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ON THE LOCATING-CHROMATIC NUMBERS OF NON-HOMOGENEOUS CATERPILLARS AND FIRECRACKER GRAPHS

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Abstract

We determine the locating-chromatic numbers of non-homogeneous caterpillars and firecracker graphs.

1. Introduction

The notion of locating-chromatic number of a graph was introduced by Chartrand et al. [8]. Let G be a finite, simple, and connected graph. Let c be a proper k-coloring of G and $\Pi = \{C_1, C_2, ..., C_k\}$ be a partition of $V(G)$ induced by c on $V(G)$, where C_i is the set of vertices receiving color i. The *color code* $c_\Pi(v)$ of v is the ordered k-tuple $(d(v, C_1), d(v, C_2), ..., d(v, C_k))$, where $d(v, C_i) = \min\{d(v, x) | x \in C_i\}$ for any i. If all distinct vertices of G

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have distinct color codes, then c is called a *locating-chromatic k-coloring* of G (*k-locating coloring*, in short). The *locating-chromatic number*, $\chi_L(G)$ is the smallest k such that G has a locating k-coloring.

Chartrand et al. [8] determined the locating-chromatic number for paths, cycles, complete multipartite graphs and double stars. Behtoei and Omoomi [7] discussed the locating-chromatic number for Kneser Graph. Specially for amalgamation of stars, Asmiati et al. [1, 4] determined locating-chromatic number for homogeneous amalgamation of stars and non-homogeneous amalgamation of stars, respectively. Furthermore, the locating-chromatic number of the operation of two graphs is discussed by Baskoro and Purwasih [6]. They determined the locating-chromatic number for a corona product of two graphs.

Chartrand et al. [8] characterized graphs have locating-chromatic number $n-1$. They also determined graphs whose locating-chromatic numbers are bounded by $n-2$. Moreover, Asmiati and Baskoro [3] characterized all maximal graphs containing cycle. In general, characterization of all tress with locating-chromatic number 3 is given in Baskoro and Asmiati [5].

Asmiati et al. [2] determined the locating-chromatic number for homogeneous firecracker graphs, Motivated by these results, we determine the locating-chromatic numbers of non-homogeneous caterpillars and firecracker graphs.

The following results were proved by Chartrand et al. in [8]. We denote the set of neighbors of a vertex v in G by $N(V)$.

Theorem 1 [8]**.** *Let c be a locating-coloring in a connected graph* $G = (V, E)$*. If u and v are distinct vertices of G such that* $d(u, w) = d(v, w)$ *for all* $w \in V(G) - \{u, v\}$, *then* $c(u) \neq c(v)$. *In particular, if u and v are non-adjacent vertices of G such that* $N(u) = N(v)$, *then* $c(u) \neq c(v)$.

Corollary 1 [8]**.** *If G is a connected graph containing a vertex adjacent to m end-vertices of G, then* $\chi_L(G) \geq m+1$.

Corollary 1 gives a lower bound for the locating-chromatic numbers of a general graph G.

2. Locating-chromatic Number of Non-homogeneous Caterpillar

In this section, we discuss about the locating-chromatic number of non-homogeneous caterpillar. Let P_m be a path with $V(P_m) = \{x_1, x_2, ..., x_m\}$ and $E(P_m) = \{x_1x_2, x_2x_3, ..., x_{m-1}x_m\}$. A *non-homogeneous caterpillar* is obtained by connecting n_i pendant vertices $(a_{ij}, j = 1, 2, ..., n_i)$ to one particular vertex x_i of path P_m, where $1 \le i \le m$, which we denote by $C(m; n_1, n_2, ..., n_m)$. The non-homogeneous caterpillar $C(m; n_1, n_2, ..., n_m)$ consists of vertex set $V(C(m; n_1, n_2, ..., n_m)) = \{x_i | 1 \le i \le m\} \cup \{a_{ij} | 1 \le i \le m, 1 \le j \le n_i\}$ and edge set $E(C(m; n_1, n_2, ..., n_m)) = \{x_ix_{i+1} | 1 \le i \le m - 1\} \cup \{x_ia_{ij} | 1 \le i \le m, 1 \le j \le n_i\}$.

