ.ÿ.ÿ, d x j ) = 1 j ! ( det [ t S ( x 1 ) .ÿ.ÿ.ÿt S ( x j ) ]) 2 ? ( d x 1 ) .ÿ.ÿ.ÿ? ( d x j ) by ? j : ( R k ) j ? R k , ( x 1 ,ÿ.ÿ.ÿ.ÿ, x j ) 7? x 1 + x 2 + ·ÿ·ÿ· + x j . By Proposition 4.2 and Expression (4.3) the modi?ed Lévy measure ? ( μ ) in (2.1) can be expressed as ? ( μ ) = (det ? ) d 0 + ? Ø , S ?{ 1 , 2 ,..., k } (det ? S ' ) ? S , (4.4) 48 Nisa et al.

where ? is a diagonal representation of S in an orthonormal basis e = ( e i ) i = 1 ,..., k (see Hassairi , 1999, page 384). Since S is the Brownian part, then it corresponds to the k - 1 normal components from the right member of (4.2); that implies r = rank ( S ) = k - 1 and det S = 0 . Therefore det ? = 0 with ? = diag ( ? 1 ,ÿ? 2 ,ÿ.ÿ.ÿ.ÿ,ÿ? k ) such that ? 1 = 0 and ? j > 0 for all j ?ÿ{ 2 ,ÿ.ÿ.ÿ.ÿ, k } . For all non-empty subsets S of { 1 , 2 ,ÿ.ÿ.ÿ.ÿ, k } there exist real numbers a S > 0 such that (det ? S ' ) ? S = ? ? ? ? ? ? ? ? ? i < S ? i ? ? ? ? ? ? ? ? S = a S [ d e 1 *ÿN (0 , 1)( e c 1 ) ] * k , (4.5) where e c 1 = ( e 2 ,ÿ.ÿ.ÿ.ÿ, e k ) denotes the induced orthonormal basis of e without component e 1 ; i.e. k - 1 is the dimension of e c 1 .

With respect to Kokonendji and Masmoudi (2006, Lemma 7) for making precise the measure ? of (4.5), it is easy to see that S 0 = { 1 } is a singleton (i.e. set with exactly one element) such that, for x = x 1 e 1 + ·ÿ·ÿ· + x k e k , x 2 1 ? ( d x ) = ßd a e 1 , with ßÿ> 0 and a , 0 . Consequently, we have the following complementary set S ' 0 = { 1 , 2 ,ÿ.ÿ.ÿ.ÿ, k }ÿ\ÿ{ 1 } . So, from (4.5) we have k th power of convolution of only one Poisson at the ?rst component e 1 and ( k - 1 )-variate standard normal.

That means K '' μ ( ? ) = K μ ( ? ) [ ˜ ? c 1 ˜ ? c ? 1 + I 0 1 k ] , with notations of (2.5). Let B ( ? ) = exp ( ? 1 + 1 2 ? k j = 2 ? 2 j ) from (4.2). Since we check that ? 2 ( K μ - B )( ? ) /?? 2 i = 0 for all i = 1 ,ÿ.ÿ.ÿ.ÿ, k , Proposition 4.3 allows that ( K μ 1 - B )( ? ) is an a ? ne function on R k and therefore K μ ( ? ) = exp ? ? ? ? ? ? ? ? ? 1 + 1 2 k ? j = 2 ? 2 j ? ? ? ? ? ? ? ? + u ? ? + b , for ( u , b ) ? R k × R . Hence F = F ( μ ) is of normal-Poisson type with t = 1 . This completes the proof of the theorem. _ A reformulation of Theorem 4.1, by changing the canonical parameterization into the mean parameterization, is stated in the following theorem without proof. Theorem 4.2.

Let F t = F ( μ t ) be an in?nitely divisible NEF on R k such that 1. M F t = { m ? R k ; m 1 > 0 and m j ? R with j = 2 ,ÿ.ÿ.ÿ.ÿ, k } , and Characterizations of Multivariate Normal-Poisson Model 49 2. det V F t ( m ) = m k 1 . Then F t is of normal-Poisson type. Theorem 4.1 can be viewed as the solution to a particular Monge-Ampère equation (4.1). Whereas Theorem 4.2 is interesting for generalized variance estimation of the model.

5 Conclusion In this paper we described some properties of normal-Poisson model. Then we showed that the characterization of normal-Poisson model by variance function was obtained through analytical calculations and using some properties of NEF. Also, the charac- terization of normal Poisson model by generalized variance which is the solution to a speci?c Monge-Ampère equation: det K '' µ ( ? ) = exp ( k × ? ? ˜ ? c 1 ) on R k can be solved using the in?nite divisibility property of normal-Poisson.

Acknowledgment The authors would like to thank the Editor and Referees for their valuable and con- structive comments that led to a considerably improved manuscript.
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Generalized Va riance Estimations of Normal-Po isson Models Célestin C. Kokonendji and Khoirin Nisa Abstract This chapter presents three estimations of generalized variance (i.e., determinant of covariance matrix) of normal-Poisson models: maximum likelihood (ML) estimator, uniformly minimum variance unbiased (UMVU) estimator, and Bayesian estimator. First, the de?nition and some properties of normal-Poisson models are established.

Then ML, UMVU, and Bayesian estimators for generalized variance are derived. Finally, a simulation study is carried out to assess the performance of the estimators based on their mean square error (MSE). Keywords Covariance matrix • Determinant • Normal stable Tweedie • Maxi- mum likelihood • UMVU • Bayesian estimator Introduction In multivariate analysis, generalized variance (i.e., determinant of covari- ance matrix) has important roles in the descriptive analysis and inferences.

It is the measure of dispersion within multivariate data which explains the variability and the spread of observations. Its estimation usually based on the determinant of the sample covariance matrix. Many studies related to the generalized variance estimation have been done by some researchers; see, e.g., [ 1–3] under normality and non-normality hypotheses.

A normal-Poisson model is composed by distributions of random vector X D (X1, X2, : : : , Xk)T with k > 1, where Xj is a univariate Poisson variable, and ( X1, : : : , Xj _1, Xj C1, : : : , Xk) given Xj are k-1 real independent Gaussian variables with variance Xj. It is a particular part of normal stable Tweedie (NST) models [ 4] with p D 1 where p is the power variance parameter of distributions within the Tweedie family.

This model was introduced in [ 4] for the particular case of normal-Poisson with j D 1. Also, normal-Poisson is the only NST model which has a discrete component, and it is correlated to the continuous normal parts. C.C. Kokonendji (_) • K. Nisa Laboratoire de Mathématiques de Besançon, University of Franche-Comté, Besançon, France e-mail: celestin.kokonendji@univ-fcomte.fr ; khoirin.nisa@univ-fcomte.fr  K. Chen (ed.), Forging Connections between Computational Mathematics and Computational Geometry , Springer Proceedings in Mathematics & Statistics 124, DOI 10.5176/2251-1911_CMCGS14.29_21 247 celestin.kokonendji@univ-fcomte.fr 248 C.C. Kokonendji and K.

Nisa In literature, there is also a model known as Poisson-Gaussian [ 5–7] which is completely different from normal-Poisson. For any value of j, a normal-Poisson j model has only one Poisson component and k-1 normal (Gaussian) components, while a Poisson-Gaussian j model has j Poisson components and k-j Gaussian components.

Poisson-Gaussian is also a particular case of simple quadratic natural exponential family (NEF) [ 5] with variance function VF( m) D Diag k(m1, : : : , mj, 1, : : : , 1), where m D (m1, : : : , mk) is the mean vector and its generalized variance function is det VF( m) D m1, : : : , mj. The estimations of generalized variance of Poisson-Gaussian can be seen in [ 8, 9].

Motivated by generalized variance estimations of Poisson-Gaussian, we present our study on multivariate normal-Poisson models and the estimations of their generalized variance using ML, UMVU, and Bayesian estimators. N o r m a l - Po i s s o n M o d e l s In this section, we establish the de?nition of normal-Poisson j models as generaliza- tion of normal-Poisson 1 model which was introduced in [ 4], and then we give some properties. De?nition 2.1

For a k-dimensional normal-Poisson random vector X D (X1, X2, : : : , Xk)T with k > 1, it must hold that 1. Xj follows a univariate Poisson distribution. 2. X 1 ; : : : ; X j _ 1 ; X j C 1 ; : : : ; X k _ DW X c ? X j are independent normal variables with mean 0 and variance Xj, i.e., X c ? X j _ i :i :d : N 0; X j _. In order to satisfy the second condition, we need Xj > 0, but in practice it is possible to have xj D 0 in the Poisson sample.

In this case, the corresponding normal components are degenerated as •0 which makes their values become 0s. The NEF Ft D F( • t) of a k-dimensional normal-Poisson random vector X is generated by _ t . d x / D t x j x j Š_ _ 1 2_ x j _ . k _ 1 / = 2 exp 0 _ t _ 1 2x j X ` ¤ j x 2 ` 1 I x j 2N n f 0 g i x j d x j _ Y ` ¤ j d x ` ; for a ?xed power of convolution t >0, where I A is the indicator function of the set A and i x j is the Dirac measure at xj.. Since t > 0, then • t: D • *t is an in?nitely divisible measure.

The cumulant function which is the log of the Laplace transform of t, i.e., K _ t .™ / D l og Z R k e xp ™ T x_ _ t . d x /, is given by celestin.kokonendji@univ-fcomte.fr Generalized Variance Estimations of Normal-Poisson Models 249 K _ t .™ / D t e x p 0 _ j C 1 2 X ` ¤ j _ 2 ` 1 : (1) The function K _ t .™ / in ( 1) is ?nite for all ™ in the canonical domain: , . _ t / D 8 ™ 2 R k I ™T Q ™ c WD _ j C X ` ¤ j _ 2 ` = 2 < 0 9 with ™ D .

