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ON SUB-EXACT SEQUENCES

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Abstract

We introduce and study the notion of a sub-exact sequence.

1. Introduction

Let R be a ring and let $A \xrightarrow{f} B \xrightarrow{g} C$ be an exact sequence of R-modules, i.e.,

$$Im\, f = Ker\, g (= g^{-1}(0)).$$

Davvaz and Parnian-Garamaleky [1] introduced the concept of quasi-exact sequences by replacing the submodule 0 by a submodule $U \subseteq C$. A sequence

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of R-modules and R-homomorphisms $A \xrightarrow{f} B \xrightarrow{g} C$ is quasi-exact at B or U-exact at B if there exists a submodule U in C such that $Im\, f = g^{1}(U)$.

Anvariyeh and Davvaz [2] proved further results about quasi-exact sequences and introduced a generalization of Schanuel lemma. Moreover, they obtained some relationships between quasi-exact sequences and superfluous (or essential) submodules.

Furthermore, Davvaz and Shabani-Solt [3] introduced a generalization of some notions in the homological algebra. They gave a generalization of the Lambek lemma, Snake lemma, connecting homomorphism and exact triangle and they established new basic properties of the U-homological algebra. In [4], Anvariyeh and Davvaz studied U-split sequences and established several connections between U-split sequences and projective modules.

In this paper, we introduce a new notion of an exact sequence which is called a *sub-exact sequence*. A sub-exact sequence is a generalization of an exact sequence. Let K, L, M be R-modules and X be a submodule of L. The triple (K, L, M) is said to be *X-sub-exact at L* if there is a homomorphism making $K \to X \to M$ exact at X. We collect all submodules X of L such that the triple (K, L, M) is X-sub-exact at L, which we denote by $\sigma(K, L, M)$. In this paper, we investigate whether $\sigma(K, L, M)$ is closed under submodules, products and extensions. Moreover, we provide necessary condition for $\sigma(K, L, M)$ so that it has a maximal element.

2. Main Result

Definition. Let K, L, M be R-modules and X be a submodule of L. Then the triple (K, L, M) is said to be *X-sub-exact at L* if there exist R-homomorphisms f and g such that the sequence of R-modules and R-homomorphisms

$$K \xrightarrow{f} X \xrightarrow{g} M$$

is exact.

Example 2.1. Let $K = 4\mathbb{Z}$, $L = \mathbb{Z}$ and $M = \mathbb{Z}/4\mathbb{Z}$ be $\mathbb{Z}$-modules. Then the triple $(4\mathbb{Z}, \mathbb{Z}, \mathbb{Z}/4\mathbb{Z})$ is $4\mathbb{Z}$-sub-exact at $\mathbb{Z}$ since there are the identity $i : 4\mathbb{Z} \to 4\mathbb{Z}$ and canonical homomorphism (projection) $\pi : 4\mathbb{Z} \to \mathbb{Z}/4\mathbb{Z}$ such that the sequence $4\mathbb{Z} \xrightarrow{i} 4\mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/4\mathbb{Z}$ is exact at $4\mathbb{Z}$.

Now, we give an example where the sequence $K \to L \to M$ is not exact, but the triple (K, L, M) is X-sub-exact, for some submodule X of L.

Example 2.2. Let $K = \mathbb{Z}_2$, $L = \mathbb{Z}_2 \oplus \mathbb{Z}_3$ and $M = 0$ be $\mathbb{Z}$-modules. Then the triple $(\mathbb{Z}_2, \mathbb{Z}_2 \oplus \mathbb{Z}_3, 0)$ is $\mathbb{Z}_2$-sub-exact at $\mathbb{Z}_2 \oplus \mathbb{Z}_3$ since for the homomorphism $i : \mathbb{Z}_2 \to \mathbb{Z}_2 \oplus \mathbb{Z}_3$, given by $i(a) = (a, 0)$, for every $a \in \mathbb{Z}_2$, the sequence

$$\mathbb{Z}_2 \xrightarrow{i} \mathbb{Z}_2 \oplus \mathbb{Z}_3 \to 0$$

is sub-exact at $\mathbb{Z}_2 \oplus \mathbb{Z}_3$.

But, we cannot define an epimorphism p from $\mathbb{Z}_2$ to $\mathbb{Z}_2 \oplus \mathbb{Z}_3$.