Let K_{1, n_i}, with the vertex x_i as the center, be a subgraph of $C(m; n_1, n_2, ..., n_m)$. We denote the set of vertices and edges by $V(K_{1, n_i}) = \{a_{ij} | 1 \le j \le n_i\} \cup \{x_i\}$ and $E(K_{1, n_i}) = \{x_ia_{ij} | 1 \le j \le n_i\}$, respectively. Thus, $C(m; n_1, n_2, ..., n_m)$ contains m stars K_{1, n_i} with x_i as a center. If $n_{\max} = \max\{n_1, n_2, ..., n_m\}$, then subgraph $K_{1, n_{\max}}$ is called *the maximum star subgraph* in the non-homogeneous caterpillar $C(m; n_1, n_2, ..., n_m)$. If there are p subgraphs $K_{1, n_{\max}}$, then every subgraph, from left to right, are denoted by $K^i_{1, n_{maks}}$, where $1 \le i \le p$.

Definition 1. Let $K_{1, n_i}, K_{1, n_j} \subset C(m; n_1, n_2, ..., n_m)$, where $1 \le i \ne j \le m$. If $n_i = n_j \ne n_{\max}$, such that

(1) $d(x_i, x_m) = d(x_j, x_m)$, with x_m is the center of $K_{1, n_{\max}}$, or

(2) $d(x_i, x_0) = d(x_j, x_p)$, with x_0 and x_p, $x_0 \neq x_p$ are the centers of $K_{1, n_{\max}}$,

then subgraphs K_{1, n_i} and K_{1, n_j} are called *star subgraphs* with the same distance.

Theorem 2. *Let* $K_{1, n_{\max}}$ *be the maximum star subgraph of* $C(m; n_1, n_2, ..., n_m)$ *and p be the number of subgraphs* $K_{1, n_{\max}}$. *Then for* $n_{\max} \geq 2$, *the locating-chromatic number of non-homogeneous caterpillar* $C(m; n_1, n_2, ..., n_m)$ *is*

$$\chi_L(C(m; n_1, n_2, ..., n_m)) = \begin{cases} n_{\max} + 1, & \text{if } p \leq n_{\max} + 1, \\ n_{\max} + 2, & \text{if } p > n_{\max} + 1. \end{cases}$$

Proof. First we determine the trivial lower bound of

$$C(m; n_1, n_2, ..., n_m)$$

for $p \leq n_{\max} + 1$. Since the number of leaves in a maximal subgraph is $n_{\max}$, by Corollary 1, $\chi_L(C(m; n_1, n_2, ..., n_m)) \geq n_{\max} + 1$, for $p \leq n_{\max} + 1$.

Next, we determine the upper bound of $C(m; n_1, n_2, ..., n_m)$ for $p \leq n_{\max} + 1$. Consider the $n_{\max} + 1$-coloring c on $C(m; n_1, n_2, ..., n_m)$ as follows:

a. Find the number of subgraphs $K_{1, n_{\max}}$ and denote it by p. Denote each of the subgraphs from left to right as $K^i_{1, n_{\max}}$, where $1 \leq i \leq p$, respectively.

b. Vertices $x_i \in K^i_{1, n_{\max}}$, where $1 \leq i \leq p$ are colored by $1, 2, 3, ..., p$, respectively.

c. Leaves in $K^i_{1, n_{\max}}$, where $1 \leq i \leq p$ are colored by $\{1, 2, 3, ..., n_{\max} + 1\} \setminus \{c(x_i)\}$.

d. Let A_1 be an open interval before $K^i_{1,n_{\max}}$, A_{k+1} be an open interval between $K^i_{1,n_{\max}}$ and $K^{k+1}_{1,n_{\max}}$, where $1 \le k \le p-1$, and A_{p+1} be an open interval after $K^p_{1,n_{\max}}$.

