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The Locating-Chromatic Number of Subdivision Firecracker Graphs

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Abstract

The locating-chromatic number of a graph was combined two graph concept, coloring vertices and partition dimension of a graph. In this paper, we discuss about locating-chromatic number of a subdivision firecracker graphs.

Keywords: graph, color code, locating-chromatic number

1. Introduction

The locating-chromatic number of a graph was introduced by Chartrand *et al.* [1] in 2002, with derived two graph concept, coloring vertices and partition dimension of a graph. Let $G = (V, E)$ be a connected graph and c be a proper k-coloring of G with color $1,2, \dots, k$. Let $\Pi = \{C_1, C_2, \dots, C_k\}$ be a partition of $V(G)$ which is induced by coloring c. The color code $c_\Pi(v)$ of v is the ordered k-tuple $(d(v, C_1), d(v, C_2), \dots, d(v, C_k))$ where $d(v, C_i) = \min \{d(v, x) | x \in C_i\}$ for any i. If all distinct vertices of G have distinct color codes, then c is called k-locating coloring of G. The locating-chromatic number, denoted by $\chi_L(G)$, is the smallest k such that G has a locating k-coloring. A vertex $u \in S_i$, for some i, is dominant if $d(u, S_j) = 1$ for $j \neq i$.

Chartrand *et al*.[2] determined the locating chromatic number for some graph classes, such as on P_n is a path of order $n \geq 3$ then $\chi_L(P_n) = 3$; for a cycle C_n if n is odd $\chi_L(C_n) = 3$, and if n even $\chi_L(C_n) = 4$; for double star graph $(S_{a,b})$, $1 \leq a \leq b$ and $b \geq 2$, obtained $\chi_L(S_{a,b}) = b + 1$. Asmiati *et al*. [3] obtained the locating chromatic number of amalgamation of stars and furhemore, Asmiati [4] found for non homogeneous amalgamation of stars. Next, Asmiati [5] investigated the locating chromatic number for banana tree.

A Firecracker graph $F_{n,k}$, namely the graph obtained by the concatenation of n stars S_k by linking one leaf from each star. Let $V(F_{n,k}) = \{x_i, m_i, l_{ij}|\ i \in [1, n];\ j \in [1, k-2]\}$ and $E(F_{n,k}) = \{x_i, m_{i+1}|\ \ i \in [1, n]\} \cup \{x_i, m_i, l_{ij}| i \in [1, n];\ j \in [1, k-2]\}$. If we give subdivision one vertex y_i in edge x_i, m_i , we denote $F^*_{n,k}$ with n, k natural numbers.

Asmiati *et al*.[6] investigated the locating-chromatic number of firecracker graph $F_{n,k}$, for $n \geq 2$, $\chi_L(F_{n,k}) = 4$;for $k \geq 5$, $\chi_L(F_{n,k}) = k - 1$for $2 \leq n \leq k - 1$ and $\chi_L(F_{n,k}) = k$ otherwise. Next, Asmiati [7] determined the locating chromatic number of $F^*_{n,k}$ with n, k natural numbers. Besides that in this paper, we will discuss about the locating chromatic number of subdivision firecracker graphs.

The following theorem is basic to determine the lower bound of the locating chromatic of a graph. The set of neighbours of a vertex s in G, denoted by $N(s)$.

Theorem 1.1 : Chartrand *et al*.[1]
Let G be a connected graph and c is a locating coloring in G. If u and s are distinct vertices of G such that d(u,w)=d(s,w) for all $w \in V(G) - \{u, s\}$, *then* $c(u) \neq c(s)$. *In particular, if u and s are adjacent to the same vertices in G such that* $N(u) \neq N(s)$, *then* $c(u) \neq c(s)$.

Corollary 1.1 : Chartrand *et al*.[1]
If G is a connected graph containing a vertex adjacent to k leaves of G, then $\chi_L(G) \geq k + 1$.

2. Main Results

In this section we will discuss the locating chromatic number of subdivision firecracker graphs, namely $F^{s*}_{n,k}$. $F^{s*}_{n,k}$is a graph obtained from subdivision graph $F^{*}_{n,k}$ as much as s ≥ 2 even vertices on each side $x_i y_i$ and $y_i m_i$ for every $i \in [1, n]$. Thus $x_i y_i$ and $y_i m_i$ become a path for every $i \in [1, n]$. Let path $x_i y_i = \{x_i, a_{i1}, a_{i2}, \dots, a_{ir}, y_i\}$ for every $r \in [1, s]$ and s ≥ 2 even, path $y_i m_i = \{y_i, b_{i1}, b_{i2}, \dots, b_{ir}, m_i\}$ for every $r \in [1, s]$ and s ≥ 2 even.