_ 1 ; _ _ _ ; _ k / T and Q ™ c WD _ 1 ; : : : ; _ j _ 1 ; _ j D 1; _ j C 1 ; : : : ; _ k _ T : (2) The probability distribution of normal-Poisson j is P .™ I t / . d x / D exp °™ T x _ K _ t .™ / _ t . d x / which is a member of NEF F . _ t / D f P .™ I t / I ™ 2 , . _ t / g. From ( 1), we can calculate the ?rst derivative of the cumulant function that produces a k-vector as the mean vector of F _ t and also its second derivative which is a k _ k matrix that represents the covariance matrix. Using notations in ( 2), we obtain K 0 t .™ / D K _ t .™ / _ Q ™ c and K 00 t .™ / D K _ t .™

/ j Q ™ c Q ™ c T C I 0 j k k with I 0 j k D Diag k 1; : : : ; 1; 0 j ; 1; : : : ; 1 _. The cumulant function presented in ( 1) and its derivatives are functions of the canonical parameter ™. For practical calculation, we need to use the mean parameterization: P . m I F t / WD P .™ . m / I _ t / with ™( m) is the solution in ™ of equation m D K 0 t .™

/ : The variance function of a normal-Poisson j model which is the variance- covariance matrix in term of mean parameterization is obtained through the second derivative of the cumulant function, i.e., V F t . m / D K 00 t Œ™ . m / • : Then we have V F t . m / D 1 m j mm T C Diag k m j ; : : : ; m j ; 0 j ; m j ; : : : ; m j _ (3) with mj > 0 and m ` 2 R ; ` ¤ j .

For j D 1, the covariance matrix of X can be expressed as below celestin.kokonendji@univ-fcomte.fr 250 C.C. Kokonendji and K. Nisa V F t . m / D 2 m 1 ? m 2 : : : m j : : : m k _ _ _ ? _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ m 2 ? m_ 1 1 m 2 C m 1 : : : m _ 1 1 m 2 m j : : : m _ 1 1 m 2 m k : ? : : : : : : m j ? m_ 1 1 m j m 2 : : : m _ 1 1 m 2 C m 1 : : : m _ 1 1 m j m k : ? : : : : : : m k ? m_ 1 1 m k m 2 : : : m _ 1 1 m k m j : : : m _ 1 1 m 2 C m 1 3 : Indeed, for the covariance matrix above, one can use the following particular Schur representation of the determinant det _ a T a A _ D det A _ _ 1 aa T _ (4) with the non-null scalar D m 1, the vector a D . m 2 ; _ _ _ ; m k / T , and the . k _ 1 / _ . k _ 1 / matrix A D _ 1 aa T C m 1 I k _ 1 ; where I j D Diag j .

1; _ _ _ ; 1 / is the j _ j unit matrix. Consequently, the determinant of the covariance matrix V F t . m / for j D 1 is det V F t . m / D m k Then, it is trivial to show that for j 2f 1, : : : ,k g, the generalized variance of normal-Poisson j model is given by det V F t . m / D m k (5) with m j > 0; m ` 2 R ; ` ¤ j: ( 5) expresses that the generalized variance of normal-Poisson models depends mainly on the mean of the Poisson component (and the dimension space k >1) .

Among NST models, normal-gamma which is also known as gamma-Gaussian is the only model that has been characterized completely; see [ 5] or [ 10 ] for characterization by variance function and [ 11 ] for characterization by generalized variance function. For normal-Poisson models, here we give our result regarding to characterization by variance function and generalized variance.

We state the results in the following theorems without proof. Theorem 2.1 Let k 2f 2, 3, : : : g and t > 0. If an NEF Ft satis?es ( 3), then, up to af?nity, Ft is of normal-Poisson model. Theorem 2.2 Let Ft D F( • t) be an in?nitely divisible NEF on R k such that 1. The canonical domain ,( ) D Rk 2. det K 00 .™ / D t e x p _ k _ ™ T Q ™ c _ celestin.kokonendji@univ-fcomte.fr Generalized Variance Estimations of Normal-Poisson Models 251 for ™ and Q ™ c given in ( 2). Then, up to af?nity and power convolution, Ft is of normal-Poisson model.

All the technical details of proofs will be given in our article which is in preparation. In fact, the proof of Theorem 2.1 obtained by algebraic calculations and by using some properties of NEF is described in Proposition 2.1 below. An idea to proof Theorem 2.2 can be obtained using the in?nite divisibility property of normal-Poisson for which this proof is the solution to the particular Monge– Ampère equation [ 12 ]: det K 00 .™ / D t e xp _ k _ ™ T Q ™ c _ .

Gikhman and Skorokhod [ 13 ] showed that if • is an in?nitely divisible measure, then there exist a symmetric nonnegative de?nite d _ d matrix † with rank k-1 and a positive measure a on Rk such that K 00 .™ / D † C Z R k xx T exp ™ T x_ _ . d x / : The Lévy–Khintchine formula of in?nite divisibility distribution is also applied. Proposition 2.1 Let _ and Q _ be two ¢-?nite positive measures on Rk such that F D F( • ), Q F D F . Q _ /, and m 2 M F : 1.

If there exists ( d,c) 2RkxR such that Q _ . d x / D e xp n d T x E C c o _ . d x / ; then F D Q F W , . Q _ / D , . _ / _ d and K Q _ .™ / D K _ .™ C d / C c ; for m D m 2 M F ; V Q F . m / D V F . m / ; and det V Q F . m / D det V F . m / : 2. If Q _ D ¥ _ _ is the image measure of • by the af?ne transformation ¥ . x / D Ax C b ; where A is a k _ k nondegenerate matrix and b 2Rk, then , .

Q _ / D A T , . _ / and K Q _ .™ / D K _ A T ™_ C b T ™ I for m D Am C b 2 ¥ . M F / ; V Q F . m / D AV F ¥ _ 1 . m /_ A T ; and det V Q F . m / D .det A / 2 det V F . m / : 3. If Q _ D _ _ t is the t-th convolution power of • for t > 0, then , . Q _ / D , . _ / and K Q _ .™ / D t K _ .™ / I for m D t m 2 t M F ; V Q F . m / D t V F _ ¥t _ 1 . m / ; and det V Q F . m / D tkdet VF( m). Proposition 2.1

shows that the generalized variance function det VF( m) of F is invariant for any element of its generating measure (Part 1) and for the af?ne transformation ¥ . x / D Ax C b such that det A D ? 1, particularly for a translation x ! x C b (Part 2). A reformulation of Theorem 2.2, by changing the canonical parameterization into mean parameterization, is stated in the following theorem. Theorem 2.3

Let Ft D F( • t) be an in?nitely divisible NEF on R k such that 1. mj > 0 and m ` 2 R with ` ¤ j 2. det V F . m / D m k : Then Ft is of normal-Poisson type. celestin.kokonendji@univ-fcomte.fr 252 C.C. Kokonendji and K. Nisa Theorem 2.3 is equivalent to Theorem 2.2. The former is used for the estimation of generalized variance, and the latter is used for characterization by generalized variance.

Generalized Va riance Estimations Here we present three methods for generalized variance estimations of normal- Poisson models P . m I F t / 2 F t D F . _ t /, and then we report the result of our simulation study. Consider X 1 ; _ _ _ ; X n be random vectors i.i.d. from P( m; Ft) of normal-Poisson models, and we denote X D .

X 1 C _ _ _ C X n / = n D X 1 ; _ _ _ ; X k _ T as the sample mean with positive j-th component X j : The followings are ML, UMVU, and Bayesian generalized variance estimators. Maximum Likelihood Estimator Proposition 3.1 The ML estimator of det V F t . m / D m k is given by T n ; t D det V F t _ X _ D X j _ k : (6) Proof The ML estimator above is easily obtained by replacing mj in ( 5) with its ML estimator X j . ut Uniformly Minimum Variance Unbiased Estimator Proposition 3.2 The UMVU estimator of det V F t .

m / D m k is given by U n ; t D n _ k C 1 X j n X j _ 1_ : : : n X j _ k C 1_ ; if n X j _ k : (7) Proof This UMVU estimator is obtained using intrinsic moment formula of univariate Poisson distribution as follows: E Œ X . X _ 1 / : : : . X _ k C 1 / • D m k : Letting Y D n X j gives the result that ( 7) is the UMVU estimator of ( 5), because, by the completeness of NEFs, the unbiased estimation is unique. So, we deduced the desired result. _ celestin.kokonendji@univ-fcomte.fr Generalized Variance Estimations of Normal-Poisson Models 253 A deep discussion about ML and UMVU methods on generalized variance estimations can be seen in [ 9] for NEF and [ 4] for NST models. Bayesian Estimator Proposition 3.3

Under assumption of prior gamma distribution of mj with param- eter ’ > 0 and “ > 0, the Bayesian estimator of det V F t . m / D m k is given by B n ; t ; ? ; ? D ? C n X j ? C n ! k : (8) Proof Let X 1j ; _ _ _ ; X nj given mj are Poisson( mj) with probability mass function P _ X ij D x ij ? m j _ D m x ij j x ij Š e _ m j D p _ x ij ? m j _ : Assuming that mj follows gamma( ?, ?), then the prior probability distribution function of mj is given by f " m j I ? ; ? _ D ? ? ? . ? / m ? _ 1 j e _ ? m j for m j > 0 and ? ; ? > 0 where ( ?) is the gamma function: ? .