e. Let $T = \{$all combinations $(n_{\max})$ from $n_{\max}+1$ color$\}$, such that $T = \{T_1, T_2, \ldots, T_{n_{\max}+1}\}$ with $T_i \in T$ are color combination not containing the color i.

f. Identify subgraph K_{1,n_i} in the interval as defined in item *d*.

g. If K_{1,n_i} lies in interval A_1 or A_2, then every vertex of K_{1,n_i} is colored by colors that are associated with T_1, respectively.

h. If K_{1,n_i} lies in interval A_k, where $3 \le k \le p+1$, then every vertex of K_{1,n_i} is colored by colors that correspond with T_{k-1}, respectively.

i. If K_{1,n_i} and K_{1,n_j} with $n_i = n_j$ have the same distance from the maximum star subgraph and $\{c(a_{il}) \mid l = 1, 2, \ldots, n_i\} = \{c(a_{jl}) \mid l = 1, 2, \ldots, n_j\}$, then x_i and x_j should be given different colors. Vice versa, if $c(x_i) = c(x_j)$, then $\{c(a_{il}) \mid l = 1, 2, \ldots, n_i\} \ne \{c(a_{jl}) \mid l = 1, 2, \ldots, n_j\}$.

We show that the color codes for all vertices in $C(m; n_1, n_2, \ldots, n_m)$ for $p \le n_{\max}+1$, are different. Let $u, v, u \ne v$ be the leaves, where $u \in V(K_{1,n_i})$, $v \in V(K_{1,n_j})$, and $c(u) = c(v)$.

- If $n_i = n_j = n_{\max}+1$, then $c_\Pi(u) \ne c_\Pi(v)$ because their color codes differ in the ordinate of colors x_i and x_j.
- If K_{1,n_i} and K_{1,n_j} lie in different intervals, say A_p and A_q, then $c_\Pi(u) \ne c_\Pi(v)$ because they have different distances from C_p and C_q.

- If K_{1,n_i} and K_{1,n_j} lie in the same interval, say A_p but they do not have the same distance, then $c_\Pi(u) \neq c_\Pi(v)$ because they have different distance to C_p. But, if they have the same distance, their color codes differ in the ordinate of colors x_i and x_j.

- If one of $\{n_i, n_j\}$ is $n_{\max}$, say $n_i = n_{\max}$ and $n_j < n_{\max}$, then the color codes of u and v differ in color of leaves of K_{1,n_i} not contained in K_{1,n_j}.

- If $x_i \in V(K_{1,n_j})$ and v have the same color, then $c_\Pi(x_i)$ contains at least two components of value 1, whereas $c_\Pi(v)$ contains exactly one component of value 1. Thus, $c_\Pi(x_i) \neq c_\Pi(v)$.

From all the above cases, we see that the color codes for all vertices in $C(m; n_1, n_2, ..., n_m)$ for $p \leq n_{\max} + 1$, are different, thus c is a locating-coloring. So, $\chi_L(C(m; n_1, n_2, ..., n_m)) \leq n_{\max} + 1$, for $p \leq n_{\max} + 1$.

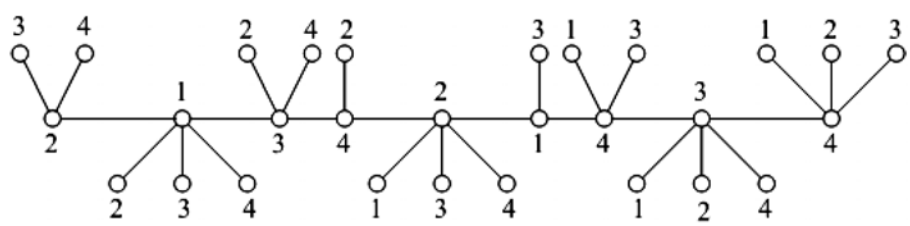

Figure 1. A minimum locating-coloring of $C(9; 2, 3, 1, 1, 3, 1, 2, 3, 3)$.