Remark 2.3. Since the sequence $K \to \{0\} \to M$ is exact, the triple (K, L, M) is $\{0\}$-sub-exact for any R-modules K, L, M.

Remark 2.4. Let K be an R-module.

(a) Since there are the identity $i : K \to K$ and zero homomorphism $\theta : K \to K$ such that the sequence $K \xrightarrow{i} K \xrightarrow{\theta} K$ is exact at K, the triple (K, K, K) is K-sub-exact at K.

(b) Since the identity $i : K \to K$ is surjective, the sequence $K \xrightarrow{i} K \xrightarrow{\theta} 0$ is exact at K. So, the triple $(K, K, 0)$ is K-sub-exact at K.

(c) Let V be a direct summand of K. We can define an epimorphism $p : K = V \oplus V' \to V$ such that the sequence $K \xrightarrow{p} V \to 0$ is exact at V. Hence, the triple $(K, K, 0)$ is V-sub-exact at K.

(d) Let U be a submodule of K. Then the triple $(U, K, K/U)$ is K-sub-exact and U-sub-exact at K.

(e) The triples $(K, 0, K)$ and $(0, 0, K)$ are 0-sub-exact at 0.

(f) The triple $(0, 0, K)$ is V-sub-exact at K, for every submodule V of K since there is the inclusion $i : V \to K$ such that the sequence $0 \to V \xrightarrow{i} K$ is exact at V.

Let K, L, M be R-modules. We define

$$\sigma(K, L, M) = \{X \leq L \mid (K, L, M)\ X\text{-sub-exact at } L\}.$$

Then $\sigma(K, L, M) \neq \emptyset$ since $0 \in \sigma(K, L, M)$.

Proposition 2.5. *Let* $K_i,\ L_i,\ M_i,\ i = 1, 2$ *be families of* R*-modules. If* $X_1 \in \sigma(K_1, L_1, M_1)$ *and* $X_2 \in \sigma(K_2, L_2, M_2)$, *then* $X_1 \times X_2 \in \sigma(K_1 \times K_2, L_1 \times L_2, M_1 \times M_2)$.

Proof. Since $X_1 \in \sigma(K_1, L_1, M_1)$ and $X_2 \in \sigma(K_2, L_2, M_2)$, there are R-homomorphisms f_1, g_1, f_2 and g_2 such that the sequences $K_1 \xrightarrow{f_1} X_1 \xrightarrow{g_1} M_1$ and $K_2 \xrightarrow{f_2} X_2 \xrightarrow{g_2} M_2$ are exact. We define:

$$f : K_1 \times K_2 \to X_1 \times X_2,$$

where $f((k_1, k_2)) = (f_1(k_1), f_2(k_2))$, for every $(k_1, k_2) \in K_1 \times K_2$ and

$$g : X_1 \times X_2 \to M_1 \times M_2,$$

where $g((x_1, x_2)) = (g_1(x_1), g_2(x_2))$, for every $(x_1, x_2) \in X_1 \times X_2$. So, the sequence

$$K_1 \times K_2 \xrightarrow{f} X_1 \times X_2 \xrightarrow{g} M_1 \times M_2$$

is exact. Therefore, $X_1 \times X_2 \in \sigma(K_1 \times K_2, L_1 \times L_2, M_1 \times M_2)$. □

As a corollary, for any index set Λ, we obtain:

Corollary 2.6. *Let* K_λ, L_λ, M_λ *be families of R-modules and* X_λ *be a submodule of* L_λ, *for every* $\lambda \in \Lambda$. *If* $X_\lambda \in \sigma(K_\lambda, L_\lambda, M_\lambda)$, *for every* $\lambda \in \Lambda$, *then* $\Pi_{\lambda\in\Lambda} X_\lambda \in \sigma(\Pi_{\lambda\in\Lambda} K_\lambda, \Pi_{\lambda\in\Lambda} L_\lambda, \Pi_{\lambda\in\Lambda} M_\lambda)$.

Proof. We define

$$f = \Pi_{\lambda\in\Lambda} f_\lambda : \Pi_{\lambda\in\Lambda} K_\lambda \to \Pi_{\lambda\in\Lambda} X_\lambda$$

and

$$g = \Pi_{\lambda\in\Lambda} g_\lambda : \Pi_{\lambda\in\Lambda} X_\lambda \to \Pi_{\lambda\in\Lambda} M_\lambda.$$

Hence, the sequence $\Pi_{\lambda\in\Lambda} K_\lambda \xrightarrow{f} \Pi_{\lambda\in\Lambda} L_\lambda \xrightarrow{g} \Pi_{\lambda\in\Lambda} M_\lambda$ is exact.