? / D Z 1 x ? _ 1 e _ x d x : Using the Bayes theorem, the posterior distribution of mj given an observation sequence can be expressed as f _ m j ? x ij I ? ; ? _ D p _ x ij ? m j _ f " m j I ? ; ? _ Z m j >0 p _ x ij ? m j _ f " m j I ? ; ? _ d m j D . ? C 1 / ? C x ij ?" ? C x ij _ m ? C x ij _ 1 j e _ . ? C 1 / m j which is a gamma density with parameters ? 0 D x ij C ? and ? 0 D 1 C ?.

Then with random sample X 1j, : : : , X nj , the posterior will be gamma " ? C n X j ; ? C n_ : The Bayesian estimator of mj is given by the mean of the posterior distribution, i.e., b m b D ? C n X j ? C n , and then this concludes the proof. _ The choice of ' and " depends on the information of mj. Notice that for any positive value c 2 . 0; 1 / ; if ? D c X j and ? D c, then the Bayesian estimator is the same as ML estimator.

In practice, the parameter of prior distribution of celestin.kokonendji@univ-fcomte.fr 254 C.C. Kokonendji and K. Nisa mj must be known or can be assumed con?dently before the generalized variance estimation. One can see, e.g., [ 14 –16 ] for more details about Bayesian inference on mj (univariate Poisson parameter).

Simulation Study In order to look at the performances of ML, UMVU, and Bayesian estimators of the generalized variance, we have done a Monte Carlo simulation using R software [17 ]. We have generated k D 2, 4, 6, 8 dimensional data from multivariate normal- Poisson distribution F( • t) with mj D 1. Fixing j D 1, we set several sample sizes n varied from 5 until 300, and we generated 1,000 samples for each sample size.

For calculating the Bayesian estimator, in this simulation we assume that the parameters of prior distribution depend on sample mean of Poisson component, X j , and the dimension k. Then we set three different prior distributions: gamma X j ; k _ ; gamma X j ; k = 2 _ ; and gamma X j ; k =3 _ : We report the results of the generalized variance estimations using the three methods in Table 1.

From these values, we calculated the mean square error (MSE) of each method over 1,000 data sets using this following formula M S E _ ^ G V _ D 1 1; 000 1;000 X i D 1 _ ^ G V i _ m k _ 2 where ^ G V is the estimate of mk using each method. From the values in Table 1, we can observe different performances of ML estimator ( Tn,t), UMVU estimator ( Un,t), and Bayesian estimator ( Bn,t, ', ") of the generalized variance.

The values of Tn,t and Bn,t, ', " converge, while the values of Un,t do not, but Un,t which is the unbiased estimator always approximate the parameter ( m k D 1) and closer to the parameter than Tn,t and Bn,t, ′, ″ for small sample sizes n _ 25. For all methods, the standard error of the estimates decreases when the sample size increases.

The Bayesian estimator with gamma X j ; k = 2 _ prior distribution, i.e., B n ; t ; X j ; k = 2 , is exactly the same as Tn,t for k D 2. This is because in this case, the Bayesian and ML estimators of m1 are the same (i.e., c D 1). The goodness of Bayesian estimator depends on the parameter of prior dis- tribution, ′ and ″.

From our simulation, the result shows that smaller parameter ″ gives greater standard error to the estimations in small sample sizes, and the accuracy of Bn,t, ′, ″ with respect to ″ varies with dimensions k. However, they are all asymptotically unbiased. There are more important performance characterizations for an estimator than just being unbiased. The MSE is perhaps the most important of them.

It captures the celestin.kokonendji@univ-fcomte.fr Generalized Variance Estimations of Normal-Poisson Models 255 Tabl e 1 The expected values (with standard error) of Tn,t, Un,t, and Bn,t, ?, ? with m 1 D 1 and k 2 f 2 ; 4 ; 6; 8 g (target values m k D 1) k D 2n Tn,t Un,t B n ;t ; X j ;k B n ;t ; X j ;k = 2 B n ;t ; X j ;k =3 k C 1 1.2790 (1.3826) 0.9533 (1.2050) 0.8186 (0.8849) 1.2790 (1.3826) 1.5221 (1.6454) k C 5 1.1333 (0.8532) 0.9915 (0.8000) 0.8955 (0.6742) 1.1333 (0.8532) 1.2340 (0.9290) k C 10 1.1121 (0.6295) 1.0276 (0.6056) 0.9589 (0.5428) 1.1121 (0.6295) 1.1714 (0.6631) 25 1.0357 (0.4256) 0.9959 (0.4175) 0.9604 (0.3946) 1.0357 (0.4256) 1.0628 (0.4367) 60 1.0090 (0.2526) 0.9924 (0.2505) 0.9767 (0.2445) 1.0090 (0.2526) 1.0201 (0.2553) 100 1.0086 (0.1988) 0.9986 (0.1979) 0.9890 (0.1950) 1.0086 (0.1988) 1.0153 (0.2002) 300 0.9995 (0.1141) 0.9962 (0.1140) 0.9929 (0.1134) 0.9995 (0.1141) 1.0017 (0.1144) k D 4n Tn,t Un,t B n ;t ; X j ;k B n ;t ; X j ;k = 2 B n ;t ; X j ;k =3 k C 1 2.3823 (4.6248) 0.9460 (2.5689) 0.4706 (0.9135) 1.2859 (2.4964) 1.9190 (3.7254) k C 5 1.6824 (2.4576) 0.9531 (1.6995) 0.5890 (0.8605) 1.1491 (1.6786) 1.4756 (2.1555) k C 10 1.4664 (1.6345) 1.0027 (1.2456) 0.7072 (0.7882) 1.1328 (1.2626) 1.3430 (1.4969) 25 1.2711 (1.0895) 1.0169 (0.9327) 0.8212 (0.7039) 1.0930 (0.9368) 1.2079 (1.0353) 60 1.0978 (0.5682) 0.9961 (0.5288) 0.9060 (0.4689) 1.0287 (0.5324) 1.0741 (0.5559) 100 1.0589 (0.4209) 0.9983 (0.4028) 0.9419 (0.3744) 1.0180 (0.4046) 1.0451 (0.4154) 300 1.0273 (0.2305) 1.0071 (0.2271) 0.9874 (0.2215) 1.0138 (0.2275) 1.0228 (0.2295) k D 6n Tn,t Un,t B n ;t ; X j ;k B n ;t ; X j ;k = 2 B n ;t ; X j ;k =3 k C 1 4.7738 (13.9827) 0.9995 (4.7073) 0.2593 (0.7594) 1.2514 (3.6655) 2.3548 (6.8972) k C 5 2.9818 (6.2595) 0.9958 (2.7565) 0.3689 (0.7743) 1.1825 (2.4823) 1.8446 (3.8723) k C 10 2.2232 (4.0454) 1.0124 (2.2131) 0.4733 (0.8612) 1.1406 (2.0756) 1.5778 (2.8709) 25 1.6399 (2.2478) 0.9555 (1.4833) 0.5708 (0.7824) 1.0513 (1.4410) 1.3076 (1.7923) 60 1.2479 (0.9978) 0.9827 (0.8226) 0.7778 (0.6220) 1.0283 (0.8222) 1.1319 (0.9051) 100 1.1830 (0.7646) 1.0235 (0.6800) 0.8853 (0.5722) 1.0517 (0.6798) 1.1151 (0.7207) 300 1.0530

(0.3758) 1.0022 (0.3608) 0.9539 (0.3404) 1.0119 (0.3612) 1.0322 (0.3684) k D 8n Tn,t Un,t B n ;t ; X j ;k B n ;t ; X j ;k = 2 B n ;t ; X j ;k =3 k C 1 8.5935 (31.9230) 0.8677 (5.4574) 0.1232 (0.4576) 1.0535 (3.9134) 2.5038 (9.3010) k C 5 4.7573 (12.5015) 0.8468 (3.0478) 0.1856 (0.4878) 1.0065 (2.6448) 1.9345 (5.0836) k C 10 3.6816 (9.0892) 1.0394 (3.2258) 0.2994 (0.7392) 1.1394 (2.8130) 1.8789 (4.6387) 25 2.9055 (6.3150) 1.1341 (2.9623) 0.4314 (0.9377) 1.2129 (2.6362) 1.7675 (3.8416) 60 1.6201 (1.8804) 1.0511 (1.3062) 0.6794 (0.7885) 1.1035 (1.2807) 1.3059 (1.5156) 100 1.2890 (1.0907) 0.9850 (0.8667) 0.7541 (0.6381) 1.0199 (0.8630) 1.1308 (0.9569) 300 1.1056 (0.5378) 1.0086 (0.4968) 0.9199 (0.4474) 1.0213 (0.4967) 1.0578 (0.5145) bias and the variance of the estimator.

For this reason, we compare the quality of the estimators using MSE in Table 2 which are presented graphically in Figs. 1, 2, 3, and 4. From these ?gures, we conclude that all estimators become more similar when the sample size increases. For small sample sizes, B n ; t ; X j ; k always has the smallest MSE, while Tn,t always has the greatest MSE (except for k D 2).