Next, we show the lower bound of $C(m; n_1, n_2, ..., n_m)$ for $p > n_{\max} + 1$. By Corollary 1, we have that $\chi_L(C(m; n_1, n_2, ..., n_m)) \geq n_{\max} + 1$. However, we will show that $(n_{\max} + 1)$ colors are not enough. For a contradiction, assume that there exists a $n_{\max} + 1$-locating coloring c on $C(m; n_1, n_2, ..., n_m)$ for $p > n_{\max} + 1$. Since $p > n_{\max} + 1$, there are two i, j, $i \neq j$, such that $\{c(a_{ih}) | h = 1, 2, ..., n_{\max}\} = \{c(a_{jh}) | h = 1, 2, ..., n_{\max}\}$.

Therefore the color codes of x_i and x_j are the same, a contradiction. So, $\chi_L(C(m; n_1, n_2, ..., n_m)) \geq n_{\max} + 2,$ for $p > n_{\max} + 1.$

To show that $\chi_L(C(m; n_1, n_2, ..., n_m)) \leq (n_{\max} + 2),$ consider the locating-coloring c on $C(m; n_1, n_2, ..., n_m)$ as follows:

- $c(l_{11}) = n_{\max} + 2.$

- The color of vertices x_i are:

$$c(x_i) = \begin{cases} 1 & \text{if } i \text{ odd,} \\ 2 & \text{if } i \text{ even.} \end{cases}$$

- Leaves, for $n_i = 1,$ $c(n_i) = 3,$ whereas for $n_i \geq 2,$ $\{c(a_{ij}) \mid j = 1, 2, ..., n_i\},$ are colored by $S \subseteq \{1, 2, ..., n_{\max} + 1\} \backslash \{c(x_i)\}$ for any $i.$

Since there is only one vertex at the end of the longest path, which is colored by $n_{\max} + 2,$ the color codes of all vertices are different. Therefore, c is the locating-chromatic coloring on $C(m; n_1, n_2, ..., n_m),$ and so $\chi_L(C(m; n_1, n_2, ..., n_m)) \leq n_{\max} + 2,$ for $p > n_{\max} + 1.$

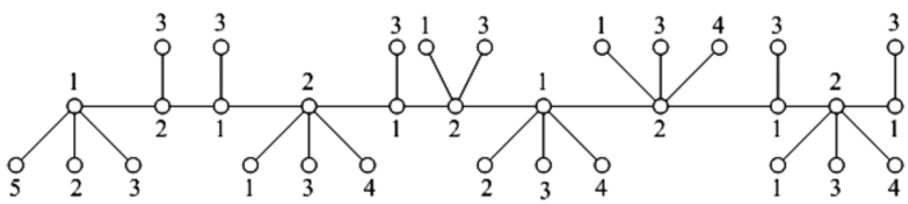

Figure 2. A minimum locating-coloring of C(11; 3, 1, 1, 3, 1, 2, 3, 3, 1, 3, 1).

3. Locating-chromatic Number of Non-homogeneous Firecracker Graphs

A non-homogeneous firecracker graph, $F_{n, (k_1, k_2, ..., k_n)}$ is obtained by the concatenation of n stars $S_{k_i},$ $i = 1, 2, ..., n$ by linking one leaf from each star.

Let $V(F_{n, (k_1, k_2, ..., k_n)}) = \{x_i, m_i, l_{ij} \mid i = 1, 2, ..., n; \; j = 1, 2, ..., k_i - 2\},$

and $E(F_{n,(k_1, k_2, \dots, k_n)}) = \{x_i x_{i+1} | i = 1, 2, \dots, n-1\} \cup \{x_i m_i, m_i l_{ij} | i = 1, 2, \dots, n; j = 1, 2, \dots, k_i - 2\}$. If $k_{\max} = \max\{k_1, k_2, \dots, k_n\}$, then the subgraph $S_{k_{\max}}$ is *the maximum star subgraph* of non-homogeneous firecracker $F_{n,(k_1, k_2, \dots, k_n)}$. If there are p subgraphs $S_{k_{\max}}$, then every subgraph, from left to right, is denoted by $S^i_{k_{\max}}$, where $1 \le i \le p$.