Therefore, $\Pi_{\lambda\in\Lambda} X_\lambda \in \sigma(\Pi_{\lambda\in\Lambda} K_\lambda, \Pi_{\lambda\in\Lambda} L_\lambda, \Pi_{\lambda\in\Lambda} M_\lambda)$. □

In case $K = 0$, we have the following properties:

Proposition 2.7. *Let L, M be two R-modules and* X_1, X_2 *be submodules of L. If* $X_1, X_2 \in \sigma(0, L, M)$, *then* $X_1 \cap X_2 \in \sigma(0, L, M)$.

Proof. Since $X_1, X_2 \in \sigma(0, L, M)$, there are R-homomorphisms f_1 and f_2 such that the sequences: $0 \to X_1 \xrightarrow{f_1} M$ and $0 \to X_2 \xrightarrow{f_2} M$ are exact. So, f_1 and f_2 are monomorphisms. We define $f = f_1|_{X_1 \cap X_2}$. Hence, f is a monomorphism. So, the sequence $0 \to X_1 \cap X_2 \xrightarrow{f} M$ is exact. Therefore, $X_1 \cap X_2 \in \sigma(0, L, M)$. □

As a corollary, we obtain:

Corollary 2.8. *Let L, M be two R-modules and* X_λ *be a submodule of L, for every* $\lambda \in \Lambda$. *If* $X_\lambda \in (0, L, M)$, *for every* $\lambda \in \Lambda$, *then* $\bigcap_{\lambda\in\Lambda} X_\lambda \in \sigma(0, L, M)$.

Proof. We define $f : \bigcap_{\lambda \in \Lambda} X_\lambda \to M$, where $f = f_\mu|_{\bigcap_{\lambda \in \Lambda} X_\lambda}$, for some $\mu \in \lambda$. Hence, by Proposition 2.7, the sequence

$$0 \to \bigcap_{\lambda \in \Lambda} X_\lambda \xrightarrow{f} M$$

is exact. Therefore, $\bigcap_{\lambda \in \Lambda} X_\lambda \in \sigma(0, L, M)$. □

Following example shows that if $X_1 \in \sigma(K, L, M)$ and $X_2 \subset X_1$, then X_2 does not necessarily belong to $\sigma(K, L, M)$.

Example 2.9. Let $\mathbb{Q}$ be a $\mathbb{Z}$-module. Since there is the identity $i : \mathbb{Q} \to \mathbb{Q}$, where $i(a) = a$, for every $a \in \mathbb{Q}$, the sequence $\mathbb{Q} \xrightarrow{i} \mathbb{Q} \to 0$ is exact. Hence, $\mathbb{Q} \in \sigma(\mathbb{Q}, \mathbb{Q}, 0)$. But, we already know that the only $\mathbb{Z}$-module homomorphism from $\mathbb{Q}$ to $\mathbb{Z}$ is zero homomorphism, then there is no homomorphism f such that the sequence $\mathbb{Q} \xrightarrow{f} \mathbb{Z} \to 0$. Hence, $\mathbb{Z} \notin \sigma(\mathbb{Q}, \mathbb{Q}, 0)$.

Proposition 2.10. *Let K, L, M be R-modules and* X_1, X_2 *be submodules of L, where* $X_2 \subset X_1$. *If* $X_1 \in \sigma(K, L, M)$ *and* X_2 *is a direct summand of* X_1, *then* $X_2 \in \sigma(K, L, M)$.

Proof. Since $X_1 \in \sigma(K, L, M)$, there are R-homomorphisms f_1 and g_1 such that the sequence

$$K \xrightarrow{f_1} X_1 \xrightarrow{g_1} M$$

is exact.

Since X_2 is a direct summand of X_1, there exists X_3 a submodule of X_1 such that $X_1 = X_2 \oplus X_3$. Hence, for every $x_1 \in X_1$, $x_1 = x_2 + x_3$, for some $x_2 \in X_2$ and $x_3 \in X_3$. Then we define R-homomorphism

$$p : X_1 = X_2 \oplus X_3 \to X_2,$$

where $p(x_1) = p(x_2 + x_3) = x_2 \in X_2$.