For n _25, Un,t is preferable than Tn,t. In this situation, the difference between Un,t and Tn,t increases when the dimension increases and also the difference between Tn,t and Bn,t, ', ". celestin.kokonendji@univ-fcomte.fr 256 C.C. Kokonendji and K. Nisa Tabl e 2 The mean square error of Tn,t, Un,t, and Bn,t, ?, ? of Table 1 k D 2n MSE (Tn,t) MSE (Un,t) M S E .B

n ;t ; X j ;k / M S E .B n ;t ; X j ;k = 2 / M S E .B n ;t ; X j ;k =3 / k C 1 1.9894 1.4542 0.8159 1.9894 2.9800 k C 5 0.7458 0.6401 0.4654 0.7458 0.9179 k C 10 0.4088 0.3675 0.2963 0.4088 0.4690 25 0.1824 0.1743 0.1573 0.1824 0.1947 60 0.0639 0.0628 0.0603 0.0639 0.0656 100 0.0396 0.0391 0.0381 0.0396 0.0403 300 0.0130 0.0130 0.0129 0.0130 0.0131 k D 4n MSE (Tn,t) MSE (Un,t) M S E .B n ;t ; X j ;k / M S E .B n ;t ; X j ;k = 2 / M S E .B n ;t ; X j ;k =3 / k C 1 23.2999 6.6019 1.1149 6.3136 14.7231 k C 5 6.5055 2.8904 0.9093 2.8398 4.8724 k C 10 2.8891 1.5514 0.7071 1.6118 2.3585 25 1.2604 0.8702 0.5274 0.8862 1.1151 60 0.3324 0.2797 0.2287 0.2843 0.3146 100 0.1806 0.1622 0.1435 0.1640 0.1746 300 0.0539 0.0516 0.0492 0.0519 0.0532 k D 6n MSE (Tn,t) MSE (Un,t) M S E .B n ;t ; X j ;k / M S E .B

n ;t ; X j ;k = 2 / M S E .B n ;t ; X j ;k =3 / k C 1 209.7568 22.1589 1.1254 13.4989 49.4073 k C 5 43.1085 7.5980 0.9979 6.1952 15.7078 k C 10 17.8618 4.8981 1.0191 4.3278 8.5761 25 5.4622 2.2020 0.7964 2.0790 3.3071 60 1.0571 0.6769 0.4362 0.6769 0.8366 100 0.6181 0.4629 0.3406 0.4647 0.5327 300 0.1440 0.1302 0.1180 0.1306 0.1368 k D 8n MSE (Tn,t) MSE (Un,t) M S E .B n ;t ; X j ;k / M S E .B n ;t ; X j ;k = 2 / M S E .B n ;t ; X j ;k =3 / k C 1 1,076.7380 29.8009 0.9782 15.3177 88.7698 k C 5 170.4059 9.3124 0.9012 6.9951 26.7168 k C 10 89.8046 10.4076 1.0373 7.9326 22.2895 25 43.5105 8.7931 1.2025 6.9949

15.3466 60 3.9204 1.7088 0.7246 1.6509 2.3907 100 1.2732 0.7515 0.4676 0.7452 0.9327 300 0.3003 0.2469 0.2066 0.2472 0.2681 In this simulation, B n ; t ; X j ; k is the best estimator because of its smallest MSE, but in general we cannot say that Bayesian estimator is much better than ML and UMVU estimators since it depends on the prior distribution parameters. In fact, one would prefer Un,t as it is the unbiased estimator with the minimum variance.

However, if in practice we know the information about prior distribution of mj, we can get a better estimate (in the sense of having a lower MSE) than Un,t by using Bn,t, ', ". celestin.kokonendji@univ-fcomte.fr Generalized Variance Estimations of Normal-Poisson Models 257 0.0 3 9 14 25 60 100 300 Sample Size 0.5 1.0 1.5 2.0 2.5 Mean Square Error ML -k BAYES-k/2 BAYES-k/3 F i g.

1 MSE plot of Tn,t, Un,t, B n ;t ; x j ;k ;B n ;t ; x j ;k = 2 ; and B n ;t ; x j ;k =3 for k D 2 C o n c l u s i o n In this chapter, we have established the de?nition and properties of normal-Poissonj models as a generalization of normal-Poisson 1 and showed that the generalized variance of normal-Poisson models depends mainly on the mean of the Poisson component.

The estimations of generalized variance using ML, UMVU, and Bayesian estimators show that UMVU produces a better estimation than ML estimator, while compared to Bayesian estimator, UMVU is worse for some choice of prior distribution parameters, but it can be much better for other cases. However, all methods are consistent estimators, and they become more similar when the sample size increases. celestin.kokonendji@univ-fcomte.fr 258 C.C.

Kokonendji and K. Nisa 0 5 9 14 25 60 100 300 Sample Size 5 10 15 20 Mean Square Error ML -k BAYES-k/2 BAYES-k/3 F i g. 2 MSE plot of Tn,t, Un,t, B n ;t ; x j ;k ;B n ;t ; x j ;k = 2 ; and B n ;t ; x j ;k =3 for k D 4 0 7 11 16 25 60 100 300 Sample Size 50 100 150 200 Mean Square Error ML -k BAYES-k/2 BAYES-k/3 F i g. 3 MSE plot of Tn,t, Un,t, B n ;t ; x j ;k ;B n ;t ; x j ;k = 2 ; and B n ;t ; x j ;k for k D 6 celestin.kokonendji@univ-fcomte.fr Generalized Variance Estimations of Normal-Poisson Models 259 0 13 9 18 25 60 100 300 Sample Size 200 400 600 800 1000 Mean Square Error ML -k BAYES-k/2 BAYES-k/3 F i g. 4 MSE plot of Tn,t, Un,t, B n ;t ; x j ;k ;B n ;t ; x j ;k = 2 ; and B n ;t ; x j ;k =3 for k D 8 References 1. Hassairi, A.:

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INSIST Vol. 2 No. 1, April 2017 (1 - 5) http://insist.unila.ac.id/ DOI: 10.23960/ins.v2i1.23 Received: 20/08/2016 Accepted: 19/01/2017 Published online: 01/04/2017 eISSN: 2502 -8588 Abstract —In this paper, a robust procedure for estimating parameters of regression model when generalized estimating equation (GEE) applied to longitudinal data that contains outliers is proposed.

The method is called ‘iteratively reweighted least trimmed square’ (IRLTS) which is a combination of the iteratively reweighted least square (IRLS) and least trimmed square (LTS) methods. To assess the proposed method a simulation study was conducted and the result shows that the method is robust against outliers. Keywords —GEE, IRLS, LTS, longitudinal data, regression model. I.

INTRODUCTION ONGITUDINAL studies are increasingly common in many scientific research areas, for example in the social, biomedical, and economical fields. In longitudinal studies, multiple measurements are taken on the same subject at different points in time. Thus, observations for the same subject are correlated. The analysis of data resulting from such studies often becomes complicated due to the within- subject correlation.

This correlation must be considered for any appropriate analysis method. Generalized linear models (GLM) as described by McCullagh and Nelder [1] is a standard method used to fit regression models for univariate data that are presumed to follow an exponential family distribution. The association between the response variable and the covariates is given by the link function.

GLM assume that the observations are independent and do not consider any correlation between the outcome of the n observations. Liang and Zeger [2] introduced an approach to this correlation problem using GEE to extend GLM into a regression setting with correlated observations within subjects. The GEE method of Liang and Zeger gives consistent estimators of the regression parameter.

The parameter estimates are consistent even when the variance structure is miss-specified under mild regularity conditions. However, problems can occurs when data contain outliers. The method is not robust against outliers since it is based on score equations from the quasi likelihood method of estimation.

The working correlation matrix would be affected by the outliers and also the parameter estimates. In this situation, we need a robust method that can minimize the effect of outliers. In recent years, a few authors have considered robust methods for longitudinal data analysis.

For example, Qaqish and Preisser [3] proposed a resistant version of the GEE using M-type estimation by involving down-weighting influential 1Department of Mathematics, Faculty of Mathematics and Natural Sciences, University of Lampung, Jalan Prof. Soemantri Brojonegoro No. 1, Bandar Lampung, Indonesia. *Correspondence to Khoirin Nisa, email: khoirin.nisa@fmipa.unila.ac.id. Tel.:+62 721 701609; fax: +62 721 702767. data points.

Gill [4] proposed a robustified likelihood based on multivariate normal distribution. Jung and Ying [5] proposed an adaptation of the Wilcoxon-Mann-Whitney method of estimating linear regression parameters for use in longitudinal data analysis under the working independence model. And recently, Abebe et al . [6] proposed a robust GEE using iterated reweighted rank-based estimation.

In this paper, we adopt the LTS [7] method for robust linear regression in the sense of trimming the data for estimating the regression coefficients so that the observations with high residuals are not included in the parameter estimation. In Section 2 we present a brief review of GEE. In Section 3 we describe our proposed method IRLTS. In Section 4 we discuss some results from our simulation study. II.

GENERALIZED ESTIMATING EQUATION AND IRLS METHOD Let Yij , j = 1, ..., mi, i = 1, ... , n represent the jth measurement on the ith subject. There are mi measurements on subject i and N= ? = n i i m 1 total measurements. Assume that the marginal distribution of y ij is of the exponential class of distributions and is given by: { } f f q q fq ,( )(/)( exp ),,( yc a b y yf + - = where a(.), b(.), and c(. ) are given, q is the canonical parameter, and f is the dispersion parameter.

Let the vector of measurements on the ith subject be Yi = [Yi1, … , Yimi]T with corresponding vector of means µi = [ µi1, … ,µ i mi]T and Xi = [X i1, … ,X i mi]T be the mixp matrix of covariates. In general, the components of Yi are correlated but Yi and Yk are independent for any i ? k. To model the relation between the response and covariates, we can use a regression model similar to the generalized linear models: g(µi) = h i = Xi ß where µi = E( Yi|Xi), g is a specified link function, and ß = [ß1, … ,ßp]T is a vector of unknown regression coefficients to be estimated.