Definition 2. Let $S_{k_i}, S_{k_j} \subset F_{n,(k_1, k_2, \dots, k_n)}$, where $1 \le i \ne j \le m$. If $n_i = n_j \ne n_{\max}$, such that

(1) $d(x_i, x_m) = d(x_j, x_m)$, with x_m is the center of $S_{k_{\max}}$, or

(2) $d(x_i, x_0) = d(x_j, x_p)$, with x_0 and x_p, $x_0 \ne x_p$ are the centers of $S_{k_{\max}}$,

then subgraphs S_{k_i} and S_{k_j} are called star subgraphs with the same distance.

Theorem 3. *Let $S_{k_{\max}}$ be the maximum star subgraph of $F_{n,(k_1, k_2, \dots, k_n)}$ and p be the number of subgraphs $S_{k_{\max}}$. Then the locating-chromatic number of $F_{n,(k_1, k_2, \dots, k_n)}$, for $n_{\max} \ge 2$ is*:

$$\chi_L(F_{n, k_1, k_2, \dots, k_n}) = \begin{cases} k_{\max} - 1, & \text{if } p \le k_{\max} - 1, \\ k_{\max}, & \text{if } p > k_{\max} - 1. \end{cases}$$

Proof. We determine the trivial lower bound of $p \le k_{\max} - 1$. Since the number of leaves in the maximal subgraph is $k_{\max} - 2$, by Corollary 1, we have $\chi_L(F_{n,(k_1, k_2, \dots, k_n)}) \ge k_{\max} - 1$, for $p \le k_{\max} - 1$.

Consider $(k_{\max} - 1)$-coloring c on $F_{n, k_1, k_2, \dots, k_n}$ as follows:

a. Find the number of subgraphs $S_{k_{\max}}$ and denote it by p. Denote each of the subgraphs from left to right by $S^i_{k_{\max}}$, where $1 \le i \le p$.

b. Vertices $x_i \in S^i_{k_{\max}}$, where $1 \le i \le p$, are colored by $3, 4, 5, ..., n, 2, 3,$ respectively.

c. Vertices $m_i \in S^i_{k_{\max}}$, where $1 \le i \le p$, are colored by $1, 2, 3, ..., p$, respectively.

d. Leaves in $S^i_{k_{\max}}$, where $1 \le i \le p$ are colored by $\{1, 2, 3, ..., (k_{\max} - 1)\} \backslash \{c(x_i)\}$.

e. Let A_1 be an open interval before $S^i_{k_{\max}}$, A_{t+1} be an open interval between $S^t_{k_{\max}}$ and $S^{t+1}_{k_{\max}}$, where $1 \le t \le p - 1$, and A_{p+1} be an open interval after $S^p_{k_{\max}}$.

f. Let $T = \{$all combinations $(k_{\max} - 2)$ from $(k_{\max} - 1)$ color$\}$, such that $T = \{T_1, T_2, ..., T_{k_{\max}-1}\}$ with $T_i \in T$ be combinations not containing color i.

g. Identify subgraph S_{k_i} in the interval as defined in item d.

h. If S_{k_i} lies in the interval A_1 or A_2, then every vertex of S_{k_i} is colored by colors that correspond with T_1, respectively.

i. If S_{k_i} lies in the interval A_k, where $3 \le k \le p + 1$, then every vertex of S_{k_i}, is colored by colors that correspond with T_{k-1}, respectively.

j. Let S_{k_i} and S_{k_j}, where $k_i = k_j$ have the same distance from the maximum star subgraph. If $c(x_i) = c(x_j)$, then $c(m_i) = c(m_j)$.