Theorem 2.1.
Let $F^{s*}_{n,k}$ be a subdivision firecracker graphs. Then,:
i. $\chi_L(F^{s*}_{n,4}) = 4$; n ≥ 2
ii. For $k \geq 5$

$$\chi_L(F^{s*}_{n,k}) = \begin{cases} k-1 & ,1 \leq n \leq k-1 \\ k & ,\text{otherwise.} \end{cases}$$

Proof:
(i) By Corollary 1.1, we have $\chi_L(F^{s*}_{n,4}) \geq 3$. For a contradiction, assume we have 3-locating coloring on $F^{s*}_{n,4}$ for $n \geq 2$, namely 1, 2, and 3. So, $\{c(m_1), c(l_{11}), c(l_{12})\} = \{c(m_2), c(l_{21}), c(l_{22})\} = \{1,2,3\}$. If we assign $c(m_1)$ is one of {1,2,3} then color of b_1 is same with one of colors $\{l_{11}, l_{12}\}$. Therefore there are two vertices have the same color codes, a contrary. So, $\chi_L(F^{s*}_{n,4}) \geq 4$ for $n \geq 2$.

Next, we determine the upper bound of $F^{s*}_{n,4}$ for $n \geq 2$. Assign the 4-coloring c on $F^{s*}_{n,4}$ as follows :

- $c(x_i) = 3$ for odd *i* and $c(x_i) = 2$ for even *i*.
- $c(a_{ir}) = c(y_i)$ for odd *r* and $c(a_{ir}) = c(x_i)$ for even *r*.
- $c(y_i) = 2$, for odd *i* and $c(y_i) = 1$ for even *i*.
- $c(b_{ir}) = c(m_i)$ for odd *r* and $c(a_{ir}) = c(y_i)$ for even *r*.
- $c(m_i) = 3$ for odd *i* and $c(m_i) = 2$ for even *i*.
- For all vertices l_{ij} , define

$$c(l_{ij}) = \begin{cases} 4 \text{ if } i = 1, j = 1 \\ 1 \text{ if } i \geq 2, j = 1 \\ 2 \text{ for odd } i, j = 2 \\ 3 \text{ for even } i, j = 2 \end{cases}$$

The coloring c will create the partition Π on $V(F^{s*}_{n,4})$. We shall show that the color codes of all vertices in $F^{s*}_{n,4}$ are different. For odd i, we have $c_\Pi(x_i) = (2,1,0, i+2+2s)$ and for even i, $c_\Pi(x_i) = (1,0,1, i+2+2s)$. For y_1, $c_\Pi(y_1) = (3+s, 0,1,2+s)$ and for $i \geq 2$, we have $c_\Pi(y_i) = (3,0,1, i+3+3s)$ for odd i, $c_\Pi(y_i) = (0,1,2, i+3+3s)$ for even i. For m_i, we have $c_\Pi(m_1) = (4,1,0,1)$ and for $i \geq 2$, we have $c_\Pi(m_i) = (1,1,0, i+4+4s)$ for odd i, $c_\Pi(m_i) =$

$(1,0,1, i+4+4s)$ for even *i*. For vertices l_{ij}, we have $c_\Pi(l_{11}) = (5+3s, 2,1,0)$ and $c_\Pi(l_{12}) = (5+3s, 0,1,2)$. For $i \geq 2$, $c_\Pi(l_{i1}) = (0,1,2, i+5+4s)$, $c_\Pi(l_{ij}) = (0,2,1, i+5+4s)$ for odd *i*, $c_\Pi(l_{ij}) = (2,1,0, i+5+4s)$ for even *i*. For $c(a_{ir}) = c(y_i)$ for odd r and $c(a_{ir}) = c(x_i)$ for even r, for $c(b_{ir}) = c(m_i)$ for odd r and $c(b_{ir}) = c(y_i)$ for even r, for some $r \in [1, s]$ and s $\geq$ 2 even.

Let $u, v \in V(F_{n,4}^{s*})$ and $c(u) = c(v)$ then:

- If $u = a_{ih}$ and $v = a_{jl}$ for some i, j, h, l and $i \neq j$, then $c_\Pi(\mathrm{u}) \neq c_\Pi(\mathrm{v})$.
- If $u = b_{ih}$ and $v = b_{jl}$ for some i, j, h, l and $i \neq j$, then $c_\Pi(u) \neq c_\Pi(v)$.
- If $u = a_{ih}$ and $v = b_{jl}$ for some i, j, h, l and $i \neq j$, then $c_\Pi(\mathrm{u}) \neq c_\Pi(\mathrm{v})$.