The GEE for estimating the p×1 vector of regression parameters ß is is given by : ( ) ( ) [ ] 0 1 1 = - ¶ ¶ =? = - n i i i i T i S ß µ Y V ß µ ß (1) where Vi be the covariance matrix of Yi modeled as Vi= ( ) 2/1 2/1 i i A a R A l , Ai is a diagonal matrix of variance function V(µ ij ), and R(a ) is the working correlation matrix of Yi indexed by a vector of parameters a . Solutions to (2) are obtained by alternating between estimation of l , a and q .

There are several specific choices of the form of working correlation matrix Ri(a ) commonly used to model Robust Estimation of Generalized Estimating Equation when Data Contain Outliers Khoirin Nisa 1,* , Netti Herawati 2 L 2 INSIST Vol. 2 No. 1, April 2017 (1 - 5) the correlation matrix of Y i. A few of the choices are shown below, one can refer to [1] for additional choices.

The dimension of the vector a, which is treated as a nuisance parameter, and the form of the estimator of a are different for each choice. Some typical choices are: 1. Ri(a ) = R0 , a fixed correlation matrix. For R0 = I, the identity matrix, the GEE reduces to the independence estimating equation. 2. Exchangeable: k j Y Y Cor ikij $^{1}$ = , ) ,( a . 3. Autoregressive-1: || ) ,( kj ikij Y Y Cor - =a . 4. Unstructured: jk ikij Y Y Cor a = ) ,( .

Solving for ß is done with iteratively reweighted least squares (IRLS). The following is the algorithm for fitting the specified model using GEEs [3] : 1. Compute an initial estimate of GEE ß ˆ , for example with an ordinary generalized linear model assuming independence. 2. A current estimate GEE ß ˆ is updated by regressing the working response vector ) ˆ ( ˆ µ y ß µ ß X Z - ¶ ¶ + = * on X.

A new estimate new ß ˆ is obtained by : ** - * = Z W X X W X ß T T 1 ) ( ˆ new (2) where * W is a block diagonal weight matrix whose ith block is the mixmi matrix 1 1 1 ˆ - - - - * ? ? ? ? ? ? ¶ ¶ ? ? ? ? ? ? ? ¶ ¶ = ß µ )A a (R A ß µ W 1 i i i i i i . 3. Use new ß ˆ to update * = = HZ ß X ? new ˆ ˆ , where * - * = W X X) W X(X H T 1 T . 4. Iterate until convergence. III.

ITERATIVELY REWEIGHTED LEAST TRIMMED SQUARE ALGORITHM First let us briefly recall that the robust estimation of regression parameters using LTS method is given by:

? = = h i i LTS e 1 2 min arg ˆ ß (3) which is based on the ordered absolute residuals | | ...| ||| 2 1 n e e e £ £ £ . LTS estimation is calculated by minimizing the h ordered squares residuals, where h can be chosen within 4 1 4 3 1 2 + + £ £ + p n h n , with n and p being sample size and number of parameters respectively. When h = [n/2], , LTS locates that half of the observations which has the smallest estimated variance.

In that case, the breakdown point is 50%. When h is set to the sample size, LTS and ordinary least square (OLS) coincide. In [7] Rousseeuw and Leroy shows n1/2 consistency and asymptotic normality of LTS in the location–scale model. Víšek [8] extends this to the regression model with random regressors, the proof for ?xed regressors is in later series of his papers: [9][10].

When n is very small, it is possible to generate all subsets of size h to determine which one minimizes the LTS criterion. Rousseeuw and Leroy computation of LTS based on subsets of size k requires _ = __ __ subsets which is usually still too large for realistic applications. When n is small enough: 1. Select h. 2. Generate all possible subsets with k observations, and compute the regression coef?cients, say _ 1
, ... , _ _ 1
. 3.

Compute the residuals using all n observations, and from this the LTS criterion. 4. The LTS estimate corresponds to the _ _ _
 that minimizes the objective function (3). Rousseeuw and van Driessen [11] propose a fast algorithm for computing LTS. The trick is to iterate a few steps on a large number of starting values, and keep the 10 (say) most promising ones.

These are then used for full iteration, yielding the ?nal estimate. The resulting algorithm makes LTS estimation faster. Our proposed procedure is a combination of IRLS and LTS methods. IRLTS estimator involves computing the hyperplane that minimizes the sum of the smallest h squared residuals and use the weighted least square estimation for ß in each iteration. To motivate our estimator and following the fast LTS algorithm [11], it is convenient to write IRLTS algorithm with involving the residuals as follow.

Concentration-step: 1. Choose h observations. 2. Compute ß ˆ based on h observations using IRLS method . 3. Use the estimate ß ˆ to calculate residuals: ij ij ij Y e m ˆ - = based on equation ) ˆ ( ˆ 1 ß X µ i - =g i of n observations. 4. Sort | | ij e for j = 1, ..., mi, i = 1, ... , n in ascending order: | |...| || | 12 11 ij e e e £ £ £ . 5.

Choose h observations which have the lowest h residuals, we denote the h observations as subset H. The repetitions of concentration-step will produce an iteration process.

IRLTS algorithm: 1. Choose h observations. 2. Compute ß ˆ based on h observations using IRLS by (2), we obtain ) ˆ ( ˆ 1 ß X µ i - =g i . 3. Calculate residuals: ij ij ij Y e m ˆ - = of n observations. 4.

Sort | | ij e in ascending order: ij e e e | ...| || | 12 11 £ £ £ | 5. Choose h1 observations which have the lowest h1 residuals, we denote as subset H 1. 6. Run concentration-step on H 1 twice, and we obtain H 1*. 7. Repeat step 1- step 6 for ? ? ? ? ? ? ? ? h n times 3 INSIST Vol. 2 No. 1, April 2017 (1 - 5) 8. From the ? ? ? ? ? ? ? ? h n results, choose the best 10 subsets H q, q=1,...,10. 9. Run concentration-step on the best 10 subsets H q until convergence. 10.

Choose the best subset H. IV. SIMULATION STUDY To look at the performance of the proposed method, we have done a simulation study by generating N=1000 observations from 200 subjects with 5 repeated measures. The model for data generation is as follows: uij = ß0+ß1 xij where ß0=ß1=1, i=1,2,... 200 and j=1,2,..,5. The covariates xij are i.i.d.

from a uniform distribution Unif(1,5). For this longitudinal data the normal distributed model is used. We generated data based on the underlying true correlation structures as exchangeable (EXCH) and autoregressive-1 (AR1) with a=0.3 and 0.7. We considered data without outliers ( e = 0%) as well as contaminated data ( e = 10%, 20% and 30%).

The contamination is generated from normal distribution N(100,1), we set two cases for the contamination, i.e. randomly spread over the sample (case A) and randomly spread over the half upper xij values of the sample (case B). For each scenario 1000 Monte Carlo data sets were generated.

We evaluated the results using relative efficiency (RE) of IRLTS to IRLS and the mean square error (MSE) of b ˆ which we define as __ _____/____ = ___ _ __ _____

____ _ __ ____
!" and #$_ = " "%%% ? '_ __ (
- __* +, with 0 = 0,1 "%%% (2" , where Var (.) is the variance. We provide the expected values (E), and the relative efficiency resulted from our simulation in Table I - Table IV and the MSEs in Table V- Table VI.

The efficiency of IRLTS and IRLS for clean data (i.e. when e = 0%) is almost equal since RE ~ 1 for each case, but IRLTS is more efficient than IRLS when data contain outliers. The parameter estimates of IRLS are much more influenced by the outliers than the parameter estimates of IRLTS.

From the expected values we can see that the more outliers contained in the data the larger the deviation of IRLS estimates from the parameter (i.e. 1 1 0 = =b b ), while the parameter estimates of IRLTS are almost stable and close to the parameter. Table 1. Simulation Result for Longitudinal Data with Exchangeable Correlation Matrix with 3.0 = a Case Coeff.

e ) ˆ ( IRLS E b ) ˆ ( IRLTS E b RE (IRLTS/IRLS) Case A 0 ˆ b 0% 1.00500 1.00499 1.00178 10% 10.08781 1.02474 0.00294 20% 17.55050 1.08504 0.00401 30% 23.73423 1.15940 0.00657 1 ˆ b 0% 0.99846 0.99848 1.00278 10% 0.99604 0.99948 0.00267 20% 0.98073 0.99775 0.00393 30% 0.94544 0.99577 0.00649 Case B 0 ˆ b 0% 1.02179 1.01050 0.99466 10% -5.20456 1.00195 0.00577 20% -10.30572 0.95625 0.01606 30% -14.27400 0.84177 0.03562 1 ˆ b 0% 0.99318 0.99693 0.99369 10% 5.95079 0.98675 0.00483 20% 9.99122 0.99425 0.05797 30% 13.21288 1.04650 0.11281 Table 2. Simulation Result for Longitudinal Data with Exchangeable Correlation Matrix with 7.0 = a Case Coeff.

e ) ˆ ( IRLS E b ) ˆ ( IRLTS E b RE (IRLTS/IRLS) Case A 0 ˆ b 0% 1.01266 1.01187 1.01763 10% 9.93676 1.01672 0.00433 20% 17.31940 1.05446 0.00505 30% 23.68107 1.16184 0.00769 1 ˆ b 0% 0.99600 0.99613 1.01882 10% 1.04154 1.00103 0.00403 20% 1.05223 1.00661 0.00478 30% 0.95609 0.99443 0.00770 Case B 0 ˆ b 0% 1.05174 1.01922 0.99052 10% -5.20125 1.00911 0.00837 20% -10.28303 0.98818 0.00528 30% -14.25209 0.83684 0.04304 1 ˆ b 0% 0.98159 0.99245 0.99948 10% 5.93523 0.98438 0.00722 20% 9.96316 0.97828 0.00489 30% 13.21954 1.05368 0.12544 4 INSIST Vol. 2 No. 1, April 2017 (1 - 5) Table 3.