So, we construct a homomorphism $f : K \to X_2$, where $f = p \circ f_1$. We can see this in the following commutative diagram:

$$\begin{tikzcd}
K \arrow[d, "f_1"'] \arrow[dr, dashed, "f"] & \\
X_1 = X_2 \oplus X_3 \arrow[r, "p"'] \arrow[d, "g_1"'] & X_2 \\
M &
\end{tikzcd}$$

Now, let $g = g_1|_{X_2}$. We will show that $Ker\, g = Im f$.

(a) Let $x \in Ker g \subseteq X_2$. Then $g(x) = g_1(x) = 0$. Hence, $x \in Ker\, g_1$. Since $Im f_1 = Ker g_1$, there is $k \in K$ such that $f_1(k) = x$. Then $f(k) = (p \circ f_1)(k) = p(f_1(k)) = p(x) = x$. This implies, $Ker\, g \subseteq Im f$.

(b) Let $x \in Im f \subseteq X_2$. We have $k \in K$ such that $f(k) = x$. Then $x = f(k) = (p \circ f_1)(k) = f_1(k)$. Hence, $x \in Im f_1 = Ker g_1$. Therefore, $g_1(x) = 0$. Since $x \in X_2$, $g(x) = g_1(x) = 0$. So that $x \in Ker\, g$. Hence, $Im f \subseteq Ker g$.

We conclude that $Im f = Ker g$. So, the sequence $K \xrightarrow{f} X_2 \xrightarrow{g} M$ is exact. Therefore, $X_2 \in \sigma(K, L, M)$. □

As a corollary of Proposition 2.10, we obtain:

Corollary 2.11. *Let K, L, M be R-modules and L be a semisimple R-module. If* $L \in \sigma(K, L, M)$, *then* $X \in \sigma(K, L, M)$, *for any submodule X of L.*

Proof. Let X be any submodule of L. Since L is a semisimple module, X is complemented. Hence, there is a submodule X' of L such that $X \oplus X' \simeq L$. Since $L \in \sigma(K, L, M)$, by Proposition 2.10, $X \in \sigma(K, L, M)$. □

Proposition 2.12. *If there are R-homomorphisms f and g such that the sequence* $K \xrightarrow{f} L \xrightarrow{g} M$ *is exact, then L is the maximal element in* $\sigma(K, L, M)$, *i.e., for every* $C \in \sigma(K, L, M)$, *if* $H \subseteq C$, *then* $H = C$.

Proof. It is obvious. □

This example illustrates Proposition 2.12.

Example 2.13. Let $8\mathbb{Z}$, $\mathbb{Z}$ be $\mathbb{Z}$-modules. We define $f : 8\mathbb{Z} \to \mathbb{Z}$, where $f(8a) = a$, for every $8a \in 8\mathbb{Z}$, and $g : \mathbb{Z} \to 0$ is zero homomorphism. We have the exact sequence $8\mathbb{Z} \xrightarrow{f} \mathbb{Z} \xrightarrow{g} 0$. Hence, $\mathbb{Z} \in \sigma(8\mathbb{Z}, \mathbb{Z}, 0)$. So, $\mathbb{Z}$ is the maximal element of $\sigma(8\mathbb{Z}, \mathbb{Z}, 0)$.

This proposition shows the relation between maximal submodule of L and maximal element of $\sigma(K, L, M)$.

Proposition 2.14. *Let K, L, M be R-modules. We assume that* $L \notin \sigma(K, L, M)$. *Consider the following assertions*:

(1) *There exists a maximal submodule* $H \subset L$ *such that* $H \in \sigma(K, L, M)$.

(2) *There exists* $H \in \sigma(K, L, M)$ *such that H is the maximal element in* $\sigma(K, L, M)$ (*i.e., for every* $C \in \sigma(K, L, M)$, *if* $H \subseteq C$, *then* $H = C$).

Then (1) $\Rightarrow$ (2).