Since the color codes of all vertices in $F_{n,4}^{*}$ are different, thus *c* is locating chromatic coloring. So $\chi_L(F_{n,4}^{s*}) \leq 4$, n $\geq$ 2. As a result, we have $\chi_L(F_{n,4}^{s*}) = 4$; n $\geq$ 2.

(ii) Next, we will show that for $k \geq 5$, $\chi_L(F_{n,k}^{s*}) = k$ if $n \geq k$ and $\chi_L(F_{n,k}^{s*}) = k-1$, if $1 \leq n \leq k-1$. To show this, let we consider the following two cases:

Case 1. For $k \geq 5$ and $1 \leq n \leq k-1$.
Since each vertex m_i is adjacant to $(k-2)$ leaves, by Corollary 1.1, we have $\chi_L(F_{n,k}^{s*}) \geq k-1$. Next, we will show that $\chi_L(F_{n,k}^{s*}) \leq k-1$ for $k \geq 5$ and $n \leq k-1$. Define a $(k-1)$-coloring c of $F_{n,k}^{s*}$ as follows. Assign $c(m_i) = i$, for $i \in [1, n]$ and all the leaves: $\{l_{ij} | j = 1,2, \ldots, k-2\}$ by $\{1,2, \ldots, k-1\} \backslash \{i\}$ for any *i*. However $c(y_i) = 2$, for odd i and 1 for even i, $c(x_i) \neq c(m_i)$ for $i \in [1,2, \ldots, k-1]$. For $c(a_{ir}) = c(y_i)$ for odd r and $c(a_{ir}) = c(x_i)$ for even r, for every $r \in [1, s]$ and s ≥ 2 even. As a result, coloring c will create a partition $\prod = \{U_1, U_2, \ldots, U_{k-1}\}$ on $V(F_{n,k}^{s*})$, where U_i is the set of all vertices with color *i*.

We show that the color codes for all vertices in $F_{n,k}^{s*}$ for $k \geq 5$ and $n \leq k-1$ are different. Let $u, v \in V(F_{n,k}^{s*})$and $c(u) = c(v)$. Then, consider the following cases:

- If $u = l_{ij}, v = l_{jl}$ for some i, j, h, l and $i \neq j$, then $c_\Pi(u) \neq c_\Pi(v)$ since $d(u, U_i) \neq d(v, U_i)$.
- If $u = l_{ih}, v = m_j$ for some $i, j, h,$ and $i \neq j$, then *u* must be dominant vertex and v is not . So $c_\Pi(u) \neq c_\Pi(v)$.
- If $u = l_{ih}, v = y_j$ for some $i, j, h,$ and $i \neq j$, then there exactly one set in Π which has the distance 1 from *u* and there is at least two set in Π which has the distance 1 from *v*. Thus $c_\Pi(u) \neq c_\Pi(v)$.
- If $u = l_{ih}, v = x_j$ for some $i, j, h,$ and $i \neq j$, then there exactly one set in Π which has the distance 1 from u and there is at least two set in Π which has the distance 1 from *v*. Thus $c_\Pi(u) \neq c_\Pi(v)$.
- If $u = l_{ij}, v = a_{jh}$ for some $i, j, h,$ and $i \neq j$, then there exactly one set in Π which has the distance 1 from u and there is at least two set in Π which has the distance 1 from *v*. Thus $c_\Pi(u) \neq c_\Pi(v)$.

- If $u = m_i,\ v = y_j$ for some i, j and $i \neq j$, then *u* must be dominant vertex and *v* is not. So $c_\Pi(u) \neq c_\Pi(v)$.
- If $u = m_i,\ v = x_j$ for some i, j and $i \neq j$, then *u* must be dominant vertex and *v* is not. So $c_\Pi(u) \neq c_\Pi(v)$.
- If $u = m_i,\ v = a_{jh}$ for some i, j,h and $i \neq j$, then *u* must be dominant vertex and *v* is not. So $c_\Pi(u) \neq c_\Pi(v)$.
- If $u = y_i$ and $v = x_j$ for some $i,\ j$ and $i \neq j$, then $c_\Pi(u) \neq c_\Pi(v)$ since $d(u, U_i) \neq d(v, U_i)$.
- If $u = y_i$ and $v = a_{jh}$ for some $i,\ j,h$ and $i \neq j$, then $c_\Pi(u) \neq c_\Pi(v)$ since $d(u, U_i) \neq d(v, U_i)$.
- If u $= x_i$ and $v = x_j$ then $i = 1$ and $j = n$. So $c_\Pi(u) \neq c_\Pi(v)$.
- If $u = a_{ih}$ and $v = a_{jl}$ for some i, j, h, l and $i \neq j$, then $c_\Pi(u) \neq c_\Pi(v)$.