Simulation Result for Longitudinal Data with Autoregressive-1 Correlation Matrix with 3.0 = a Case Coeff. e ) ˆ ( IRLS E b ) ˆ ( IRLTS E b RE (IRLTS/IRLS) Case A 0 ˆ b 0% 0.99848 0.99744 1.03077 10% 10.04370 1.02103 0.00247 20% 17.60751 1.07994 0.00359 30% 23.63332 1.15166 0.00621 1 ˆ b 0% 1.00043 1.00079 1.04004 10% 1.00954 1.00069 0.00217 20% 0.96174 0.99879 0.00356 30% 0.97699 0.99897 0.00625 Case B 0 ˆ b 0% 1.01800 1.00963 1.00481 10% -5.40516 0.99210 0.00113 20% -10.07597 0.94809 0.00393 30% -14.18714 0.85628 0.05029 1 ˆ b 0% 0.99341 0.99621 1.00108 10% 5.98712 0.99180 0.00541 20% 9.89834 0.99916 0.01079 30% 13.19889 1.04016 0.16313 Table 4. Simulation Result for Longitudinal Data with Autoregressive-1 Correlation Matrix with 7.0 = a Case Coeff.

e ) ˆ ( IRLS E b ) ˆ ( IRLTS E b RE (IRLTS/IRLS) Case A 0 ˆ b 0% 1.00024 1.00079 1.07947 10% 9.93814 1.00872 0.00412 20% 17.52048 1.07392 0.00486 30% 23.77262 1.16751 0.00790 1 ˆ b 0% 0.99960 0.99939 1.06915 10% 1.04489 1.00404 0.00379 20% 0.98595 1.00065 0.00481 30% 0.94520 0.99331 0.00785 Case B 0 ˆ b 0% 1.03666 1.01330 0.99589 10% -5.18276 1.01138 0.00618 20% -10.37403 0.97097 0.00484 30% -14.34357 0.80114

0.04723 1 ˆ b 0% 0.98765 0.99545 0.99343 10% 5.92778 0.98511 0.00464 20% 10.00611 0.98523 0.00461 30% 13.25732 1.07041 0.14581 The consistency of the estimators is assessed through their MSEs (see Table V and Table VI).

When data contain outliers, the MSEs of IRLTS are relatively small compared to the MSEs of the classical GEE (IRLS). From the result we conclude that IRLTS is robust against outliers. Table 5. Mean Square Error of Parameter Estimates for Data with Exchangeable Correlation Matrix Case Coeff. e 3.0 = a 7.0 = a IRLS IRLTS IRLS IRLTS Case A 0 ˆ b 0% 0.01664 0.01666 0.03141 0.03194 10% 90.44780 0.02372 88.48682 0.03765 20% 291.41893 0.07749 283.90429 0.09182 30% 544.84787 0.20944 540.15839 0.22414 1 ˆ b 0% 0.00165 0.00165 0.00299 0.00305 10% 0.87157 0.00233 0.94041 0.00378 20% 1.91855 0.00754 1.92692 0.00924 30% 3.08263 0.02001 2.82052 0.02174 Case B 0 ˆ b 0% 0.01726 0.01680 0.03467 0.03206 10% 41.61809 0.01801 41.89402 0.02886 20% 136.26362 0.13754 133.06421 0.03051 30% 240.63699 0.28655 240.56976 0.36852 1 ˆ b 0% 0.00171 0.00167 0.00346 0.00318 10% 24.88252 0.00197 24.75140 0.00309 20% 82.04616 0.06984 80.97956 0.00361 30% 149.95039 0.09196 150.19042 0.11242 Table 6.

Mean Square Error of Parameter Estimates for Data with Autoregressive-1 Correlation Matrix Case Coeff. e 3.0 = a 7.0 = a IRLS IRLTS IRLS IRLTS Case A 0 ˆ b 0% 0.01309 0.01349 0.02329 0.02515 10% 89.85427 0.02033 87.75463 0.03249 20% 292.90715 0.06774 289.43715 0.08572 30% 539.61490 0.19286 543.38625 0.22402 1 ˆ b 0% 0.00131 0.00136 0.00231 0.00247 10% 0.88483 0.00192 0.86341 0.00328 20% 1.85020 0.00659 1.77790 0.00856 30% 2.98370 0.01864 2.71468 0.02134 Case B 0 ˆ b 0% 0.01354 0.01337 0.02704 0.02577 10% 87.20910 0.05218 42.39676 0.02589 20% 164.20519 0.16609 135.05622 0.02836 30% 237.06421 0.34325 243.38623 0.41559 1 ˆ b 0% 0.00135 0.00133 0.00273 0.00258 10% 27.51679 0.01437 24.80085 0.00262 20% 86.50801 0.07905 81.76371 0.00323 30% 149.53554 0.11950 151.10469 0.13076 31 INSIST Vol. 2 No. 1, April 2017 (1 - 5) V.

C ONCLUSION Our proposed method have two different iterations in its procedure, one is the iteration for the estimation of regression parameter using IRLS method, and the other iteration is for selecting the best subset H for calculating the parameter estimate. We have shown that this procedure can minimize the effect of outliers on parameter estimation; IRLTS can produce a relatively efficient and consistent estimator compared to the classical GEE (IRLS). Base on the MSE, IRLTS performs much better than the classical GEE.

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www.arpnjournals.com ON GENERALIZED VARIANCE OF NORMAL-POISSON MODEL AND POISSON VARIANCE ESTIMATION UNDER GAUSSIANITY Khoirin Nisa1,*, Célestin C. Kokonendji2, Asep Saefuddin3, Aji Hamim Wigena3, I Wayan Mangku4 1 Department of Mathematics, University of Lampung, Bandar Lampung, Indonesia 2Laboratoire de Mathématiques de Besançon, Université Bourgogne Franche-Comté, France 3 Department of Statistics, Bogor Agricultural University, Bogor, Indonesia 4 Department of Mathematics, Bogor Agricultural University, Bogor, Indonesia * Corresponding author: khoirin.nisa@fmipa.unila.ac.id ABSTRACT As an alternative to full Gaussianity, multivariate normal-Poisson model has been recently introduced.

The model is composed by a univariate Poisson variable, and the remaining random variables given the Poisson one are real independent Gaussian variables with the same variance equal to the Poisson component. Under the statistical aspect of the generalized variance of normal-Poisson model, the parameter of the unobserved Poisson variable is estimated through a standardized generalized variance of the observations from the normal components. The proposed estimation is successfully evaluated through simulation study.

Keywords: Covariance matrix, determinant, exponential family, generalized variance, in?nitely divisible measure.

# INTRODUCTION

Normal-Poisson model is a special case of normal stable Tweedie (NST) models which were introduced by Boubacar Maïnassara and Kokonendji [1] as the extension of normal gamma [2] and normal inverse Gaussian [3] models.

The NST family is composed by distributions of random vector X = (X1, ..., Xk)T where Xj is a univariate (non-negative) stable Tweedie variable and (X1, ..., Xj-1, Xj+1,...,Xk)T =: ?? ?? ?? given Xj are k-1 real independent Gaussian variables with variance Xi, for any ?xed j ( {1, 2, ..., k}. Several particular cases have already appeared in different contexts; one can refer to [1] and references therein.

Normal-Poisson is the only NST model which has a discrete component and it is correlated to the continuous normal parts. Similar to all NST models, this model was introduced in [1] for a particular case of j that is j=1. For a normal-Poisson random vector X as described above, Xj is a univariate Poisson variable.

In literatures, there is a model called "Poisson Gaussian" [4][5] which is also composed by Poisson and normal distributions. However, normal-Poisson and Poisson Gaussian are two completely different models. Indeed, for any value of j ( {1, 2, ..., k}, a normal-Poissonj model has only one Poisson component and k-1 Gaussian components, while a Poisson-Gaussianj model has j Poisson components and k-j Gaussian components which are all independent.

Normal-Poisson is also different from the purely discrete Poisson-normal model of Steyn [6] which can be defined as a multiple mixture of k independent Poisson distributions with parameters m1, m2, ... , mk and those parameters have a multivariate normal distribution. Hence, the multivariate Poisson-normal distribution is a multivariate version of the Hermite distribution [7]. Generalized variance (i.e.

the determinant of covariance matrix expressed in term of mean vector) has important roles in statistical analysis of multivariate data. It was introduced by Wilks [8] as a scalar measure of multivariate dispersion and used for overall multivariate scatter. The uses of generalized variance have been discussed by several authors.

In sampling theory, it can be used as a loss function on multiparametric sampling allocation [9]. In the theory of statistical hypothesis testing, generalized variance is used as a criterion for an unbiased critical region to have the maximum Gaussian curvature [10]. In the descriptive statistics, Goodman [11] proposed a classification of some groups according to their generalized variances.

In the last two decades the generalized variance has been extended for non-normal distributions in particular for natural exponential families (NEFs) [12][13]. Three generalize variance estimators of normal-Poisson models have been introduced (see [14]). Also, the characterization by variance function and by generalized variance of normal-Poisson have been successfully proven (see [15]).

In this paper, a new statistical aspect of normal Poisson model is presented, i.e. the Poisson variance estimation under only observations of normal components leading to an extension of generalized variance term i.e. the "standardized generalized variance".