Proof. Let H be a maximal submodule of L. Assume that $H \in \sigma(K, L, M)$. Since H is a maximal submodule of L, for every $C \in \sigma(K, L, M)$, if $H \subseteq C$, then $H = C$. Hence, H is the maximal element in $\sigma(K, L, M)$. □

But, the converse is not always true. For example, let $K = M = 0$ and $L = \mathbb{Z}_6$ be $\mathbb{Z}$-modules. We get $\sigma(0, \mathbb{Z}_6, 0) = \{0\}$. So, $0 \subset \mathbb{Z}_6$ is the maximal element in $\sigma(0, \mathbb{Z}_6, 0)$. But, 0 is not a maximal submodule of $\mathbb{Z}_6$.

The properties of Noetherian module are in [5]. M is Noetherian if and only if every non-empty set of (finitely generated) submodules of M has a maximal element.

Proposition 2.15. *Let K, L, M be R-modules and L be Noetherian. If $U \in \sigma(K, L, M)$, then there is a maximal element W in $\sigma(K, L, M)$ which contains U.*

Proof. Let $U \in \sigma(K, L, M)$. If U is a maximal element in $\sigma(k, L, M)$, then it is clear.

If not, let

$$U \subset U' \subset U'' \subset \cdots$$

be an ascending chain of submodules of a module L in $\sigma(K, L, M)$. Since L is Noetherian, there is a maximal element $W \in \sigma(K, L, M)$ which contains U. □

Let M be an R-module. A finite chain of submodules

$$0 = M_0 \subset M_1 \subset \cdots \subset M_k = M, \quad k \in \mathbb{N} \tag{1}$$

is called a *normal series* of M. A normal series (1) is a composition series of M if all factors M_i / M_{i-1} are simple modules. The number k is said to be the *length* of the normal series and the factor modules M_i / M_{i-1}, $1 \leq i \leq k$ are called its *factors* [5]. So, any finitely generated semisimple module has a finite length or equivalently, it is Noetherian. As a corollary of Proposition 2.15, we obtain:

Corollary 2.16. *Let K, L, M be R-modules and L be a finitely generated semisimple module. If $U \in \sigma(K, L, M)$, then there is a maximal element W in $\sigma(K, L, M)$ which contains U.*

Proof. Let K, L, M be R-modules and L be a finitely generated semisimple module. Since any finitely generated semisimple module is Noetherian, by Proposition 2.15, if $U \in \sigma(K, L, M)$, then there is a maximal element W in $\sigma(K, L, M)$ which contains U. □

However, $\sigma(K, L, M)$ may have more than one maximal element.

Example 2.17. Let $A = \{2a \mid a \in \mathbb{Z}_6\} = \{0, 2, 4\}$ and $B = \{3a \mid a \in \mathbb{Z}_6\} = \{0, 3\}$ be $\mathbb{Z}$-modules. If we take $K = 0$, $L = \mathbb{Z}_6$ and $M = A \times B$ as $\mathbb{Z}$-modules, then $\sigma(K, L, M) = \{0, \{0, 2, 4\}, \{0, 3\}\}$. Since we cannot define a monomorphism from $\mathbb{Z}_6$ to M, $\mathbb{Z}_6 \notin \sigma(K, L, M)$. So, the maximal elements of $\sigma(K, L, M)$ are $\{0, 2, 4\}$ and $\{0, 3\}$. Furthermore, $\{0, 2, 4\}$ is not isomorphic to $\{0, 3\}$. So, we can conclude that two elements of $\sigma(K, L, M)$ are not necessarily unique up to isomorphism.

3. Conclusion

Let K, L, M be R-modules. The collection of all submodules X of L such that the triple (K, L, M) is X-sub-exact denoted by $L(\sigma(K, L, M))$ is not closed under submodules. But, if a submodule of L is a direct summand of any element of $\sigma(K, L, M)$, then this submodule is contained in $\sigma(K, L, M)$. Therefore, if L is semisimple and $L \in \sigma(K, L, M)$, then any submodule of L is contained in $\sigma(K, L, M)$. Moreover, $\sigma(K, L, M)$ is not closed under extensions.

If there are R-module homomorphisms f and g such that the sequence $K \xrightarrow{f} L \xrightarrow{g} M$ is exact, then $\sigma(K, L, M)$ has a maximal element. If not, then the set $\sigma(K, L, M)$ has a maximal element if L is Noetherian. Furthermore, $\sigma(K, L, M)$ may have more than one maximal element. But, any two elements of $\sigma(K, L, M)$ are not necessarily unique up to isomorphism.

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