For all the above cases, we can see that the color codes of all vertices in $F_{n,k}^{s*}$ for $k \geq 5$,$n \leq k - 1$ are different. Thus $\chi_L(F_{n,k}^{s*}) \leq k - 1$ for $1 \leq n \leq k - 1$. So, we have, $\chi_L(F_{n,k}^{s*}) = k - 1$ for $1 \leq n \leq k - 1$.

Case 2. For $k \geq 5$ and $n \geq k$.
Since each vertex m_i is adjacant to $(k-2)$ leaves, by Corollary 1.1, we have $\chi_L(F_{n,k}^{s*}) \geq k - 1$ for $n \geq k$. For a contradiction, assume we have $(k-1)$-locating coloring c on $F_{n,k}^{s*}$ for $k \geq 5$ and $n \geq k$. Since $n \geq k$, then there are i, j , $i \neq j$ such that $\{c(l_{it}) \mid t \in [1, k-2]\} = \{c(l_{jp}) \mid p \in [1, k-2]\}$. Therefore the color codes of m_i and m_j are the same, a contradiction. As a result, $\chi_L(F_{n,k}^{s*}) \geq k$ for $k \geq 5$ and $n \geq k$.

Next, we determine the upper bound of $F_{n,k}^{s*}$ for $k \geq 5,\ \ n \geq k$. To show that $\chi_L(F_{n,k}^{s*}) \leq k,\ \ k \geq 5$ and $n \geq k$, consider the locating coloring c on $F_{n,k}^{s*}$ as follows:

- $c(x_i) = 1$ for odd *i* and $c(x_i) = 3$ for even *i*.
- $c(a_{ir}) = c(y_i)$ for odd *r* and $c(a_{ir}) = c(x_i)$ for even *r*.
- $c(y_i) = 2$, for every *i*.
- $c(b_{ir}) = c(m_i)$ for odd *r* and $c(b_{ir}) = c(y_i)$ for even *r*.
- $c(m_i) = 1$, for every *i*.
- If $A = \{1,2, \dots, k\}$, define

$$\{c(l_{ij}) | j = 1,2, \dots, k-2\} = \begin{cases} A \backslash \{1, k-1\}, & if\ i = 1 \\ A \backslash \{1, k\}, & \text{otherwise.} \end{cases}$$

The coloring c will create a partition $V(F_{n,k}^{s*})$. We shall show that the color codes of all vertices in $F_{n,k}^{s*}$ are different.

$$c_\Pi(x_1) = \begin{cases} 0 & , \text{ for } i^{th} \text{component} \\ 1 & , \text{ for } 2^{nd} \text{ and } 3^{rd} \text{component} \\ 4 + 4s & , \text{ for } (k-1)^{th} \text{component} \\ 3 + 2s & , \text{ otherwise .} \end{cases}$$

For $i \geq 2$ odd

$$c_\Pi(x_i) = \begin{cases} 0 & , \text{ for } i^{th} \text{component} \\ 1 & , \text{ for } 2^{nd} \text{ and } 3^{rd} \text{component} \\ i+2+2s & , \text{ for } k^{th} \text{component} \\ 3+2s & , \text{ otherwise .} \end{cases}$$

For $i \geq 2$ even

$$c_\Pi(x_i) = \begin{cases} 0 & , \text{ for } 1^{st} \text{ and } 2^{nd} \text{component} \\ 1 & , \text{ for } 3^{rd} \text{component} \\ i+2+2s & , \text{ for } k^{th} \text{component} \\ 3+2s & , \text{ otherwise.} \end{cases}$$

$$c_\Pi(y_1) = \begin{cases} 1 & , \text{ for } i^{th} \text{ component} \\ 0 & , \text{ for and } 2^{nd} \text{component} \\ 5+3s & , \text{ for } (k-1)^{th} \text{component} \\ i+3+s & , \text{ for } k^{th} \text{component} \\ 2+s & , \text{ otherwise} \end{cases}$$

For $i \geq 2$ odd

$$c_\Pi(y_i) = \begin{cases} 1 & , \text{ for } 1^{st} \text{ component} \\ 0 & , \text{ for } 2^{nd} \text{component} \\ \mathrm{i}+3+\mathrm{s} & , \text{ for } k^{th} \text{component} \\ 2+\mathrm{s} & , \text{ otherwise .} \end{cases}$$