NORMAL POISSON MODELS The family of multivariate normal-Poissonj models for all j ( {1, 2, ..., k} and fixed positive integer k>1 is defined as follows. Definition 1. For X = (X1, ..., Xk)T a k-dimensional normal-Poisson random vector, it must hold that Xj is a univariate Poisson random variable, and ?? ?? ?? :=(X1, ..., Xj-1, Xj+1,...,Xk)T given Xj follows the (??-1)-variate normal Nk-1(0, XjIk-1) distribution, where Ik-1 = diagk-1(1, ..., 1) denotes the (k-1)× (k-1) unit matrix.

In order to satisfy the second condition we need Xj > 0. But in practice it is possible to have Xj = 0 in the Poisson component. In this case, the corresponding normal components are degenerated as the Dirac mass (0 which makes their values become 0s. We have shown that zero values in Xj do not affect the estimation of the generalized variance of normal-Poisson [16].

From Definition 1, for a fixed power of convolution t > 0 and given j ( {1, 2, ..., k}, denote Ft;j=F((t;j) the multivariate NEF of normal-Poisson with (t :=( t *, the NEF of a k-dimensional normal-Poisson random vector X is generated by / where 1A is the indicator function of the set A. Since t>0 then (t;j is known to be an infinitely divisible measure; see, e.g., Sato [17].

The cumulant function of normal-Poisson is obtained from the logarithm of the Laplace transform of (t;j , i.e. ?? ( ??;?? (??)=log ?? ?? exp ?? T ?? ( ??;?? (????) and the probability distribution of normal-Poissonj which is a member of NEF is given by P(??; ( ??;?? )(????) = exp ?? T ??- ?? ( ??;?? ?? ( ??;?? (????) The mean vector and the covariance matrix of Ft;j can be calculated using the first and the second derivatives of the cumulant function, i.e.: ??= ??' ( ??;?? (??) and ?? ?? ??;?? (??)= ?? '' ( ??;?? ?? ?? .

For practical calculation we need to use the following mean parameterization: P ??; ?? ??;?? ??? ?? ?? ; ( ??;?? , where ?? ?? is the solution in ?? of the equation ??= ?? ' ( ??;?? ?? . Then for a fixed j ( {1, 2, ..., k}, the variance function (i.e the variance-covariance matrix in term of mean parameterization) is given by ?? ?? ??;?? ?? = 1 ?? ?? ?? ?? T +diag( ?? ?? , ..., ?? ?? , 0 ?? , ?? ?? , ..., ?? ?? ) (2) on its support ?? ?? ??;?? = {μ ( Rk; μj > 0 and μl ( R for

I ?j}.

(3) For j = 1, the covariance matrix of X can be expressed as follows: / Indeed, for the covariance matrix above one can use the Schur complement [18] of a matrix block to obtain the following representation of determinant / with the non-null scalar ( = (1, the vector aT=((2, ..., (k) and the (k-1) ×(k-1) matrix A = (-1aaT + (1Ik-1, where Ij= =diag(1, ..., 1) is the j×j unit matrix.

Consequently, the determinant of the covariance matrix for j = 1 is det ?? ?? ??;1 ?? = ?? 1 ?? with ????? ?? ??;1 Then, it is trivial to show that for j ( {1, 2, ..., k} the generalized variance of normal-Poissonj model is given by det ?? ?? ??;1 ?? = ?? ?? ?? with ????? ?? ??;?? (5) Equation (5) expresses the generalized variance of normal-Poisson model depends only on the mean of the Poisson component and the dimension space k >1.

CHARACTERIZATIONS AND GENERALIZED VARIANCE ESTIMATIONS Among NST models, normal-Poisson and normal-gamma are the only models which are already characterized by generalized variance (see [19] for characterization of normal-gamma by generalized variance). In this section we present the characterizations of normal-Poisson by variance function and by generalized variance, then we present three estimations of generalized variance by maximum likelihood (ML), uniformly minimum variance unbiased (UMVU) and Bayesian methods.

Characterization The characterizations of normal-Poisson models are stated in the following theorems without proof. Theorem 1 Let k ( {2,3, ...} and t>0. If an NEF Ft;j satisfies (2) for a given j ( {1,2, ...,k}, then up to affinity, Ft;j is a normal-Poissonj model. Theorem 2 Let Ft;j=F( ( ??;?? ) be an infinitely divisible NEF on Rk (k>1) such that ?? ( ??;?? = ?? ?? and det ?? '' ( ??;?? ?? = ?? exp?(??× ?? T ?? ?? ?? ) for ?? =((1, .... (k)T and ?? ?? ?? =((1,..., (j-1, 1, (j+1, ..., (k)T. Then Ft,j is of normal-Poisson type. All technical details of proofs can be seen in [15].

In fact, the proof of Theorem 1 is established by analytical calculations and using the well-known properties of NEFs described in Proposition 3 below. Proposition 3 Let ( and ( be two (-finite positive measures on Rk such that F=F(( )$, ?? =??( ( ) and µ(MF. (i) If there exists (d,c)( ?? ?? ×?? such that ( (????) = exp (dTx+c)((dx), then ??= ?? : ?? ?? = ?? ?? -?? and ?? ?? ?? = ?? ?? ??+?? +??; for ?? =?? ? ?? ?? , ?? ?? ?? = ?? ?? (??) and det ?? ?? ?? = det ?? ?? ?? .

(ii) If ?? =??*?? with ?? ?? =????+??, then: ?? ?? = ?? T ?? ?? and ?? ?? ?? = ?? ?? ?? T ?? + ?? T ??; for ?? =????+?? ? ?? ?? ?? , ?? ?? ( ?? )=?? ?? ?? ?? -1 ?? ?? T and det ?? ?? ?? = det ?? ?? det ?? ?? (??) . (iii) If ?? = ?? * is the t-th convolution power of ?? for t>0, then, for

?? =???? ? ?? ?? ?? , ?? ?? ( ?? )=?? ?? ?? ?? -1 ?? and det ?? ?? ( ?? )= ?? ?? det ?? ?? ?? The proof of Theorem 2 is obtained by using the infinite divisibility property of normal-Poisson, also applying two properties of determinant and affine polynomial.

The infinite divisibility property used in the proof is provided in Proposition 4 below. Proposition 4 If ?? is an infinitely divisible measure on Rk, then there exist a symmetric non-negative definite d×d matrix ( with rank r(k and a positive measure ( on Rk such that ??" ?? ?? =S+ ?? ?? ?? ?? T exp ?? T ?? ??(????). See, e.g. [20, page 342].

The above expression of ??" ?? ?? is an equivalent of the Lévy-Khinchine formula [17]; thus, ( comes from a Brownian part and the rest ??" ?? ?? = ?? ?? ?? ?? T exp ?? T ?? ??(????) corresponds to jumps part of the associated Lévy process through the Lévy measure (. Generalized Variance Estimators Let X1, ..., Xn be random vectors i.i.d.

with distribution P((;Ft;j) in a normal-Poissonj model Ft;j=F((t;j) for fixed j ( {1, 2, ..., k}. Denoting ?? = ?? 1 +...+ ?? ?? ?? = ( X 1 ,..., X ?? ) T the sample mean. Maximum Likelihood Estimator The ML generalized variance estimator of normal Poisson model det ?? ?? ??;?? ?? = ?? ?? ?? is given by ?? ??,??;?? = det ?? ?? ??;?? ?? = ?? ?? ?? . (6) The ML estimator (6) is directly obtained from (5) by substituting µj with its ML estimator ?? ?? .

For all p=1, Tn,t;j is a biased estimator of det ?? ?? ??;?? ?? with a given quadratic risk with tedious calculation of explicit expression or infinite. Uniformly Minimum Variance Unbiased Estimator The UMVU generalized variance estimator of normal Poisson models det ?? ?? ??;?? ?? = ?? ?? ?? is given by ?? ??,??;?? = ?? -??+1 ?? ?? ?? ?? ?? -1 ...(?? ?? ?? -??+1) , if ?? ?? ?? =?? (7) The UMVU estimator of det ?? ?? ??;?? ?? is deduced by using intrinsic moment formula of univarite Poisson distribution as follows ?? ?? ?? ?? ?? -1 ... ?? ?? -??+1 = ?? ?? ?? . Indeed, letting ?? ?? =?? ?? ?? gives the result that (7) is the UMVU estimator of (5).

Because, by the completness of NEF, the unbiased estimator is unique. Bayesian Estimator Under assumption of prior gamma distribution of µj with parameter ( > 0 and ( > 0, the Bayesian estimator of det ?? ?? ??;?? ?? = ?? ?? ?? is given by ?? ??,??;??,??,?? = ??+?? ?? ?? ??+?? ?? .

(8) To show this, let Xj1, ..., Xjn given µj be Poisson distribution with mean µj, then the probability mass function is given by ?? ?? ???? | ?? ?? = ?? ?? ?? ???? ?? ???? ! exp ( -?? ?? ) ? ?? ???? ?? ?? Assuming that µj follows gamma((,(), then the prior probability distribution function of µj is written as ?? ?? ?? ,??,?? = ?? ?? ??(??) ?? ?? ??-1 exp -???? ?? , ? ?? ?? >0 with G ?? ? 0 8 ?? ??-1 ?? -?? ???? .

Using the classical Bayes theorem, the posterior distribution of ?? ?? given an observation xji can be expressed as ?? ?? ?? | ?? ???? ;??,?? = ?? ?? ???? ?? ?? ?? ?? ?? ,??,?? ?? ?? >0 ?? ?? ???? ?? ?? ?? ?? ?? ,??,?? ?? ?? ?? = (??+1) ??+ ?? ???? ??(??+ ?? ???? ) ?? ?? ??+ ?? ???? -1 exp -(??+1) ?? ?? which is the gamma density with parameters ('=(+xji , ('=( +1. Then with random sample Xj1, ..., Xjn the posterior will be gamma(??+?? ?? ?? , ??+?? ) .