Next, we show that the color codes for all vertices in $F_{n, k_1, k_2, ..., k_n}$ for $p \le k_{\max} - 1$, are different. Consider two distinct vertices $u \in V(S_{k_i})$ and $v \in V(S_{k_j})$, where $c(u) = c(v)$.

- If $k_i = k_j = k_{\max}$, then $c_\Pi(u) \neq c_\Pi(v)$ because their color codes differ in ordinate of colors m_i and m_j.

- If S_{k_i} and S_{k_j} lie in different intervals, say A_p and A_q, then $c_\Pi(u) \neq c_\Pi(v)$ because they have different distances from C_p and C_q.

- If S_{k_i} and S_{k_j} lie in the same interval, say A_p but they do not have the same distance, then $c_\Pi(u) \neq c_\Pi(v)$ because they have different distance to C_p. But, if they have the same distance, their color codes differ in ordinates m_i and m_j.

- If one of $\{k_i, k_j\}$ is $k_{\max}$, say $k_i = k_{\max}$ and $n_j < k_{\max}$, then the color codes of u and v differ in color of leaf of S_{k_i}, not contained in S_{k_j}.

- If $x_i \in V(K_{1,n_j})$ and v lie in the same interval, say A_k, where $c(x_i) = c(v)$, then $c_\Pi(x_i)$ and $c_\Pi(v)$ have different distance from C_k. So, $c_\Pi(x_i) \neq c_\Pi(v)$. But, if they lie in different intervals, say A_r and A_s, then $c_\Pi(x_i) \neq c_\Pi(v)$ because they have different distances from C_r and C_s.

From all above cases, we see that the color codes for all vertices in $F_{n,k_1,k_2,\ldots,k_n}$, for $p \leq k_{\max} - 1$ are different, thus c is a locating-coloring. So, $\chi_L(F_{n,(k_1,k_2,\ldots,k_n)}) \leq k_{\max} - 1$, for $p \leq k_{\max} - 1$.

We will show the lower bound for $p > k_{\max} - 1$. By Corollary 1, we have that $\chi_L(F_{n,(k_1,k_2,\ldots,k_n)}) \geq k_{\max} - 1$. However, we will show that $k_{\max} - 1$ colors are not enough. For a contradiction that there exists a $(k_{\max} - 1)$-locating coloring c on $F_{n,(k_1,k_2,\ldots,k_n)}$, for $p > k_{\max} - 1$. Since $p > k_{\max}$

-1, there are two $i, j,$ $i \neq j,$ such that $\{c(l_{ih}) | h = 1, 2, ..., k_{\max} - 2\} = \{c(l_{jl}) | l = 1, 2, ..., k_{\max} - 2\}$. Therefore, the color codes of m_i and m_j are the same, a contradiction. So, $\chi_L(F_{n,(k_1,k_2,...,k_n)}) \geq k_{\max}$ for $p > k_{\max} - 1$.

Next, we determine the upper bound of $F_{n,(k_1,k_2,...,k_n)}$ for $p > k_{\max} - 1$. To show that $\chi_L(F_{n,(k_1,k_2,...,k_n)}) \leq k_{\max}$, consider the locating-coloring c on $F_{n,(k_1,k_2,...,k_n)}$ as follows:

- $c(x_i) = 1$ if i is odd and $c(x_i) = 3$ if i is even.
- $c(m_1) = k_{\max}$ and $c(m_i) = 2$ for i otherwise.
- If $A = \{1, 2, ..., k_{\max}\}$, define:

$$\{c(l_{ij}) | j = 1, 2, ..., k_i - 2\} = \begin{cases} A \backslash \{1, k_{\max}\} & \text{if } i = 1, \\ A \backslash \{2, k_{\max}\} & \text{otherwise.} \end{cases}$$

It is easy to verify that the color codes of all vertices are different. Therefore, c is a locating-chromatic coloring on $F_{n,(k_1,k_2,...,k_n)}$, and so $\chi_L(F_{n,(k_1,k_2,...,k_n)}) \leq k_{\max}$, for $p > k_{\max} - 1$. This completes the proof.

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