For $i \geq 2$ even

$$c_\Pi(y_i) = \begin{cases} 1 & , \text{ for } 1^{st} \text{and } 3^{rd} \text{ component} \\ 0 & , \text{ for } 2^{nd} \text{ component} \\ \mathrm{i}+3+\mathrm{s} & , \text{ for } k^{th} \text{ component} \\ 2+\mathrm{s} & , \text{ otherwise .} \end{cases}$$

$$c_\Pi(m_1) = \begin{cases} 0 & , \text{ for } 1^{st} \text{ component} \\ 6+4\mathrm{s} & , \text{ for } (k-1)^{th} \text{ component} \\ 1 & , \text{ otherwise.} \end{cases}$$

For $\mathrm{i} \geq 2$

$$c_\Pi(m_i) = \begin{cases} 0 & , \text{ for } 1^{st} \text{ componen} \\ \mathrm{i}+4+4\mathrm{s} & , \text{ for } k^{th} \text{ component} \\ 1 & , \text{ otherwise .} \end{cases}$$

For $j = 1,2, \dots, k-2$

$$c_\Pi(l_{1\mathrm{j}}) = \begin{cases} 1 & , \text{ for } 1^{st} \text{ component} \\ 0 & , \text{ for } j^{th} \text{ component} \\ 7+4s & , \text{ for } (k-1)^{th} \text{ component} \\ 2 & , \text{ otherwise .} \end{cases}$$

$$c_\Pi(l_{1k-2}) = \begin{cases} 1 & , \text{ for } 1^{st} \text{ component} \\ 7+4s & , \text{ for } (k-1)^{th} \text{ component} \\ 0 & , \text{ for } k^{th} \text{ component} \\ 2 & , \text{ otherwise.} \end{cases}$$

For $i \neq 1$,$j = 1,2,\ldots,k-2$

$$c_\Pi\left(l_{l_{ij}}\right) = \begin{cases} 1 & , \text{ for } 1^{st} \text{ component} \\ 0 & , \text{ for } j^{th} \text{ component} \\ i+5+4s & , \text{ for } k^{th} \text{ component} \\ 2 & , \text{ otherwise .} \end{cases}$$

Let c(a_{ir}) $= c(y_i)$ and $c(b_{ir}) = c(m_i)$ for odd r ; $c(a_{ir}) = c(x_i)$ and $c(b_{ir}) = c(y_i)$ for even r, $r \in [1,s]$ and $s \geq 2$ even. Let $u,v \in V(F_{n,k}^{s*})$ and $c(u) = c(v)$ then :

- If $u = a_{ih}$ and $v = a_{jl}$ for some i,j,h,l and $i \neq j$, then $c_\Pi(u) \neq c_\Pi(v)$.
- If $u = b_{ih}$ and $v = b_{jl}$ for some i,j,h,l and $i \neq j$, then $c_\Pi(u) \neq c_\Pi(v)$.
- If $u = a_{ih}$ and $v = b_{jl}$ for some i,j,h,l and $i \neq j$, then $c_\Pi(u) \neq c_\Pi(v)$.

Since the color codes of all vertices are different, thus c is locating chromatic coloring in $F_{n,k}^{s*}$, so $\chi_L(F_{n,k}^{s*}) \leq k, n \geq k$. Thus, we have $\chi_L(F_{n,k}^{s*}) = k$ for $n \geq k$.■

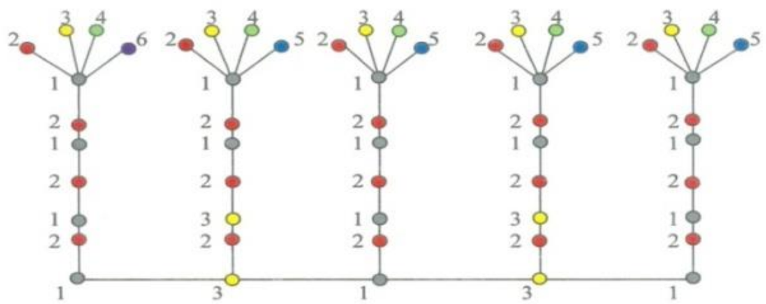

Fig.(1): A minimum locating coloring of $F_{6,5}^{2*}$

3. Conclusion

The locating chromatic number of subdivision firecracker graphs, namely $F_{n,k}^{s*}$ is 4 for $n \geq 2$, whereas for $k \geq 5$ and $1 \leq n \leq k-1$, we have $\chi_L(F_{n,k}^{s*}) = k-1$ and k for otherwise.

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