Since Bayesian estimator of μj is given by the expected value of the posterior distribution i.e. ??+?? ?? ?? ??+?? , then this will lead to (8). MAIN RESULT Poisson Variance Estimation under Gaussianity For a given random vector X = (X1, ..., Xk)T on Rk of normal-Poissonj , we now assume that only k-1 normal terms ?? ?? ?? of X are observed: ?? ??1 ?? , ..., ?? ???? ?? and, therefore, Xj is an unobserved Poisson random effect. Note that j is fixed in {1,2,...,k}.

Assuming t=1 and following [1] with X having mean vector μ=( μ1, ..., μk)T ( ?? ?? 1;?? and covariance matrix V=V(μ), then ?? ?? ?? follows a (k-1)-variate normal distribution, denoted by ?? ?? ?? ~ Nk-1 (?? ?? ?? , ?? ?? ?? ?? ?? ), (9) with ?? ?? ?? = ( ?? 1 ,..., ?? ??-1 , ?? ??+1 ,..., ?? ?? ) T . The (k-1)×(k-1)-matrix ?? ?? ?? (which does not depend on ?? ?? ?? ) is symmetric and positive definite such that det ?? ?? ?? =1 or ?? ?? ?? = ?? ??-1 .

Thus, without loss of generality, Xj in (9) can be a univariate Poisson variable with parameter μj >0 which is known to be at the same time the mean and the variance. It follows that the unit generalized variance of ??= ( ?? ?? , ?? ?? ??T ) T is easily deduced as ?? ?? ?? . Hence, the Poisson parameter (j of Xj can be estimated through generalized variance estimators of normal observations in the sense of ``standardized generalized variance" [21]: / or / with ?? ?? ?? =( ?? ??1 ?? +...+ ?? ???? ?? )/?? and ?? l =( ?? l1 +...+ ?? l?? )/??.

This statistical aspect of normal-Poissonj models in (9) points out the flexibility of these models compared with the classical multivariate normal model Nk-1 (?? ?? ?? ,??), where the generalized variance det( is replaced to the random effect ?? ?? ?? ?? ?? . In fact, for ?? ?? ?? = ?? ??-1 in (9) with estimation ?? ?? of (10) which corresponds to Part 2 of Definition 1, one has a kind of conditional homoscedasticity under the assumption of normality. However, we here have to handle the presence of zeros in the sample of Xj when the Poisson parameter μj is close to zero.

More precisely and without loss of generality, within the framework of one-way analysis of variance and keeping the previous notations, since there are at least two normal components to be tested, so the minimum value of k is 3 (or k= 3) for representing the number of levels k-1. Simulation Study We present empirical analyses through simulation study to evaluate the consistency of ?? ?? .

In order to apply this point of view, one can refer to [21] for a short numerical illustration; or in the context of multivariate random effect model, it can be used as the distribution of the random effects when they are assumed to have conditional homoscedasticity. Using the standardized generalized variance estimation in (10) we assume that the Poisson component is unobservable and we want to estimate ?? ?? based on observations of normal components. In this simulation, we fixed j=1 and we set some sample sizes n = 30, 50, 100, 300, 500, 1000.

We consider k=3, 4, 6, 8 to see the effects of k on the standardized generalized variance estimations. Moreover, to see the effect of zero values proportion within Xj, we also consider small mean (variance) values on the Poisson component i.e. ?? ?? = 0.5, 1, 5, because P(Xj=0)=exp(-μj). We generated 1000 samples for each case.

From the resulted ?? ?? values of the generated samples we obtained the expected values and variance of ?? ?? i.e. E( ?? ?? ) and Var( ?? ?? ) respectively. Then we calculated their MSE using the following formula MSE( ?? ?? ) = [E( ?? ?? )-μj]2+ Var( ?? ?? ), where E( ?? ?? )= 1 1000 ??=1 1000 ?? ?? (??) and Var( ?? ?? )= 1 999 ??=1 1000 ?? ?? (??) -??( ?? ?? ) 2 We report the expected values and MSE of ?? ?? in Table 1 - Table 3. Table 1.

The expected values and MSE of ?? ?? with 1000 replications for n ( {30,50,100,300,500,1000}, k ( {3,4,6,8}, and μj = 0.5. k _n _E( ?? ?? ) _MSE( ?? ?? ) _ _3 _30 _0.473270 _0.039251 _ _ _50 _0.487402 _0.023864 _ _ _100 _0.491117 _0.010882 _ _ _300 _0.495814 _0.004058 _ _ _500 _0.496612 _0.002540 _ _ _1000 _0.499035 _0.001158 _ _4 _30 _0.465915 _0.031980 _ _ _50 _0.488503 _0.019574 _ _ _100 _0.491804 _0.009975 _ _ _300 _0.494617 _0.003457 _ _ _500 _0.496200 _0.002019 _ _ _1000 _0.498271 _0.000968 _ _6 _30 _0.452953 _0.026781 _ _ _50 _0.478994 _0.015763 _ _ _100 _0.483284 _0.007801 _ _ _300 _0.495324 _0.002713 _ _ _500 _0.496771 _0.001562 _ _ _1000 _0.497542 _0.000771 _ _8 _30 _0.454636 _0.023539 _ _ _50 _0.468367 _0.014280 _ _ _100 _0.482915 _0.007374 _ _ _300 _0.495749 _0.002395 _ _ _500 _0.499078 _0.001542 _ _ _1000 _0.499199 _0.000726 _ _ Table 2.

The expected values and MSE of ?? ?? with 1000 replications for n ( {30,50,100,300,500,1000}, k( {3,4,6,8} , and μj = 1. k _n _E( ?? ?? ) _MSE( ?? ?? ) _ _3 _30 _0.962617 _0.095854 _ _ _50 _0.983720 _0.055901 _ _ _100 _0.993564 _0.029386 _ _ _300 _0.994837 _0.010214 _ _ _500 _0.997781 _0.005969 _ _ _1000 _0.998467 _0.003125 _ _4 _30 _0.955849 _0.078891 _ _ _50 _0.973454 _0.049405 _ _ _100 _0.981452 _0.023848 _ _ _300 _0.992874 _0.007467 _ _ _500 _0.996215 _0.004848 _ _ _1000 _1.001149 _0.002456 _

_6 _30 _0.944165 _0.058871 _ _ _50 _0.972215 _0.033577 _ _ _100 _0.985437 _0.017781 _ _ _300 _0.992045 _0.006229 _ _ _500 _0.995822 _0.003725 _ _ _1000 _0.998113 _0.001752 _ _8 _30 _0.944031 _0.052258 _ _ _50 _0.973103 _0.032476 _ _ _100 _0.981169 _0.015210 _ _ _300 _0.992240 _0.005135 _ _ _500 _0.998451 _0.002981 _ _ _1000 _0.999042 _0.001400 _ _ From the results in the tables we can see that when the sample size (n) increases, the expected values of ?? ?? converge to the target value (µj) for all µj values we consider here. Also, the MSE of ?? ?? decrease when sample size increase for all dimension k, this means that ?? ?? is consistent.

The simulation results with moderate sample sizes produce very good performances of ?? ?? . Note that the presence of zeros in the samples of the Poisson component does not affect the estimation of µj. For a clear description of the performance of ?? ?? , we provide the bargraphs of MSE of ?? ?? in Figure 1 - Figure 3.

The figures show that MSE value decrease when the sample size increase. From the result we conclude that ?? ?? is a consistent estimator of µj. Notice that ?? ?? produce smaller MSE for larger dimension. / Figure 1. Bargraph of MSE( ?? ?? ) for µj = 0.5 Table 3. The Expected Values and MSE of ?? ?? with 1000 replications for n ( {30,50,100,300,500,1000}, k ( {3,4,6,8} , and µj = 5.

k _n _E( ?? ?? ) _MSE( ?? ?? ) _ _3 _30 _4.886415 _1.120641 _ _ _50 _4.942883 _0.690184 _ _ _100 _4.984949 _0.356851 _ _ _300 _4.987437 _0.118193 _ _ _500 _4.992459 _0.064591 _ _ _1000 _5.006692 _0.035814 _ _4 _30 _4.856583 _0.928853 _ _ _50 _4.921017 _0.511915 _ _ _100 _4.950201 _0.269422 _ _ _300 _4.983517 _0.086223 _ _ _500 _4.988398 _0.050144 _ _ _1000 _4.988551 _0.025137 _ _6 _30 _4.852608 _0.589918 _ _ _50 _4.926390 _0.354075 _ _ _100 _4.942147 _0.175198 _ _ _300 _4.974067 _0.056670 _ _ _500 _4.995231 _0.033951 _ _ _1000 _4.996774 _0.016958 _ _8 _30 _4.838751 _0.457897 _ _ _50 _4.910668 _0.281625 _ _ _100 _4.949142 _0.135143 _ _ _300 _4.985705 _0.046987 _ _ _500 _4.990750 _0.027643 _ _ _1000 _4.998134 _0.013399 _ _
/ Figure 2. Bargraph of MSE( ?? ?? ) for µj = 1 / Figure 3.

Bargraph of MSE( ?? ?? ) for µj = 5 CONCLUSION In this paper we discussed some properties of normal-Poisson model, its characterizations by variance function and by generalized variance, and also its generalized variance estimators. Then we showed that the variance (which is also the mean) of unobserved Poisson component can be estimated through the standardized generalized variance of the (k-1) normal components. The result from simulation study gives a conclusion that ?? ?? is a consistent estimator of the Poisson variance. REFERENCES Y.

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